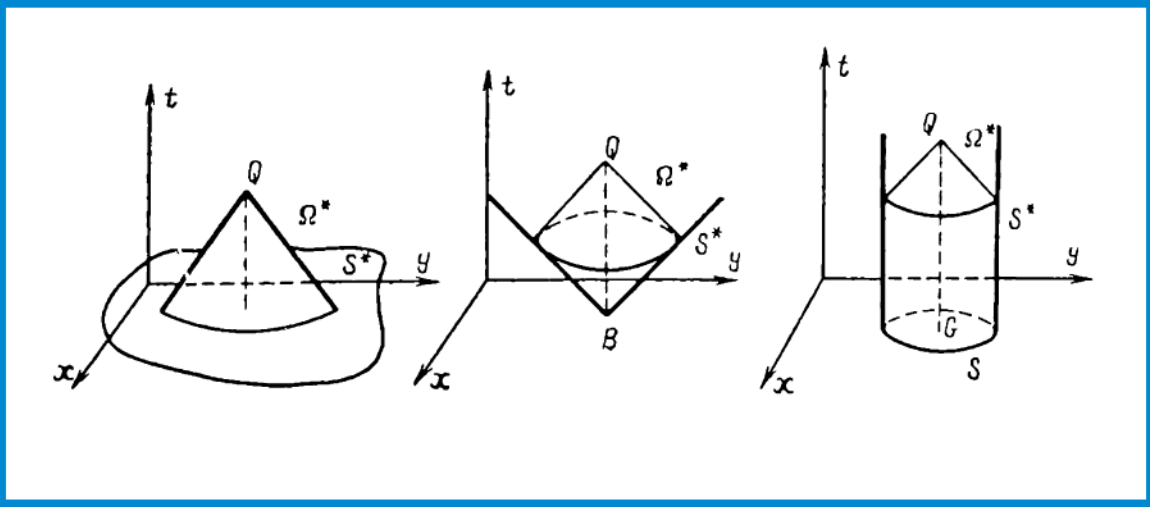
G. N. POLOZHIY

Equations of Mathematical Physics



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Preface

This book was written as a result of reworking and supplementing lecture notes for a course in equations of mathematical physics which the author has given for a number of years in the Department of Mathematics and Mechanics at the Kiev State University.

In mathematical physics one is concerned with questions which are closely connected with the study of various physical phenomena. Examples of these occur in fluid dynamics, electrodynamics, elasticity theory, heat flow, quantum mechanics, atomic physics, etc. These phenomena give rise to mathematical problems which have many elements in common, and which constitute the subject matter of mathematical physics.

Mathematical physics encompasses an extremely wide range of questions. By "equations of mathematical physics" one usually means the mathematical discipline which has for its subject matter the study of questions in physics and mechanics which are associated with partial differential equations.

The distinctive feature of equations of mathematical physics lies in the formulation of these problems in partial differential equations, which follows from physical considerations. The procedure for solving the problems is based on mathematical methods, but in every case the solution itself must have a definite physical interpretation regardless of the mathematical method used to obtain it.

For the sake of mathematical rigor and elegance, and without sacrificing any of the physical content, we have thought it expedient to depart from the traditional development of the subject, in which questions associated with hyperbolic equations and separation of variables are presented very early, as a rule. Our presentation is characterized, first of all, by the fact that the study of parabolic and hyperbolic equations is preceded by a detailed investigation of general properties of elliptic equations and the associated mathematical apparatus. Separate chapters are assigned to the examination of physical processes which lead to the fundamental equations of mathematical physics, and to general questions in the theory of partial differential equations. The method of separation of variables and the method of integral transformations are applied simultaneously to all three types of equations: hyperbolic, parabolic, and elliptic.

In every chapter we have tried to include as many problems as possible for which solutions can be obtained in closed form.

In view of definite requirements concerning the volume of this book, we have been unable to present with the same degree of thoroughness all the various questions in the extremely large group related to equations of mathematical physics. For more detailed familiarity with specific questions, we have recommended to the reader some specialized periodical literature and various monographs.

The author takes this opportunity to express his gratitude to Professor P. V. Cherpakov, Head of the Department of Mathematical Physics at Voronezh State University, and to V. M. Babich, Docent in the Department of Mathematical Physics at the Leningrad State University, both of whom read the manuscript of this book and made a number of valuable suggestions which have contributed to its improvement.

The author will be grateful to readers for any comments and suggestions.

The author

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CHAPTER 1

Physical and Mechanical Problems Which Lead to the Basic Equations of Mathematical Physics

In the present chapter, we shall look at various physical processes (from hydrodynamics, acoustics, electrodynamics, heat-flow theory, diffusion theory, etc.) which lead to the basic and most frequently encountered equations of mathematical physics.

All the physical quantities with which we shall deal are measured in the International System of Units. Only in connection with electrostatic and electromagnetic fields (see section 2, subsections 5, 6, and 7) do we use the absolute Gaussian system of units and here we will give the coefficients for converting these units into the units of the International System.

1. Basic Concepts

A *system of partial differential equations* is a collection of equations that relates the values of certain independent variables with the values of the unknown functions and their partial derivatives with respect to the independent variables.

A system of partial differential equations is said to be *complete* or *determined* if the number of unknown functions in that system and the number of equations are the same; it is said to be *underdetermined* if the number of unknown functions is greater than the number of equations; it is said to be *overdetermined* if the number of unknown functions is less than the number of equations.

Ordinarily, by a system of partial differential equations, we mean a determined system and by a partial differential equation, we mean a determined system consisting of a single equation.

The highest order of any of the derivatives of the unknown functions appearing in the system is called the *order* of the system of equations.

A system of equations is said to be *linear* if it is linear in all the unknown functions and their derivatives; it is said to be *quasilinear* if it is linear in the highest-order derivatives of the unknown functions.

A solution of a system of equations is any set of functions which, when substituted into the system of equations in place of the unknown functions, will reduce that system to an identity.

In connection with systems of partial differential equations, it is of fundamental interest to find solutions which satisfy some supplementary conditions. As a rule, these supplementary conditions are equations which give the values of the unknown functions or some of their derivatives on the boundary of the region in which we are seeking a solution, or statements specifying the singular behavior of the unknown functions. In the general case, these supplementary conditions are called *boundary conditions*. In those cases in which they specify the values of the unknown functions and their derivatives for a fixed value of one of the independent variables—for example, the time t—they are also called *initial conditions*.

Problems of finding solutions for a system of partial differential equations satisfying specified supplementary conditions are called, in general, *boundary-value problems of mathematical physics*. The formulation of boundary-value problems of mathematical physics is very closely connected with the study of specific physical problems.

Of all the possible partial differential equations, those most frequently encountered and most thoroughly studied are certain well-defined equations known as the *basic equations of mathematical physics*.

Let x, y, and z denote the coordinates in a three-dimensional rectangular Cartesian system; let t denote the time; let F be a given function of the variables x, y, z, and t; suppose a, b, and c are real constants, u an unknown function, and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ the so-called *Laplace differential operator*. Then, these basic equations of mathematical physics can be written in one of the forms

$$\Delta u = -F, \tag{1}$$

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -F, \tag{2}$$

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -F, \tag{3}$$

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + cu = -F. \tag{4}$$

Equation (1) for $F \equiv 0$ is called *Laplace's equation;* for $F \not\equiv 0$, it is called *Poisson's equation.* Equations (2), (3), and (4) are called, respectively, the *heat equation,* the *wave equation,* and the *telegraph equation.*

2. The Simplest Physical and Mechanical Problems Which Lead to the Basic Equations of Mathematical Physics

We shall show that many physical processes lead to the necessity of studying the basic equations of mathematical physics. In the course of this, we shall frequently make use of the formulas of Ostrogradskiy, Green, and Stokes.

Let $\vec{v}$ denote a vector which is continuously differentiable in some closed two- or three-dimensional region, let $\vec{n}$ denote the inner normal to the boundary of that region, let v_n denote the projection of $\vec{v}$ onto the normal $\vec{n}$. Then the formulas of Ostrogradskiy, Green, and Stokes can be written in the following manner:*

$$-\iint_S v_n dS = \iiint_G \operatorname{div} \vec{v}\, d\tau; \tag{5}$$

$$-\int_C v_n ds = \iint_D \operatorname{div} \vec{v}\, d\sigma; \tag{5'}$$

$$\int_C v_s ds = \iint_\sigma (\operatorname{curl} \vec{v})_\nu d\sigma. \tag{5''}$$

Here G is a domain in the xyz-space; S is a piecewise-smooth surface bounding G which intersects every straight line parallel to any of the coordinate axes at a finite number of points, or has a finite number of segments in common with it, C is a piecewise-smooth contour bounding a domain D of the xy-plane in the case of formula (5′) and bounding the surface σ in xyz-space in the case of formula (5″). Here, the curve C can be partitioned into a finite number of parts on each of which the coordinates of its points vary monotonically; $\vec{\nu}$ is the positive normal to the surface σ, i.e., the normal to the side from which the positive direction around C is counterclockwise; $(\operatorname{curl} \vec{v})_\nu$ is the projection of $\operatorname{curl} \vec{v}$ onto the normal $\vec{\nu}$; v_s is the projection of $\vec{v}$ onto the positive direction of the tangent to C. These conditions of applicability of formulas (5), (5′), and (5″) will be assumed to be fulfilled in what follows, and the meanings of these notations will remain unchanged throughout this chapter.

---

*The first two of these formulas are frequently referred to as Gauss' theorem, or the divergence theorem [Ed.].

1. Propagation of heat and diffusion. Diffusion with disintegration and with chain reactions. Consider an isotropic body. Let u denote its temperature, ρ its density, γ its specific heat, f the intensity of heat sources, i.e., the amount of heat liberated per unit volume in a unit of time.

Let us calculate the balance of heat in a unit of time of those particles of the body occupying a volume G. In accordance with Fourier's hypothesis, which agrees with experiment, the amount of heat entering G through a surface element ΔS is given by the formula

$$\Delta Q = -k \frac{\partial u}{\partial n} \Delta S,$$

where k is a positive proportionality constant characterizing the body and is known as the *thermal conductivity*. Consequently, the amount of heat entering G through the surface S is, by Ostrogradskiy's formula, equal to

$$-\iint_S k \frac{\partial u}{\partial n} dS = \iiint_G \operatorname{div}(k \nabla u)\, d\tau,$$

where ∇u is the gradient of the function u. The total amount of heat entering G is given by the equation

$$Q_1 = \iiint_G \operatorname{div}(k \nabla u)\, d\tau + \iiint_G f\, d\tau,$$

where the second term on the right side of the equation is the heat added as a result of the sources.

To increase the temperature of an element of volume $d\tau$ by an amount du during an interval Δt requires an amount of heat equal to

$$\gamma\, du \rho\, d\tau = \gamma\rho \frac{\partial u}{\partial t} \Delta t\, d\tau.$$

The total amount of heat per unit time which contributes to the temperature increase of the particles of the body occupying the volume G will be

$$Q_2 = \iiint_G \gamma\rho \frac{\partial u}{\partial t}\, d\tau.$$

Setting $Q_1 = Q_2$, we obtain

$$\iiint_G [\operatorname{div}(k \nabla u) + f]\, d\tau = \iiint_G \gamma\rho \frac{\partial u}{\partial t}\, d\tau.$$

Since the domain G is arbitrary, we obtain from the mean-value theorem

$$\operatorname{div}(k\nabla u) - \gamma\rho\frac{\partial u}{\partial t} = -f. \tag{6}$$

Equation (6) is the differential equation for the flow of heat in a nonhomogeneous body. When the body is homogeneous, this equation is written in the form

$$\Delta u - \frac{1}{a^2}\cdot\frac{\partial u}{\partial t} = -\frac{f}{k},\quad a = \sqrt{\frac{k}{\gamma\rho}} \tag{7}$$

and is one of the basic equations of mathematical physics, namely, the heat equation. In particular, if the heat flow is steady, i.e., independent of time, this equation becomes *Poisson's equation* and if, in addition, there are no heat sources, it becomes *Laplace's equation.*

Suppose that some medium is filled nonuniformly with a gas, or that some dissolved substance is distributed nonuniformly throughout a volume containing the solvent. Under these conditions, there will be a diffusion of the particles of the gas or substance in question from regions of high concentration to regions of lower concentration. Here, by concentration, we mean the function

$$u = \frac{dQ}{d\tau},$$

where dQ is the amount of substance or gas in an element of volume $d\tau$ filled by the solution or gas. According to Nernst's experimental law, the amount of substance or gas diffusing in a volume G in a unit of time through an element of surface ΔS is given by the formula

$$\Delta Q = -D\frac{\partial u}{\partial n}\Delta S,$$

where D is a positive coefficient known as the *coefficient of diffusion.* Consequently, the total amount of substance or gas entering G in a unit of time will be

$$-\iint_S D\frac{\partial u}{\partial n}dS + \iiint_G f\,d\tau = \iiint_G [\operatorname{div}(k\nabla u) + f]\,d\tau,$$

where the second term on the left is the amount of substance or gas appearing as a result of the sources, and f is the strength of these sources. In order to change the concentration in a volume element $d\tau$ by an amount du during a time interval Δt, an amount of gas, or of the substance, equal to

$$c\,du\,d\tau = c\,\frac{\partial u}{\partial t}\,\Delta t\,d\tau,$$

is required. Here, c is a positive constant of proportionality, known as the *coefficient of porosity.** The total amount of material contributing to the change in the concentration in a volume G in unit time will be

$$\iiint_G c\,\frac{\partial u}{\partial t}\,d\tau.$$

Now, according to the law of conservation of mass, we have

$$\iiint_G [\operatorname{div}(D\nabla u) + f]\,d\tau = \iiint_G c\,\frac{\partial u}{\partial t}\,d\tau.$$

From this we obtain, just as above, the equation for diffusion in a nonhomogeneous medium

$$\operatorname{div}(D\nabla u) - c\,\frac{\partial u}{\partial t} = -f. \tag{8}$$

In the case of a homogeneous medium, this equation takes the form

$$\Delta u - \frac{1}{a^2}\,\frac{\partial u}{\partial t} = -\frac{f}{D}, \quad \text{where} \quad a = \sqrt{\frac{D}{c}} \tag{9}$$

and coincides with the heat-flow equation.

In the diffusion of certain gases (for example, the emanation of radium), there is a disintegration of the molecules of that gas. The speed of disintegration is naturally assumed to be proportional to the concentration of the gas. Therefore, when there is disintegration, the diffusion equation (9) takes the form

$$\Delta u - \frac{1}{a^2}\,\frac{\partial u}{\partial t} - \frac{\beta}{D}\,u = -\frac{f}{D}, \tag{9'}$$

where β is a positive coefficient of proportionality characterizing the speed of the disintegration. It is easy to see this if we replace the intensity of the sources f in Eq. (9) with $f - \beta u$. In the case of steady-state diffusion, Eq. (9') leads to an equation of the following form:

---

*The coefficient of porosity is the ratio of the volume of pores filled by the solution or gas to the total volume in which these pores are distributed.

$$\Delta u - k^2 u = -\frac{f}{D}, \quad k = \sqrt{\frac{\beta}{D}}. \tag{9''}$$

Diffusion processes when there are "chain reactions" are of special interest. Chain reactions are characterized by the fact that the particles of the diffusing material or gas "multiply" upon reacting with the surrounding medium. Thus, when a neutron collides with an "active" nucleus of uranium, there occurs nuclear fission accompanied by the appearance of new neutrons, which in turn react with active nuclei and cause the appearance of new neutrons, and the process is continued. If this process is treated approximately as a diffusion process by setting the velocity of the reaction proportional to the concentration (that is, to the density of neutrons), we must take the intensity of the sources as equal to $f+\beta u$, where u is the concentration and β is a positive proportionality constant characterizing the speed of the chain reaction. Thus, the diffusion equation (9) for a chain reaction takes the form

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} + \frac{\beta}{D}u = -\frac{f}{D}. \tag{9'''}$$

In the case of steady-state diffusion in the presence of a chain reaction, Eq. (9''') reduces to the following form:

$$\Delta u + k^2 u = -\frac{f}{D}, \quad k = \sqrt{\frac{\beta}{D}}. \tag{9''''}$$

Let us show that Eq. (9'), just like Eq. (9'''), can by a simple change of unknown functions, be reduced to the heat equation. Specifically, if we set $v = ue^{-a^2ct}$ in the equation

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} + cu = -\frac{f}{k} \quad (a, c = \text{const}),$$

this equation is converted into the heat equation of the following form:

$$\Delta v - \frac{1}{a^2}\frac{\partial v}{\partial t} = -\frac{f}{k}e^{-a^2ct}.$$

2. The potential flow of an incompressible fluid. The equation of continuity. Let $\vec{v}$ be a vector denoting the velocity of the fluid particles and consider it to be a function of x, y, z, and t. Let v_x, v_y, and v_z be its component in the directions of the x-, y-, and z-axes. Let ρ denote the density of the fluid and let f denote the intensity of the sources, i.e., the amount of fluid emitted by a unit volume in unit time.

The total amount of fluid passing through a surface S into a volume G plus the amount emitted by the sources in unit time is

$$Q = \iint_S \rho v_n \, dS + \iiint_G f \, d\tau = \iiint_G [-\operatorname{div}(\rho \vec{v}) + f] \, d\tau.$$

For the density of the fluid in an element of volume $d\tau$ to increase by an amount $d\rho$ in an interval Δt, an amount of liquid $d\rho d\tau = (\partial\rho/\partial t)\,\Delta t d\tau$ is required. The total amount of fluid needed to increase the density in the volume G in a unit of time is

$$\iiint_G \frac{\partial \rho}{\partial t} d\tau.$$

By setting this amount of fluid equal to Q, just as before, we obtain the so-called *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = f. \tag{10}$$

This equation of continuity is valid regardless of the particular properties of the fluid in question and applies in general to an arbitrary continuous medium in motion.

Since $v_x = dx/dt$, $v_y = \partial y/dt$ and $v_z = dz/dt$, we can also write the equation of continuity in the form

$$\frac{d\rho}{dt} + \rho \operatorname{div} \vec{v} = f, \tag{10'}$$

and, if the fluid is incompressible (that is, if ρ = const), we obtain

$$\operatorname{div} \vec{v} = \frac{f}{\rho}. \tag{10''}$$

In the case of potential flow of a fluid $\vec{v} = -\nabla u$ or, in expanded form, $v_x = -\partial u/\partial x$, $v_y = -\partial u/\partial y$, and $v_z = -\partial u/\partial z$, where u is a function, called a *potential function* of the flow, of the variables x, y, and z. By using these equations, we obtain from the equation of continuity the equation for potential flow of an incompressible fluid

$$\Delta u = -\frac{f}{\rho}. \tag{11}$$

This is *Poisson's equation* and when $f \equiv 0$, it is *Laplace's equation*.

3. The hydrodynamical equations for an ideal fluid. By an ideal fluid, we mean a fluid in which there are no frictional forces between the particles of the fluid or, what amounts to the same thing, there are no viscous forces. Let $\vec{v}$, ρ, and f have the

same meanings as before, let p denote pressure and let $\vec{F}$ denote a body force, i.e., an external force per unit mass of fluid, for example, gravity. Let us consider those fluid particles which at time t occupy the volume G, and let us apply the law of conservation of momentum of this collection of particles. *The time derivative of the momentum of the system is equal to the resultant of the external force applied to the system.* Mathematically, when this law is applied to the indicated collection of fluid particles, it is written in the form

$$\frac{d}{dt}\iiint_G \rho\vec{v}\,d\tau = \iint_S \vec{n}p\,dS + \iiint_G \rho\vec{F}\,d\tau.$$

Here, when we evaluated the forces applied to the volume G from the side of the $\vec{n}$ normal to the surface S, we made essential use of the fact that, in an ideal fluid, there can be no tangential forces.

The quantity $\rho d\tau$ is independent of time by virtue of the law of conservation of mass. Therefore,*

$$\frac{d}{dt}\iiint_G \rho\vec{v}\,d\tau = \iiint_G \frac{d\vec{v}}{dt}\rho\,d\tau.$$

Now, from Ostrogradskiy's formula, we have

$$-\iint_S \vec{n}p\,dS = \iiint_G \nabla p\,d\tau.$$

Consequently, the law of conservation of momentum can be written in the form

$$\iiint_G \frac{d\vec{v}}{dt}\rho\,d\tau = -\iiint_G \nabla p\,d\tau + \iiint_G \rho\vec{F}\,d\tau.$$

Therefore, since G is arbitrary, we have

$$\frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho} + \vec{F}. \tag{12}$$

This is known as *Euler's equations of motion.*

---

*This equation can also be justified in a simple manner by reducing the integration over the volume at time $t+\Delta t$ to integration over the volume G by making a change of variables. We have

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\iiint_{G'}\vec{v}'\rho'\,d\tau' - \iiint_G \vec{v}\rho\,d\tau\right] = \lim_{\Delta t\to 0}\iiint_G \frac{\vec{v}'-\vec{v}}{\Delta t}\rho\,d\tau = \iiint_G \frac{d\vec{v}}{dt}\rho\,d\tau,$$

where G', $\vec{v}'$, ρ', and $d\tau'$ are, respectively, the same as G, $\vec{v}$, ρ, and $d\tau$, not at the instant t, however, but at the instant $t+\Delta t$.

If the fluid is incompressible, the equation of continuity (10″) and Euler's equations of motion (12) constitute a complete system of hydrodynamic equations for an ideal incompressible fluid. Here, the unknown functions are $\vec{v}$ and p, and the total number of equations is 4. If the liquid is compressible, the complete system of differential equations for an ideal fluid consists of the equation of continuity (10), Euler's equations of motion (12), and a so-called *equation of state* giving the relationship

$$p = \Phi(\rho) \tag{13}$$

between the pressure p and the density ρ, or else giving the relationship between the pressure p, the density ρ and the absolute temperature T:

$$p = \Phi_1(\rho, T), \quad p = \Phi_2(\rho, T). \tag{13'}$$

4. The equations of gas dynamics and acoustics. The motion of a gas may be regarded as the motion of an ideal fluid. Here, it is assumed that there are no sources of the gas and no body forces. At high velocities (higher than 300-400 km per hour), it becomes necessary to take the compressibility of the gas into consideration. Therefore, to obtain the complete system of equations of motion of the gas, we must consider not only the equation of continuity and the Euler equations of motion but also the equation of state of the gas. Under the additional hypothesis that the processes in the moving gas take place adiabatically, such an equation of state is given by

$$\frac{p}{\rho^{\varkappa}} = c = \text{const}, \quad \varkappa = \frac{c_p}{c_v},$$

where p and ρ are the same functions as in subsection 3 and where c_p and c_v are the specific heats of the gas at constant pressure and constant volume respectively (for air, $\varkappa \simeq 1.41$).

The equation of state of the gas can also be written in another form by introducing the absolute temperature T as an additional unknown. In this case, the equation of state will consist of the previously stated adiabatic equation and the perfect gas law

$$p = \rho RT,$$

where R is the gas constant.

Thus, the equations for the adiabatic motion of a gas at high velocities, known as the complete system of gas-dynamic equations, are written in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0, \quad \frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho}; \qquad \frac{p}{\rho^{\varkappa}} = c = \text{const}, \quad \varkappa = \frac{c_p}{c_v}. \tag{14}$$

For small velocities, the gas may be treated as an incompressible ideal fluid, and in this case, the system of equations—the complete system of aerodynamic equations—can, by virtue of the conclusions of subsection 3, be written in the form

$$\operatorname{div} \vec{v} = 0, \quad \frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho}. \tag{15}$$

It is natural to regard the propagation of sound waves in space as adiabatic longitudinal vibrations of particles of the gas about their equilibrium positions, in the direction of propagation of the waves. Let p_0 and ρ_0, in contrast with p and ρ, denote the pressure and density of the gas at rest. Because of the relatively high frequency of vibrations, the condensation of the gas $s = (\rho - \rho_0)/\rho_0$, the velocity vector $\vec{v}$, the derivatives $\partial\vec{v}/\partial x$, $\partial\vec{v}/\partial y$ and $\partial\vec{v}/\partial z$, and ∇s will all be sufficiently small, so that we can agree to neglect all quantities of order higher than these. Then, since $\rho = \rho_0(1+s)$ and since $p = p_0(1+s)^{\varkappa}$ (the adiabatic equation), we have

$$p \simeq p_0(1 + \varkappa s), \quad \nabla p = p_0 \varkappa (1+s)^{\varkappa - 1} \nabla s \simeq p_0 \varkappa \nabla s, \quad \frac{d\vec{v}}{dt} \simeq \frac{\partial \vec{v}}{\partial t},$$

and Euler's equations of motion take the form

$$\frac{\partial \vec{v}}{\partial t} = -a^2 \nabla s, \quad a = \sqrt{\frac{\varkappa p_0}{\rho_0}}.$$

Therefore, assuming that at the initial instant of time the vector $\vec{v}$ is derivable from a potential: $\vec{v}|_{t=0} = -\nabla \Phi(x, y, z)$, we obtain

$$\vec{v} = -\nabla u, \quad u = \Phi(x, y, z) + a^2 \int_0^t s\, dt, \quad s = \frac{1}{a^2} \frac{\partial u}{\partial t}.$$

By virtue of the equations

$$\operatorname{div} \rho \vec{v} = \rho \operatorname{div} \vec{v} + \vec{v} \nabla \rho = \rho_0 (1+s) \operatorname{div} \vec{v} + \rho_0 \vec{v} \nabla s \simeq \rho_0 \operatorname{div} \vec{v}$$

the equation of continuity may be written in the form

$$\frac{\partial s}{\partial t} + \operatorname{div} \vec{v} = 0. \tag{15'}$$

From this it follows that the potential function u satisfies the so-called *acoustic equation:**

$$\Delta u - \frac{1}{a^2} \cdot \frac{\partial^2 u}{\partial t^2} = 0, \quad a = \sqrt{\frac{\varkappa p_0}{\rho_0}}, \tag{16}$$

which is one of the basic equations of mathematical physics, namely, the wave equation. The pressure, condensation, and density of the gas satisfy this equation since they are simply expressed in terms of the potential function:

$$s = \frac{1}{a^2}\frac{\partial u}{\partial t}, \quad \rho = \rho_0\left(1 + \frac{1}{a^2}\frac{\partial u}{\partial t}\right), \quad p = p_0\left(1 + \frac{\varkappa}{a^2}\frac{\partial u}{\partial t}\right). \tag{16'}$$

5. The equations of electrostatics and of a constant electric current. Suppose in some dielectric medium a constant electric field is created by given electric charges. Let $\vec{E}$ denote the electric field intensity, ρ the charge density, and ε the dielectric constant.

Suppose that an electric charge e is concentrated at a point in a homogeneous medium (i.e., one in which ε =const). Then, according to Coulomb's law, this charge will create an electric field whose intensity is given by

$$\vec{e} = \frac{\vec{r}}{\varepsilon r^3} e = -\nabla\left(\frac{e}{\varepsilon r}\right),$$

where $\vec{r}$ is the vector connecting the point at which the charge e is located with a variable point in space and where r is the length of that vector.** The flux of the vector $\vec{\varepsilon e}$ through an arbitrary closed surface S enclosing the charge satisfies the equations

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*We note that when there are body forces, the same assumptions lead to

$$\frac{\partial \vec{v}}{\partial t} = a^2 \nabla s + \vec{F}, \quad \text{and} \quad \vec{v} = -\nabla u + \int_0^t F\,dt\,.$$

Then, by virtue of (15'), Eq. (16) should be replaced with the equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \operatorname{div}\int_0^t F\,dt.$$

**Here, all quantities are taken in the absolute electrostatic system of units, i.e., the electrostatic CGS units. In this system, the dielectric constant ε is a dimensionless quantity ≥ 1; in a vacuum, $\varepsilon = 1$. Here, a unit charge is defined on the basis of Coulomb's law as a charge that acts on an identical charge in a vacuum at a distance one centimeter away with a force of one dyne. The dimensions of electric charge are $L^{3/2} M^{1/2} T^{-1}$. The intensity of the electric field $\vec{E}$ is the force acting on a unit positive charge of electricity. As our unit of electric field intensity in the electrostatic CGS system, we take the intensity at a point in the field at which a force of one dyne is acting on a unit charge as defined above.

$$\iint_S \varepsilon e_n \, dS = \iint_{S_1} \left(\frac{\vec{r}}{r^3}\right)_n e \, dS = 4\pi e,$$

where S_1 is a sphere with center at the point charge.

A charge $\rho d\tau$ located in an element of volume $d\tau$ can be regarded as a point charge and a given electric field of intensity $\vec{E}$ as the field caused by a superposition of point charges. On the basis of this, the vector field $\vec{E}$, being the resultant of gradient fields, will be a gradient field; that is, $\vec{E} = -\nabla u$ or, what amounts to the same thing

$$\int_C E_s \, ds = 0. \tag{17}$$

Also,

$$-\iint_S \varepsilon E_n \, dS = 4\pi \iiint_G \rho \, d\tau. \tag{18}$$

In the case of a nonhomogeneous medium, i.e., for $\varepsilon \neq$ const, Eqs. (17) and (18) can be established by direct experiment. They may be regarded as a *generalization of Coulomb's law.* From a theoretical point of view, Eq. (17) may be regarded as a consequence of the principle of conservation of energy since the left member of this equation is the work done by a unit positive charge on going around a closed contour and this work cannot be different from zero because of the constancy of the given electric field. Equation (18) can be interpreted as the law of conservation of flux of the vector $\vec{\varepsilon E}$.

If we write Eqs. (17) and (18) in accordance with Stokes' and Ostrogradskiy's formulas in the forms

$$\int_C E_s \, ds = \iint_\sigma (\operatorname{curl} \vec{E})_\nu \, d\sigma, \quad \iiint_G \operatorname{div}(\varepsilon \vec{E}) \, d\tau = 4\pi \iiint_G \rho \, d\tau,$$

we immediately obtain, because of the arbitrariness of G and σ, the so-called *system of equations of electrostatics:*

$$\operatorname{curl} \vec{E} = 0, \qquad \operatorname{div}(\varepsilon \vec{E}) = 4\pi\rho. \tag{19}$$

It follows from this system that not only will the vector field $\vec{E}$ be a gradient field ($E = -\Delta u$) but the potential function u will satisfy the so-called electrostatic equation

$$\operatorname{div}(\varepsilon \nabla u) = -4\pi\rho. \tag{19'}$$

In the case of a homogeneous medium, this last equation is *Poisson's equation,* and when there are no charges, it is *Laplace's equation.*

Suppose that there is a steady electric current with volume density $\vec{j}(x, y, z)$ in a homogeneous conducting medium. If there are no current sources in the medium, we have

$$\operatorname{div} \vec{j} = 0. \tag{19''}$$

The electric field $\vec{E}$ can be determined from the current density by use of Ohm's law, which in differential form is

$$\vec{E} = \frac{\vec{j}}{\lambda}, \tag{19'''}$$

where λ is the conductivity of the medium. In the case of a steady electric current, the vector $\vec{E}$ will be a potential vector*; i.e., there exists a scalar function $u(x, y, z)$ such that

$$\vec{E} = -\nabla u.$$

In view of this, we obtain from (19'') the equation for a constant electric current

$$\Delta u = 0 \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

This equation coincides with the three-dimensional form of Laplace's equation.

6. The equation of magnetostatics. Consider a constant magnetic field created by stationary magnets or by a constant electric current in some medium. Let $\vec{H}$ be the magnetic intensity vector, $\vec{j}$ the current density, and μ the magnetic permeability of the medium.

Before going into the mathematical calculations, let us give a physical interpretation of the quantities that we have introduced and the corresponding units used in measuring them.

Consider a circular conductor along which a constant electric current is flowing and which is free to rotate about the center of the circle. Then, the direction of the vector $\vec{H}$ can be determined by reference to the positive normal to the plane containing the conductor, i.e., the normal which points to the side from which the direction of the electric current will appear counterclockwise. The magnitude of the vector $\vec{H}$ in electrostatic CGS units is given by the equation $H = c^2M/\mu i\vartheta$, where i is the current in the conductor, ϑ is the area enclosed by it, M is the moment of forces applied to the circular

*The justification for this assertion can be obtained from the general system of Maxwell's equations of electrodynamics (see subsection 7).

conductor in the position when the positive normal to its plane is perpendicular to the vector $\vec{H}$, c is a proportionality constant, and μ is a dimensionless proportionality constant called the magnetic permeability of the medium ($0<\mu<\infty$ and, for free space, $\mu=1$).

The magnetic lines of force about an infinitely long straight conductor take the form of concentric circles in planes perpendicular to the conductor and with centers in the conductor. Their directions are clockwise as viewed from points from which the current is proceeding. The magnitude of the vector $\vec{H}$ is determined experimentally for the case of free space: $H = 2kI/r$, where I is the current in the conductor, r is the distance from the conductor, and k is a proportionality constant. If we set $k = 1$ and equate I and r with the corresponding electrostatic CGS units, we obtain the unit for magnetic intensity in the electrostatic CGS system. Therefore, c^2 is uniquely determined and the dimensions of c are those of velocity in accordance with the equations $H = c^2M/i\vartheta$ and $H = 2I/r$. The constant c is called the electrodynamic constant. It has the dimensions of velocity, and its magnitude is equal to the velocity of light in vacuum. Its value has been found experimentally to be approximately $c \approx 3 \cdot 10^{10}$ cm/sec.

If we formally set $M = fl$, where f is a force acting in the direction of $\vec{H}$ and l is the length of some segment perpendicular to $\vec{H}$, we can write $H = f/\mu m$, where $m = I\vartheta/c^2l$ can be regarded as a magnetic mass. This yields one of the simpler physical interpretations of the magnitude of the vector $\vec{H}$.

In this and the following subsections, all quantities will be measured in the electrostatic CGS system just as in subsection 5 except that the magnetic intensity will be measured in magnetic CGS units. The unit of magnetic intensity in the magnetic CGS system is the Oersted (abbreviated Oe), which is numerically equal to the value of the electrodynamic constant c multiplied by unit magnetic intensity in electrostatic CGS units. Altogether, this means that the absolute Gaussian system of units is being used (see, for example, S. E. Frish and B. A. Timoreva, *Kurs obshchev fiziki* (Course in general physics), Vol. 2, GITTL, 1959, p. 307). *

As starting points for setting up differential equations which the vector $\vec{H}$ must satisfy, let us take two equations which we may consider experimentally verified:

$$\int_C H_s\, ds = \frac{4\pi}{c} \iint_\sigma j_\nu\, d\sigma, \tag{20}$$

$$\iint_S \mu H_n\, dS = 0, \tag{21}$$

where c is the electrodynamic constant.

The first of these is the *generalized Biot-Savart law*; the second is the *law of conservation of the lines of force* of the vector $\mu\vec{H}$ created by the vector $\vec{H}$ and an additional vector due to the magnetization of the medium. If we apply the Stokes and Ostrogradskiy formulas to these equations, just as before, we obtain the so-called system of magnetostatic equations

$$\operatorname{curl} \vec{H} = \frac{4\pi}{c}\vec{j}, \quad \operatorname{div}(\mu\vec{H}) = 0. \tag{22}$$

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*In the International System of Units (Systeme Internationale—SI) the current i is measured in amperes (amp): 1 amp = 3×10^9 esu = 0.1 emu; the electric charge e—in coulombs: 1 coulomb = 3×10^9 esu = 0.1 emu; the magnetic field intensity H—in ampere turns per meter: 1 amp turn/m = $4\pi(3 \times 10^7)$ esu = $4\pi \times 10^{-3}$ emu; the electric field intensity E—in volts per meter (v/m): 1 v/m = $(1/3) \times 10^{-4}$ esu = 10^6 emu; the electrical resistance R—in ohms: 1 ohm: $(1/9) \times 10^{-11}$ esu = 10^9 emu. Here esu and emu denote CGS electrostatic and electromagnetic units. For further conversion factors see, for example, M. H. Green, *International and Metric Units of Measurement*, New York, Chemical Publishing Company, 1961.

In the case in which $\vec{j}=0$, the magnetic field will be a potential field $\vec{H}=-\nabla u$ and the potential function u will satisfy the equation $\operatorname{div}(\mu \nabla u)=0$, which in the case of a homogeneous medium, coincides with Laplace's equation:

$$\Delta u = 0.$$

7. Maxwell's equations. Suppose that a magnetic field in a certain medium varies with time. Let $\vec{H}$ denote the intensity of the magnetic field and let ε and μ denote the dielectric constant and the magnetic permeability of the medium, just as in the preceding subsections. According to Faraday's law, which has been experimentally verified, the change in the magnetic field induces an electric field. Let $\vec{E}$ denote the intensity of the combination of the induced electric field and any constant original field that may have existed (if there was no field originally, $\vec{E}$ denotes the intensity of the induced field alone).

Mathematically, Faraday's law can be written in the form

$$\int_C E_s\, ds = -\frac{1}{c}\frac{\partial}{\partial t}\iint_\sigma \mu H_\nu\, d\sigma. \tag{23}$$

This equation, by virtue of Stokes' formula, enables us to write one of the equations which the vectors $\vec{E}$ and $\vec{H}$ must satisfy:

$$\operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \mu\vec{H}}{\partial t}. \tag{24}$$

For our second equation for determining $\vec{E}$ and $\vec{H}$, it would be natural to take the first of the magnetostatic equations (22), where we understand $\vec{j}$ to refer to the conduction current density caused by the vector $\vec{E}$. But then, by virtue of the general formula from vector analysis which states that $\operatorname{div}\operatorname{curl}\vec{H}=0$, $\operatorname{div}\vec{j}$ must be zero; but this cannot hold in general since the equation of continuity for the vector $\vec{j}$ implies that $\operatorname{div}\vec{j}=-\partial\rho/\partial t$, where ρ is the charge density, which is a function of time. To remove this contradiction, we take the first of formulas (22) in a somewhat modified form:

$$\operatorname{curl}\vec{H} = \frac{4\pi}{c}(\vec{j}+\vec{j}_D),$$

where $\vec{j}$ is the conduction current and $\vec{j}$ is an as yet undefined term, which we shall call the *displacement current*. Obviously, this displacement current must be chosen in such a way that $\operatorname{div}(\vec{j}+\vec{j}_D)=0$ since we wish to remove the contradiction in Eq. (22). From this and from the equation of continuity, we obtain the condition for determining the displacement current:

$$\operatorname{div}\vec{j}_D = -\frac{\partial\rho}{dt}.$$

But, according to the equation of continuity of the flux of the vector $\varepsilon\vec{E}$, we have $\operatorname{div}(\varepsilon\vec{E})=4\pi\rho$. Consequently, our last condition regarding the displacement current can be written in the form

$$\operatorname{div} \vec{j}_D = \frac{1}{4\pi} \frac{\partial}{\partial t} \operatorname{div} (\varepsilon \vec{E}).$$

This condition may be satisfied in more than one way. But to obtain a completely determined differential equation, it is most natural to define the displacement current by the equation

$$\vec{j}_D = \frac{1}{4\pi} \frac{\partial}{\partial t} \varepsilon \vec{E}.$$

Thus, we obtain the following differential equation:

$$\operatorname{curl} \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} \varepsilon \vec{E}, \tag{25}$$

where $\vec{j}$ is the conduction current, which, according to Ohm's law, is given by the equation

$$\vec{j} = \lambda \vec{E},$$

where λ is the conductivity of the medium.*

Equations (24) and (25) represent a complete system of electrodynamic equations. They are known as *Maxwell's equations*.

Sometimes, by "equations of electrodynamics," we mean a system consisting of Eqs. (24) and (25) combined with two other equations:

$$\operatorname{curl} \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} \varepsilon \vec{E}, \quad \operatorname{curl} \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mu \vec{H}, \tag{26}$$

$$\operatorname{div} (\varepsilon \vec{E}) = 4\pi\rho, \quad \operatorname{div} (\mu \vec{H}) = 0.$$

The first two of these equations coincide respectively with Eqs. (25) and (24) and the other two are compatible with them. Indeed, from the first of Eqs. (26) and the equation of continuity

$$\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

we get

$$\frac{\partial}{\partial t} \operatorname{div} (\varepsilon \vec{E}) = -4\pi \operatorname{div} \vec{j} = 4\pi \frac{\partial \rho}{\partial t}, \quad \operatorname{div} (\varepsilon \vec{E}) = 4\pi\rho + f(x, y, z).$$

From the second of Eqs. (26), we get

$$\frac{\partial}{\partial t} \operatorname{div} (\mu \vec{H}) = 0, \quad \operatorname{div} (\mu \vec{H}) = f_1(x, y, z).$$

Thus, the third and fourth of Eqs. (26) add only certain restrictions on the choice of solutions of the first two equations at the initial instant of time.

In the case of a homogeneous medium, with $\rho = 0$, by taking the curl of the first of Eqs. (26), we obtain

$$\operatorname{curl} \operatorname{curl} \frac{\partial}{\partial t} \operatorname{div} (\varepsilon \vec{E}) \operatorname{curl} \vec{E} + \frac{4\pi}{c} \lambda \operatorname{curl} \vec{E}.$$

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*Sometimes, it is necessary to add another vector to the vector or $\vec{E}$ in the last equation. This additional vector represents the electric field intensity due to external (nonelectrostatic) sources.

But according to the general formulas, curl curl $\vec{H}=\nabla \operatorname{div}\vec{H}-\Delta\vec{H}$, and, consequently, since $\operatorname{div}\vec{H}=0$, we have

$$\Delta\vec{H}-\frac{1}{a^2}\frac{\partial^2\vec{H}}{\partial t^2}-\frac{4\pi\lambda\mu}{c^2}\frac{\partial\vec{H}}{\partial t}=0,\quad a=\frac{c}{\sqrt{\varepsilon\mu}}. \tag{27}$$

In just the same way, we obtain from Eqs. (26) and the equation $\operatorname{div}\vec{E}=0$ a vector equation for $\vec{E}$:

$$\Delta\vec{E}-\frac{1}{a^2}\frac{\partial^2\vec{E}}{\partial t^2}-\frac{4\pi\lambda\mu}{c^2}\frac{\partial\vec{E}}{\partial t}=0,\quad a=\frac{c}{\sqrt{\varepsilon\mu}}. \tag{28}$$

This means that every component of the vectors $\vec{E}$ and $\vec{H}$ satisfy the *telegraph equation*

$$\Delta u-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}-\frac{4\pi\lambda\mu}{c^2}\frac{\partial u}{\partial t}=0,\quad a=\frac{c}{\sqrt{\varepsilon\mu}}. \tag{29}$$

If the medium is nonconducting ($\lambda = 0$), this equation becomes the wave equation. If the conductivity is very high, we may neglect the displacement currents in comparison with the conduction currents. By discarding the terms $(-1/a^2)(\partial^2 u/\partial t^2)$ in this equation, we reduce it to the *heat equation.*

In many rather general cases, it is possible and expedient in analogy with electrostatics and magnetostatics, to express the solutions of the electrodynamic equations (26) in terms of auxiliary potential functions. For example, suppose that a medium is homogeneous and nonconducting, that is, that $\lambda = 0$. Then, the fourth of Eqs. (26) yields

$$\mu\vec{H}=\operatorname{curl}\vec{A}, \tag{30}$$

where $\vec{A}$ is a vector determined up to an additive term which is the gradient of an arbitrary function. Correspondingly, the second of these equations can be written in the form

$$\operatorname{curl}\left(\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=0.$$

Consequently,

$$\vec{E}=-\nabla\varphi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{31}$$

where φ is an arbitrary function.

The vector potential $\vec{A}$ and the scalar potential φ introduced in this manner are not uniquely determined. Let us remove the ambiguity by requiring that the field of the vectors $\vec{H}$ and $\vec{E}$ that these potentials determine be unique. As is clear from Eqs. (30) and (31), it will be sufficient to set

$$\vec{A}=\vec{A}_0-\nabla\psi,\quad \varphi=\varphi_0+\frac{1}{c}\frac{\partial\psi}{\partial t}, \tag{32}$$

where $\vec{A}_0$ is any fixed solution of the equation curl $\vec{A}=0$, φ_0 is any function satisfying the equation $\Delta\varphi=0$, and ψ is a completely arbitrary function. Let us now choose ψ in such a way that the so-called Lorentz relation

$$\operatorname{div}\vec{A}+\frac{\mu\varepsilon}{c}\frac{\partial\varphi}{\partial t}=0. \tag{33}$$

will be satisfied. For this, we obviously need only require that the function ψ satisfy the equation

$$\Delta\psi-\frac{1}{a^2}\frac{\partial^2\psi}{\partial t^2}=\operatorname{div}\vec{A}_0+\frac{\mu\varepsilon}{c}\frac{\partial\varphi_0}{\partial t},\quad a=\frac{c}{\sqrt{\varepsilon\mu}}. \tag{34}$$

Therefore, we may assume the Lorentz relation satisfied. Now, by virtue of (30) and (31), the third and first of Eqs. (26) can be written in the form

$$\begin{gathered}-\varepsilon\operatorname{div}\left(\nabla\varphi+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)=4\pi\rho,\\ \frac{1}{\mu}\operatorname{curl}\operatorname{curl}\vec{A}=-\frac{\varepsilon}{c}\frac{\partial}{\partial t}\left(\nabla\varphi+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right).\end{gathered} \tag{34'}$$

On the basis of (33), if we use the identity $\operatorname{curl}\operatorname{curl}\vec{A}=\nabla\operatorname{div}\vec{A}-\Delta\vec{A}$, we may write these equations in the forms

$$\Delta\varphi-\frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{4\pi\rho}{\varepsilon},\quad \Delta\vec{A}-\frac{1}{a^2}\frac{\partial^2\vec{A}}{\partial t^2}=0,\quad a=\frac{c}{\sqrt{\varepsilon\mu}}. \tag{35}$$

Thus, in the present case, i.e., in a homogeneous nonconducting medium, the six unknown functions in the system of electrodynamic Eqs. (26) can, in accordance with formulas (30) and (31), be expressed in terms of four functions: the vector potential $\vec{A}$ and the scalar potential φ, which are determined from the system of four equations (35).

Following the same procedure, we can introduce vector and scalar potentials in many cases to simplify as much as possible the solution of the electrodynamic equations.

From the general electrodynamic equations, we obtain as a special case the equations of a constant electric current. Specifically, by assuming that the vectors $\vec{E}$ and $\vec{H}$ are independent of time, we obtain from the second and third of Eqs. (26) a system of equations for a constant electric current in the form

$$\operatorname{curl}\vec{E}=0,\quad \operatorname{div}(\varepsilon\vec{E})=4\pi\rho.$$

8. The equation of free electric oscillations in conductors. When an electric current flows through an infinitely long straight conductor as a result of electromagnetic oscillations, a counter-emf due to self-inductance is set up. Suppose that the conductor is placed along the x-axis and that the parameters of the conductor C, R, L, and G (standing for the capacitance, resistance, inductance, and leakage, respectively), as measured per unit length of the conductor, are continuous functions of x. If these parameters are assumed to be known, we can derive the differential equations for the unknown functions i and v (current and voltage) directly without using the general system of electrodynamic equations (26). Specifically, on the basis of Ohm's law, we can say that the voltage drop per element of conductor length dx is equal to the sum:

$$-\frac{\partial v}{\partial x}\,dx = iR\,dx + L\,\frac{\partial i}{\partial t}\,dx. \tag{36}$$

The amount of electric charge entering an element of length dx per unit time is $i|_x - i|_{x+dx} = -(\partial i/\partial x)dx$, while the amount of charge leaving the element dx in the same time is $C(\partial v/\partial t)dx + Gv\,dx$. Consequently, from the law of conservation of charge, we have

$$-\frac{\partial i}{\partial x}\,dx = C\,\frac{\partial v}{dt}\,dx + Gv\,dx. \tag{36'}$$

From this equation and from Eq. (36) we have the so-called system of *telegraph equations**

$$\begin{aligned} &\frac{\partial i}{\partial x} + C\,\frac{\partial v}{\partial t} + Gv = 0,\\ &\frac{\partial v}{\partial x} + L\,\frac{\partial i}{\partial t} + Ri = 0. \end{aligned} \tag{37}$$

In particular, if the coefficients in this system are constant, by applying the operator $\partial/\partial x$ to the first equation and the operator $C(\partial/\partial t)$ to the second and adding them, we obtain an equation for the function i:

$$\frac{\partial^2 i}{\partial x^2} - LC\,\frac{\partial^2 i}{\partial t^2} - (RC + LG)\,\frac{\partial i}{\partial t} - RGi = 0.$$

In just the same way, we obtain an equation for the function v:

$$\frac{\partial^2 v}{\partial x^2} - LC\,\frac{\partial^2 v}{\partial t^2} - (RC + LG)\,\frac{\partial v}{\partial t} - RGv = 0.$$

This means that the functions i and v both satisfy the telegraph equation

$$u_{xx} - \frac{1}{a^2}\,u_{tt} - bu_t - cu = 0, \quad a = \frac{1}{\sqrt{LC}}, \tag{38}$$

where $b = RC + LG$ and $c = RG$. By setting $w = ue^{\frac{a^2 b}{2}t}$, we reduce the last equation to

*From the point of view of general electromagnetic field theory, these equations are approximate since they can be obtained from the general equations of Maxwell only by making certain supplementary physical hypotheses, corresponding in the present derivation to giving the parameters of the conductor.

$$w_{xx} - \frac{1}{a^2} w_{tt} + k^2 w = 0, \quad a = \frac{1}{\sqrt{LC}}, \tag{39}$$

where

$$k^2 = \frac{1}{4}(a^2 b^2 - 4c),$$
$$k = \frac{1}{2} \frac{RC - LG}{\sqrt{LC}}. \tag{39'}$$

9. The string and membrane equations. Consider a string in the $x\,u$-plane, i.e., fine thread that offers no resistance to bending. Suppose that this string undergoes small transverse vibrations about an equilibrium position coincident with the x-axis. Let T denote the tension in the string, ρ its linear density, and f—a body force, i.e., a force applied to a unit mass of string in a direction perpendicular to it. In deriving an equation for the function $u = u(x, t)$, which gives the displacements of points on the string from the equilibrium position, let us agree to discard quantities which are small compared to $\partial u/\partial x$ in view of the small amplitude of the vibrations.

First of all, we note that the tension T will be uniform in the position of equilibrium by virtue of the law of equality of action and reaction. Furthermore, an element of string dx in the equilibrium position will, after being displaced from the equilibrium position, have a length equal to $ds = dx \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}$ (see Fig. 1). Thus, the elongation per unit length is $\Delta = ds/dx - 1 \simeq 0$. This means that, in accordance with Hooke's law, the tension will remain constant at all times. Let us find an expression for the sum of the projections onto the u-axis of all the forces applied to an element of string ds, and, in accordance with d'Alembert's principle, let us set this sum equal to 0. The projection of the tension at a point $x+dx$ will be

$$T \sin \alpha \Big|_{x+dx} = T \frac{u_x}{\sqrt{1 + u_x^2}} \Big|_{x+dx} \simeq T \frac{\partial u}{\partial x} \Big|_{x+dx}$$

(see Fig. 1), and the projection of the resultant of the tensile forces applied to an element ds is given by

$$T \frac{\partial u}{\partial x} \Big|_{x+dx} - T \frac{\partial u}{\partial x} \Big|_{x} = T \frac{\partial^2 u}{\partial x^2} \Big|_{x+\theta dx} dx \qquad (0 < \theta < 1).$$

If we add to this the body force applied to ds, which is equal to $f\rho dx$ and the inertial forces $(-\partial^2 u/\partial t^2) \cdot \rho dx$, we obtain, in accordance with d'Alembert's principle,

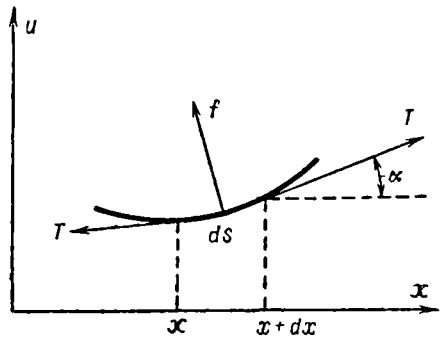

Fig. 1.

$$T\frac{\partial^2 u}{\partial x^2}\Big|_{x+\vartheta dx} dx + \rho f\, dx - \frac{\partial^2 u}{\partial t^2}\rho\, dx = 0.$$

If we divide by dx and take the limit as $dx \to 0$, we obtain the equation for small transverse vibrations of the string:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = -\frac{\rho}{T} f. \qquad (40)$$

In particular, in the case of a homogeneous string (one in which ρ = const), this equation takes the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f}{a^2}, \quad a = \sqrt{\frac{T}{\rho}}. \qquad (41)$$

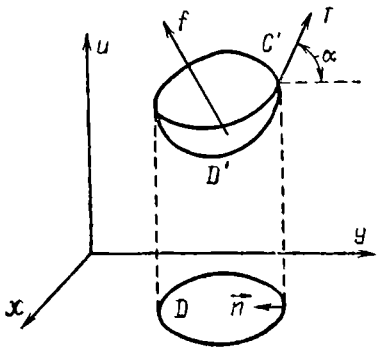

Fig. 2.

The last equation coincides with the wave equation in one space variable and is usually called the *string equation.*

Consider a membrane in a three-dimensional xyz-space, i.e., a thin film that offers no resistance to bending. Suppose that the membrane undergoes small transverse vibrations in the direction of the u-axis about its equilibrium position, which coincides with the xy-plane. Let T denote the tension in the membrane, i.e., the force per unit length applied to a line segment in the membrane in a direction perpendicular to that segment and tangent to the membrane itself.* Let ρ denote the surface density of the membrane. Let f denote the body force applied to a unit of mass of the membrane in a direction perpendicular to it. In deriving an equation for the function $u = u(x, y, t)$, which gives the displacement of the membrane from its equilibrium position, let us agree to discard quantities which are small compared to $\partial u/\partial x$ and $\partial u/\partial y$, in view of the small amplitude of the vibrations.

First of all, let us apply the law of equality of action and reaction to infinitely narrow rectangular strips of membrane in the equilibrium position. First, let us assume these strips to be parallel to the x-axis, then parallel to the y-axis. We conclude that the tension T is independent of x and y in the equilibrium position. Furthermore, every line element ds of membrane in the equilibrium position will be displaced from the equilibrium position

*By hypothesis, at a given point (x, y), the tension T is independent of the direction of the line segment passing through that point.

into an element whose length is $ds' = ds\sqrt{1+\left(\frac{\partial u}{\partial s}\right)^2}$ and will thus receive a relative elongation of $\Delta = ds'/ds - 1 \approx 0$. But, by Hooke's law, this means that the tension T will remain constant at all times. Let C' be a contour bounding some portion of the membrane D' and let C and D denote the projections of C' and D' onto the xy-plane. Let us project the resultants of the forces applied to $D'+C'$ onto the u-axis. The projection of the resultants of the tensile forces applied to C' are

$$Q = \int_C T \sin\alpha\, ds = \int_C T \frac{-\frac{\partial u}{\partial n}}{\sqrt{1+\left(\frac{\partial u}{\partial n}\right)^2}} ds \simeq -\int_C T \frac{\partial u}{\partial n} ds,$$

where $\vec{n}$ is the inner normal to the contour C (see Fig. 2). Using Green's formula, we now obtain

$$Q = \iint_D T \operatorname{div} \nabla u\, dS = \iint_D T(u_{xx} + u_{yy})\, dS.$$

To this we add the result of the body forces applied to D'

$$\iint_D \rho f\, dS$$

and the resultant of the inertial forces of the portion D' of the membrane

$$\iint_D -\rho \frac{\partial^2 u}{\partial t^2} dS.$$

We then obtain the equation

$$\iint_D \left[T(u_{xx} + u_{yy}) + \rho f - \rho \frac{\partial^2 u}{\partial t^2}\right] dS = 0. \tag{42}$$

Since D is arbitrary, we can reduce this equation by use of the mean-value theorem to the equation for small transverse vibrations of a membrane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = -\frac{\rho}{T} f. \tag{42'}$$

In the case of a homogeneous membrane, that is, one in which $\rho = \text{const}$, we obtain the equation of a membrane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f}{a^2}, \quad a = \sqrt{\frac{T}{\rho}}, \tag{43}$$

which is the wave equation in two space variables.

10. The equation of longitudinal vibrations of a thin rod. Suppose that a thin rod undergoes longitudinal vibrations about

its equilibrium position, which coincides with the x-axis. Suppose that these vibrations are sufficiently small so that the cross sections of the rod perpendicular to the x-axis remain plane surfaces and remain parallel to each other, being displaced only along the x-axis. Let E denote Young's modulus of the material of the rod, s—the area of a cross section of it, ρ—its linear density, T—the tension in the rod, f—the body force applied in the direction of the x-axis, and let $u=u(x, t)$ denote an unknown function representing the displacement of a cross section of the rod, the abscissa of the equilibrium position of which is x.

Consider an element of the rod that lies between the points x and $x+dx$ in the equilibrium position. The displacement at the point x is u, and that at the point $x+dx$ is $u+(\partial u/\partial x)\,dx$. Consequently, the relative elongation of the rod at the point whose abscissa is x is given by $\partial u/\partial x\,|_x$. By Hooke's law, the tension at that point is $T=Es\,(\partial u/\partial x\,|_x)$. The tension of the rod in the cross section whose abscissa is $x+dx$ is $Es\,(\partial u/\partial x\,|_{x+dx})$. Consequently, the resultant of the tensile forces applied to this element of the rod of length dx is

$$Es\frac{\partial u}{\partial x}\Big|_{x+dx}-Es\frac{\partial u}{\partial x}\Big|_x=\frac{\partial}{\partial x}\left(Es\frac{\partial u}{\partial x}\right)\Big|_{x+\theta dx}dx\quad(0\leqslant\theta\leqslant 1).$$

Let us add to this quantity the resultants of the other forces applied to the element of the rod contained between the cross section whose abscissas are x and $x+dx$, namely, the body forces $\rho sf\,dx$ and the inertial forces $-\rho s\,(\partial^2u/\partial t^2)\,dx$. Then, by setting this sum equal to 0, we obtain in accordance with d'Alembert's principle,

$$\frac{\partial}{\partial x}\left(Es\frac{\partial u}{\partial x}\right)\Big|_{x+\theta dx}+\rho sf\,dx-\rho s\frac{\partial^2u}{\partial t^2}dx=0.$$

If we take the limit in this equation as $dx\to 0$, we obtain the equation for longitudinal vibrations of the rod:

$$\frac{\partial}{\partial x}\left(Es\frac{\partial u}{\partial x}\right)-\rho s\frac{\partial^2u}{\partial t^2}=-\rho sf. \tag{44}$$

In particular, if $Es=\text{const}$, this equation takes the form

$$\frac{\partial^2u}{\partial x^2}-\frac{\rho}{E}\frac{\partial^2u}{\partial t^2}=-\frac{\rho}{E}f, \tag{44'}$$

and if the rod is homogeneous, this last equation coincides with the wave equation in one space variable:

$$\frac{\partial^2u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2u}{\partial t^2}=-\frac{f}{a^2},\quad a=\sqrt{\frac{E}{\rho}}. \tag{44''}$$

CHAPTER 2

General Questions in the Theory of Partial Differential Equations

This chapter is devoted to a discussion of general questions in the theory of partial differential equations from a purely mathematical point of view. We shall give a complete proof of Kovalevskaya's theorem,* which is the most important theorem in the theory of partial differential equations and which is related to the central concept of that theory, namely, the concept of a characteristic. The classification of quasi-linear systems of partial differential equations will be given in the most general form. We shall study the solution of the Cauchy problem for first-order partial differential equations.

1. Normal Systems of Equations. Kovalevskaya's Theorem

Kovalevskaya's theorem on the existence and uniqueness of a solution of the so-called Cauchy problem for systems of partial differential equations discloses certain inherent regularities in the theory of partial differential equations. It leads to the basic concept of that theory, namely, the concept of a characteristic, and it leads to the most natural presentation of the general classification of such equations.

A system of p equations with p unknown functions $u_1, u_2, \ldots, u_p$

$$\frac{\partial^{r_i} u_i}{\partial t^{r_i}} = F_i(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_j, \ldots, u_p, \ldots) \quad (i = 1, 2, \ldots, p), \tag{1}$$

where $t, x_1, x_2, \ldots, x_n$ *are independent variables, is said to be* normal *with respect to the variable* t *if the right-hand members of this system do not depend on the partial derivatives of the* $u_j (j = 1, 2, \ldots, p)$ *with respect to* t *of order higher than* $r_j - 1$, *or on the remaining derivatives of the* u_j *of order higher than* r_j.

In particular, a normal system of first-order equations is of the form

*We have chosen to use this more accurate transliteration instead of the traditional Kowalewski. The theorem is usually known as the Cauchy-Kovalevskaya theorem [Ed.].

$$\frac{\partial u_i}{\partial t} = F_i\left(t,\ x_1,\ x_2,\ \ldots,\ x_n,\ u_1,\ u_2,\ \ldots,\ u_p,\ \frac{\partial u_1}{\partial x_1},\ \ldots,\ \frac{\partial u_p}{\partial x_n}\right) \qquad (1')$$
$$(i = 1, 2, \ldots, p).$$

Here, the right-hand members do not depend on the derivatives of the unknown functions with respect to t or on their derivatives with respect to the other independent variables of order higher than the first.

For example, the string equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f}{a^2},$$

is normal with respect to the variable t (and also with respect to the variable x), but the equation

$$\frac{\partial^2 u}{\partial x\,\partial t} - \frac{\partial u}{\partial t} = 0$$

is not normal with respect to either x or t.

The basic problem for normal systems of equations is the Cauchy problem. For example, for the string equation, the Cauchy problem consists in finding the solution of this equation $u=u(x, t)$ (for $-\infty<x<\infty$ and $t>0$) which satisfies the so-called initial conditions

$$u|_{t=0} = \varphi^0(x),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi^1(x) \qquad (-\infty<x<\infty),$$

where $\varphi^0(x)$ and $\varphi^1(x)$ are given functions of x. From a physical point of view, the problem amounts to this: We are required to find the displacements of points of the string from the x-axis, and the complete subsequent history of the motion, from data prescribed at the initial instant on the displacements of the points of the string from its equilibrium position (coincident with the x-axis), and on their velocities.

In the general case for a normal system of equations, the Cauchy problem is posed as follows:

We need to find the solution of the normal system of equations (1) satisfying the following initial conditions:

$$\left.\frac{\partial^{k_i} u_i}{\partial t^{k_i}}\right|_{t=t^0} = \varphi_i^{k_i}(x_1, x_2, \ldots, x_n),\ k_i = 0, 1, \ldots, r_i - 1, \qquad (2)$$
$$(i = 1, 2, \ldots, p),$$

where t^0 is a fixed value of the variable t and $\varphi_i^{k_i}$ are given functions of the variables $x_1, x_2, \ldots, x_n$.

This means that, in solving the Cauchy problem, it is assumed that, for some fixed value $t=t^0$, the unknown functions $u_j (j=1, 2, \ldots, p)$ and their first $r_j - 1$ derivatives, respectively, with respect to t may be considered as given functions of the remaining independent variables other than t.

In particular, in the case of the normal system of first-order equations (1′), the unknown functions are assumed given for $t=t^0$ since here the initial conditions take the form

$$u_i|_{t=t^0} = \varphi_i^0(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, p), \tag{2′}$$

where the φ_i^0 are given functions of the variables $x_1, x_2, \ldots, x_n$.

Theorem (Kovalevskaya). *Suppose the initial values (2) in the Cauchy problem for a normal system of Eqs. (1) are analytic functions of the independent variables in some neighborhood of a point* $P(t^0, x_1^0, \ldots, x_n^0)$. *Then, if the right-hand members of the given system of equations are analytic functions of all their arguments in some neighborhood about those values which correspond to* P,* *then in some neighborhood of this point,* P, *there exists an analytic solution of the Cauchy problem, which is unique in the class of analytic functions.*

Proof: 1. The case of a system of first-order equations.

In the first place, if there exists an analytic solution of the Cauchy problem

$$u_i = \sum_{k_0, k_1, \ldots, k_n = 0}^{\infty} c^i_{k_0 k_1 \ldots k_n} (t - t^0)^{k_0} (x_1 - x_1^0)^{k_1} \ldots (x_n - x_n^0)^{k_n} \tag{3}$$
$$(i = 1, 2, \ldots, p),$$

where the $c^i_{k_0 k_1 \ldots k_n}$ are constants, this solution is unique. To see this, note that the coefficients $c^i_{0 k_1 \ldots k_n}$ are determined from the initial conditions $u_i|_{t=t^0} = \varphi_i^0$ by differentiating them $k_1, k_2, \ldots, k_n$ times respectively with respect to $x_1, x_2, \ldots, x_n$ at the point $P(t^0, x_1^0, \ldots, x_n^0)$. The coefficients $c^i_{1 k_1 k_2 \ldots k_n}$ are determined from the system by differentiation with respect to the variables $x_1, x_2, \ldots, x_n$ and they can be expressed in terms of the $c^i_{0 k_1 k_2 \ldots k_n}$ and the coefficients in the expansions of the F_i. Furthermore, the coefficients $c^i_{2 k_1 k_2 \ldots k_n}$ can be determined from the system by differentiation with respect to the variables $x_1, x_2, \ldots, x_n$ and once with respect to t. The coefficients $c^i_{k_0 k_1 k_2 \ldots k_n}$ are determined in just the same way but the system of equations must be differentiated $k_0 - 1$ times with respect to t. Thus, the system of equations and the initial conditions uniquely determine the coefficients in the expansions (3) and hence there cannot be two analytic solutions.

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*The values of all the arguments on which the right-hand members of the system of equations depend are completely determined at the point $P(t^0, x_1^0, \ldots, x_n^0)$ from the initial conditions; for example, the unknown functions and their partial derivatives with respect to t are determined immediately, and the remaining arguments are determined by differentiation of the initial conditions with respect to the variables $x_1, x_2, \ldots, x_n$.

To prove the existence of a solution, it will be sufficient to show that the series (3), which are obtained by the procedure just described, converge and hence constitute our solution. The characteristic feature of this procedure consists of the fact that the coefficients $c^{i}_{k_0 k_1 k_2 \dots k_n}$ are obtained from the coefficients in the expansions of the F_i and the coefficients in the expansions of the initial conditions solely by use of the operations of addition and multiplication. Therefore, to show that this process leads to convergent series, it will be sufficient to show that it leads to convergent series for some auxiliary system, the right-hand members of which dominate the functions F_i and the initial conditions of which dominate the functions φ_i^0. This means that the coefficients in the expansions of the right-hand members of the auxiliary system must be greater than the absolute values of the corresponding coefficients in the expansions of the F_i, and the coefficients of the expansions of the initial conditions of the auxiliary system must be greater than the absolute values of the corresponding coefficients in the expansions of the functions φ_i^0.

To construct such an auxiliary system, we may assume without loss of generality that $t^0 = x_1^0 = \dots = x_n^0 = 0$ and $\varphi_i^0 = 0$ since, otherwise, we need only make the change $t - t^0 \sim t$, $x_i - x_i^0 \sim x_i$, $u_i - \varphi_i^0 \sim u_i$. Similarly, we may assume without loss of generality that the constant terms in the expansions of the F_i vanish since otherwise we could make the change $u_i - A_i t \sim u_i$, where the A_i are constants.

Now, let M denote a number with the following property: For all values $t, x_1, \dots, x_n, \dots, \partial u_p/\partial x_n$, which appear in the right-hand members of the system and do not exceed some positive number ρ in absolute value, all the terms in the expansions of the F_i do not exceed M. Then, the function

$$\frac{M}{\left(1-\frac{t}{\rho}\right)\left(1-\frac{x_1}{\rho}\right)\dots\left(1-\frac{x_n}{\rho}\right)\left(1-\frac{u_1}{\rho}\right)\dots\left(1-\frac{u_p}{\rho}\right)\dots\left(1-\frac{1}{\rho}\frac{\partial u_p}{\partial x_n}\right)} - M.$$

is a majorant for all the F_i, as simple summation shows. We easily obtain

$$\frac{M}{\left(1-\frac{t}{\rho}\right)\left(1-\frac{x_1}{\rho}\right)\dots\left(1-\frac{u_p}{\rho}\right)} = M\sum_{N=0}^{\infty}\frac{1}{\rho^N}\sum_{k_0+k_1+\dots+k_{n+p}=N} t^{k_0}x_1^{k_1}\dots u_p^{k_{n+p}} \ll$$

$$\ll M\sum_{N=0}^{\infty}\frac{1}{\rho^N}\left(\frac{t}{\alpha}+x_1+\dots+u_p\right)^N = M\frac{1}{1-\dfrac{\frac{t}{\alpha}+x_1+\dots+x_n+u_1+\dots+u_p}{\rho}},$$

where $0 < \alpha < 1$.* In exactly the same way, we obtain

$$\frac{1}{\left(1-\frac{1}{\rho}\frac{\partial u_1}{\partial x_1}\right)\cdots\left(1-\frac{1}{\rho}\frac{\partial u_p}{\partial x_n}\right)} \ll \sum_{N=0}^{\infty}\frac{1}{\rho^N}\left(\frac{\partial u_1}{\partial x_1}+\ldots+\frac{\partial u_p}{\partial x_n}\right)^N = \frac{1}{1-\frac{\frac{\partial u_1}{\partial x_1}+\ldots+\frac{\partial u_p}{\partial x_n}}{\rho}}.$$

Therefore, as our auxiliary system, we may, for simplicity, take a system of the form

$$\frac{\partial z_l}{\partial x_l} = \frac{M}{\left(1-\frac{\frac{t}{\alpha}+x_1+\ldots+x_n+z_1+\ldots+z_p}{\rho}\right)\left(1-\frac{\frac{\partial z_1}{\partial x_1}+\ldots+\frac{\partial z_p}{\partial x_n}}{\rho}\right)} - M. \qquad (4)$$

We look for a solution of this system, under any set of initial conditions which majorize the zero initial conditions, in the form

$$z_1 = z_2 = \ldots = z_p = z(x),$$

where $x = t/\alpha + x_1 + x_2 + \ldots + x_n$. For such a solution, all the equations in the system (4) coincide with the single equation

$$\frac{1}{\alpha}\frac{dz}{dx} = \frac{M}{\left(1-\frac{x+pz}{\rho}\right)\left(1-\frac{np\frac{dz}{dx}}{\rho}\right)} - M$$

or

$$\frac{dz}{dx}\left(\frac{1}{\alpha}-\frac{npM}{\rho}\right) = \frac{np}{\alpha\rho}\left(\frac{dz}{dx}\right)^2 + \frac{M}{1-\frac{x+pz}{\rho}} - M. \qquad (5)$$

Let us now take α sufficiently small so that the coefficient of dz/dx in (5) will be positive. Then, the last equation can be written in the form

$$\frac{dz}{dx} = A\left(\frac{dz}{dx}\right)^2 + \Phi, \qquad A = \text{const} > 0, \qquad (5')$$

or

$$\frac{dz}{dx} = \frac{1}{2A} - \sqrt{\left(\frac{1}{2A}\right)^2 - \frac{1}{A}\Phi}, \qquad (6)$$

*The symbol $\ll$ means that the expression on the right-hand side of the symbol majorizes the expression on the left-hand side.

where $\Phi=\Phi(x, z)$ is a function of x and z that vanishes at $x=z=0$ and has positive coefficients in its expansion in a neighborhood of the point $x=z=0$.

The right-hand member of Eq. (6) is an analytic function of x and z in a neighborhood of the point $x=z=0$. Therefore, as we know from the theory of ordinary differential equations, there exists an analytic solution of Eq. (6) satisfying the condition $z|_{x=0}=0$. From Eq. (6), we note that $dz/dx\Big|_{x=0}=0$, and it follows from Eq. (5′) that

$$\frac{d^2z}{dx^2}=2A\frac{dz}{dx}\frac{d^2z}{dx^2}+\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial z}\frac{dz}{dx},$$

so that $d^2z/dx^2\Big|_{x=0}\geqslant 0$. We also see that the coefficients in the expansion of $z=z(x)$ in a neighborhood of the point $x=0$ are positive and, in particular, that $z|_{t=0}=0$ can be expanded in a series of powers of $(x_1+x_2+\ldots+x_n)$ with positive coefficients. This means that the system of functions $z_1=z_2=\ldots=z_p=z(x)$ of the variables $t, x_1, \ldots, x_n$ will be an analytic solution of the auxiliary system of Eqs. (4) under initial conditions that majorize the initial conditions of the original system since these last are assumed to be zero. From this we conclude that the previously described formal procedure for obtaining the series (3) for the original system with zero initial conditions always leads to convergent series.

Consequently, the Cauchy problem in the case of a system of first-order equations has a solution.

Proof: 2. The case of a system of equations of arbitrary order.

Let us suppose that $u_1, u_2, \ldots, u_p$ is a solution of the Cauchy problem for the system of Eqs. (1) under the initial conditions (2). Then, by introducing new unknown functions $u_i^{k_0,k_1,\ldots,k_n}$, $u_i^{0,0,\ldots,0}$ defined by

$$\frac{\partial^{k_0+k_1+\ldots+k_n}u_i}{\partial t^{k_0}\partial x_1^{k_1}\ldots\partial x_n^{k_n}}=u_i^{k_0,k_1,\ldots,k_n},\quad k_0+k_1+\ldots+k_n\leqslant r_i-1 \qquad (7)$$
$$(i=1, 2, \ldots, p),$$

we conclude that these functions will be a solution of the Cauchy problem for a normal system of first-order equations

$$\frac{\partial u_i^{k_0,k_1,\ldots,k_n}}{\partial t}=\frac{\partial u^{k_0+1,k_1,\ldots,k_j-1,\ldots,k_n}}{\partial x_j},\quad \begin{matrix}k_0+k_1+\ldots+k_n=r_i-1,\\ k_0\neq r_i-1,\end{matrix} \qquad (8a)$$

$$\frac{\partial u_i^{k_0,k_1,\ldots,k_n}}{\partial t}=u_i^{k_0+1,k_1,\ldots,k_n},\quad k_0+k_1+\ldots+k_n\leqslant r_i-2, \qquad (8b)$$

$$\frac{\partial u_i^{r_i-1,0,\ldots,0}}{\partial t}=F_i\left(t, x_1, \ldots, x_n, \ldots, \frac{\partial u_p^{k_0,k_1,\ldots,k_n}}{\partial x_n}\right) \qquad (8c)$$
$$(i=1, 2, \ldots, p)$$

with the initial conditions

$$u_i^{k_0,k_1,\ldots,k_n}\Big|_{t=t^0} = \frac{\partial^{k_0+\ldots+k_n} u_i}{\partial t^{k_0}\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\Big|_{t=t^0} = \frac{\partial^{k_1+\ldots+k_n}\varphi_i^{k_0}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}}. \quad (9)$$
$$(i = 1, 2, \ldots, p).$$

Suppose now that the functions $u_i^{k_0,k_1,\ldots,k_n}$ (where $k_0 + k_1 + \ldots + k_n \leqslant r_i - 1$, for $i = 1, 2, \ldots, p$) constitute the solution of the normal system of Eqs. (8) with the initial conditions (9). Then, the functions $u_i^{0,0,\ldots,0}$ (for $i = 1, 2, \ldots, p$) will constitute a solution of the system of Eqs. (1) under the initial conditions (2). Indeed, for $k_0 + k_1 + \ldots + k_n = r_i - 1$, where $k_0 \neq r_i - 1$, we have on the basis of (a) and (b),

$$\frac{\partial u_i^{k_0,k_1,\ldots,k_n}}{\partial t} = \frac{\partial u_i^{k_0+1,k_1,\ldots,k_j-1,\ldots,k_n}}{\partial x_j} = \frac{\partial^2}{\partial x_j \partial t} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n}.$$

Consequently, the expression

$$u_i^{k_0,k_1,\ldots,k_n} - \frac{\partial}{\partial x_j} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n}$$

is independent of t and, for $t = t^0$, this expression vanishes since, on the basis of the initial conditions (9),

$$u_i^{k_0,k_1,\ldots,k_n}\Big|_{t=t^0} = \frac{\partial^{k_1+\ldots+k_n}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}\varphi_i^{k_0} = \frac{\partial}{\partial x_j} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n}.$$

Thus, for $k_0 + k_1 + \ldots + k_n = r_i - 1$, we have

$$u_i^{k_0,k_1,\ldots,k_n} = \frac{\partial}{\partial x_j} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n};\quad u_i^{r_i-1,0,\ldots,0} = \frac{\partial}{\partial t} u_i^{r_i-2,0,\ldots,0}. \quad (10)$$

For $k_0 + k_1 + \ldots + k_n = r_i - 2$, we have from (c), on the basis of (10),

$$\frac{\partial}{\partial t} u_i^{k_0,k_1,\ldots,k_n} = u_i^{k_0+1,k_1,\ldots,k_n} = \frac{\partial}{\partial x_j} u_i^{k_0+1,k_1,\ldots,k_j-1,\ldots,k_n} =$$
$$= \frac{\partial^2}{\partial t\,\partial x_j} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n}.$$

From initial conditions (9), we obtain for $t = t^0$

$$u_i^{k_0,k_1,\ldots,k_n} = \frac{\partial}{\partial x_j} u_i^{k_0,k_1,\ldots,k_j-1,\ldots,k_n}.$$

Therefore, for $k_0 + k_1 + \ldots + k_n = r_l - 2$,

$$u_l^{k_0,k_1,\ldots,k_n} = \frac{\partial}{\partial x_j} u_l^{k_0,\ldots,k_j-1,\ldots,k_n}, \quad u_l^{r_l-2,0,\ldots,0} = \frac{\partial}{\partial t} u_l^{r_i-3,0,\ldots,0}. \tag{11}$$

In just the same way as in (10) and (11), we see that, in general, when $k_0 + k_1 + \ldots + k_n \leqslant r_l - 1$ $(i = 1, 2, \ldots, p)$, the transition from one of the functions $u_l^{k_0,k_1,\ldots,k_n}$ to another by increasing one of the indices $k_0, k_1, \ldots, k_n$ by one, means differentiation of the original function with respect to the corresponding variable among $t, x_1, x_2, \ldots, x_n$.

Consequently, the functions $u_i^{0,0,\ldots,0}$ (for $i = 1, 2, \ldots, p$) constitute a solution of the system of Eqs. (1) with initial conditions (2) since the first-order system (3) has a solution under initial conditions (9). This completes the proof of Kovalevskaya's theorem for the general case.*

2. Reduction of Quasi-linear Systems of Equations to Normal Form. Classification and Characteristics of Systems of Equations

1. The general case of quasi-linear systems of equations. Examples. A quasi-linear system of equations for the unknown functions $u_1, u_2, \ldots, u_p$, with the highest derivatives of these functions having orders $r_1, r_2, \ldots, r_p$, respectively, can be written in the form:

$$\sum_{j=1}^{p} L_{ij} u_j + f_i = 0, \quad i = 1, 2, \ldots, p. \tag{12}$$

Here L_{ij} denotes the general quasi-linear operator of order r_j with respect to the independent variables $x_1, x_2, \ldots, x_n$,

$$L_{ij} = \sum_{k_1+k_2+\ldots+k_n=r_j} A_{ij}^{k_1,k_2,\ldots,k_n} \frac{\partial^{r_j}}{\partial x_1^{k_1} \partial x_2^{k_2}, \ldots, \partial x_n^{k_n}}, \tag{13}$$

*For a summary of results obtained with regard to the existence and uniqueness of a solution of the Cauchy problem for general systems of partial differential equations, see S. L. Sobolev "Differentsial'nyye uravneniya v chastnykh proizvodnykh" (Partial differential equations) in *Matematika v SSSR za 30 let* (Thirty years of mathematics in the USSR), Moscow and Leningrad, 1948, pp. 518-544 and M. I. Vishik, A. D. Myshkis, and O. A. Oleynik, "Differentsial'nyye uravneiya v chastnykh proizvodnykh" (Partial differential equations) in *Matematika v SSSR za 40 let* (Forty years of mathematics in the USSR), GIFML, 1959.

while $A_{ij}^{k_1 k_2 \ldots k_n}$ and f_i are functions of the independent variables x_1, $x_2, \ldots, x_n$, the unknown functions $u_1, u_2, \ldots, u_p$, and their derivatives of orders not exceeding $r_i - 1$, respectively.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be new independent variables related to the variables $x_1, x_2, \ldots, x_n$ according to the equations

$$\xi_k = \psi_k(x_1, x_2, \ldots, x_n) \qquad (k = 1, 2, \ldots, n), \tag{14}$$

where the ψ_k are sufficiently many times continuously differentiable functions of their arguments with nonzero Jacobian in a neighborhood of some point

$$Q(\xi_1^0, \xi_2^0, \ldots, \xi_n^0), \quad \xi_k^0 = \psi_k(x_1^0, x_2^0, \ldots, x_n^0).$$

We may now ask, "under what conditions the system of Eqs. (12) will be a normal system with respect to the variable ξ_1 in a neighborhood of the point Q, so that Kovalevskaya's theorem can be applied to it at least when the right-hand members and the initial values are analytic functions?" Obviously, to answer this question we must perform a change of variables in (12) and then be able to solve the resulting system for the unmixed partials of $u_1, u_2, \ldots, u_p$ with respect to ξ_1 of orders $r_1, r_2, \ldots, r_p$, respectively. Let us make this change of variables, noting that these unmixed derivatives appear only in the differential operators L_{ij}, which are linear with respect to them. Let us calculate the coefficients of $(\partial^{r_i}/\partial\xi_1^{r_i})u_i$ for each of the operators L_{ij}, denoting them respectively by B_{ij}. We have

$$\frac{\partial u_i}{\partial x_1} = \frac{\partial u_1}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \ldots,$$

where the dots denote terms not containing the derivative of u_i with respect to ξ_1. Furthermore,

$$\frac{\partial^2 u_i}{\partial x_1 \partial x_2} = \frac{\partial^2 u_i}{\partial \xi_1^2}\frac{\partial \xi_1}{\partial x_1}\frac{\partial \xi_1}{\partial x_2} + \ldots,$$

where the dots indicate terms not containing the second partial derivative of the function u_i with respect to the variable ξ_1. In general, we have the equations

$$\frac{\partial^{r_i} u_i}{\partial x_1^{k_1}, \ldots, \partial x_n^{k_n}} = \frac{\partial^{r_i} u_i}{\partial \xi_1^{r_i}}\left(\frac{\partial \xi_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial \xi_1}{\partial x_n}\right)^{k_n} + \ldots,$$

where the dots indicate terms not containing the derivative $\partial^{r_i} u_i / \partial \xi_1^{r_i}$. Consequently, the coefficients B_{ij} in which we are interested are given by the equations

$$B_{ij} = \sum_{k_1 + k_2 + \ldots + k_n = r_j} A_{ij}^{k_1 k_2 \ldots k_n} \alpha_1^{k_1} \alpha_2^{k_2}, \ldots, \alpha_n^{k_n}, \tag{15}$$

where $\alpha_k = \partial\xi_1/\partial x_k$ for $k = 1, 2, \ldots, n$. The system (12) becomes, after a change of variables,

$$\sum_{j=1}^{p} \frac{\partial^{r_j} u_j}{\partial \xi_1^{r_i}} B_{ij} = -f_i^*, \qquad i = 1, 2, \ldots, p, \tag{12'}$$

where the $A_{ij}^{k_1 k_2 \ldots k_n}$ and the f_i^* are functions not depending on the highest-order partial derivatives

$$\frac{\partial^{r_i} u_i}{\partial \xi_1^{r_i}}.$$

It is clear now that for the question posed above to have an affirmative answer, that is, for it to be possible to reduce the system (12) to normal form at the point $Q(\xi_1^0, \xi_2^0, \ldots, \xi_n^0)$ with respect to the variable ξ_1, it is necessary and sufficient that the so-called characteristic determinant of that system

$$\Delta(\alpha) = \begin{vmatrix} B_{11} & B_{12} \ldots B_{1p} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ B_{p1} & B_{p2} \ldots B_{pp} \end{vmatrix} \tag{16}$$

be nonzero at that point.

The equation

$$\Delta(\alpha) = 0 \tag{17}$$

with the condition

$$|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1, \tag{17'}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are parameters, is known as the *characteristic equation*, and the planes

$$\sum_{i=1}^{n} \alpha_i (x_i - x_i^0) = 0, \tag{18}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary roots of the characteristic equation, are called the *characteristic directions*. Also, we shall call equation

$$\Delta(\varphi) =$$

$$= \begin{vmatrix} \sum\limits_{k_1+\ldots+k_n=r_1} A_{11}^{k_1\ldots k_n}\varphi_{x_1}^{k_1}\ldots\varphi_{x_n}^{k_n}\ldots & \sum\limits_{k_1+\ldots+k_n=r_p} A_{1p}^{k_1\ldots k_n}\varphi_{x_1}^{k_1}\ldots\varphi_{x_n}^{k_n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum\limits_{k_1+\ldots+k_n=r_1} A_{p1}^{k_1\ldots k_n}\varphi_{x_1}^{k_1}\ldots\varphi_{x_n}^{k_n}\ldots & \sum\limits_{k_1+\ldots+k_n=r_p} A_{pp}^{k_1\ldots k_n}\varphi_{x_1}^{k_1}\ldots\varphi_{x_n}^{k_n} \end{vmatrix} = 0 \tag{19}$$

with the condition

$$|\varphi_{x_1}|^2 + |\varphi_{x_2}|^2 + \ldots + |\varphi_{x_n}|^2 \neq 0 \tag{19'}$$

the *equation for the characteristic variables*, and we shall call any function $\varphi(x_1, x_2, \ldots, x_n)$ which satisfies this equation a *characteristic variable*. We shall refer to the surfaces

$$\varphi(x_1, x_2, \ldots, x_n) = 0, \tag{20}$$

where $\varphi(x_1, x_2, \ldots, x_n)$ is a characteristic variable, as the *characteristics* of the system of Eqs. (12), and we shall call Eq. (19) with conditions (19′) and (20) the *equation for the characteristics*.*

The concept of a characteristic is, by virtue of Kovalevskaya's theorem, closely connected with the solvability of the Cauchy problem. Specifically, let us suppose that a surface $\psi_1(x_1, \ldots, x_n) - \xi_1^0 = 0$ containing a point $Q(\xi_1^0, \ldots, \xi_n^0)$ is not a characteristic, that is, let us suppose that $\Delta(\psi_1) \neq 0$ in a neighborhood of that point. Then, the system of Eqs. (12) can be reduced to normal form with respect to the variable ξ_1. On the basis of Kovalevskaya's theorem, under additional assumptions regarding the analyticity of the right-hand members of the system and of the initial conditions given on the surface $\xi_1 = \psi_1(x_1, x_2, \ldots, x_n) = \xi_1^0$ in a neighborhood of the point Q, there exists exactly one analytic solution of the Cauchy problem.

However, if the surface $\xi_1 = \psi_1(x_1, x_2, \ldots, x_n) = \xi_1^0$ is a characteristic, all other conditions remaining the same, the Cauchy problem will either have no solution or it will have more than one solution; that is, the Cauchy problem now becomes indeterminate. Since the characteristic determinant in this case vanishes at the point Q, the left-hand members of the system (12′) will be linearly dependent, so that the right-hand members of that system must also be linearly dependent. But these right-hand members are

*Note that we define the characteristics in terms of the characteristic variables, which are introduced here as solutions of the partial differential equation (19) with condition (19′).

completely determined at Q by the initial conditions. Hence, if these initial conditions are chosen in such a way as to be linearly independent, the Cauchy problem will have no solution. On the other hand, if the initial conditions are chosen in such a way that they are linearly dependent, then at least one of the highest-order unmixed partial derivatives with respect to ξ_1 of any one of the unknown functions can be prescribed arbitrarily without contradicting the given system (12′).

Let us clarify the situation described above by means of a simple example of the Cauchy problem:

$$\frac{\partial^2 u}{\partial x\,\partial y} = 0, \quad u|_{x=0} = \varphi(y), \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \varphi_1(y). \tag{21}$$

Here, the characteristic determinant is $\Delta(\alpha) = \alpha_1\alpha_2$. Consequently, the characteristic directions and also the characteristics are the straight lines x = const and y = const. Thus, we see that the initial conditions of the Cauchy problem (21) are given on the characteristic $x=0$. Let us note now that the general form of a function satisfying the differential equation of problem (21) is

$$u = \Phi(x) + \Psi(y), \tag{21′}$$

where $\Phi(x)$ and $\Psi(y)$ are arbitrary functions of x and y respectively. Here, we may, without loss of generality, assume that $\Phi(0) = 0$. The first of the initial conditions yields

$$\Psi(y) = \varphi(y),$$

and from the second initial condition, we have

$$\left.\frac{d\Phi(x)}{dx}\right|_{x=0} = \varphi_1(y).$$

Thus, if $\varphi_1(y) \neq$ const, the Cauchy problem (21) has no solution. If $\varphi_1(y) =$ const, there will be infinitely many such solutions since in this case $d^2\Phi(x)/dx^2\,|_{x=0}$ can be arbitrary. All this is in complete accord with the theorem of Kovalevskaya that we proved earlier and with the properties of the characteristics.

Suppose that we define new independent variables

$$\xi_k = \sum_{l=1}^{n} a_{kl}x_l, \qquad k = 1, 2, \ldots, n \tag{22}$$

by performing a linear transformation with nonzero determinant on the system of differential equations (12). Suppose also that we introduce new parameters $\lambda_k (k=1, 2, \ldots, n)$ in the characteristic determinant (16) of that system by making a linear transformation which, for definiteness, we shall call the transformation associated with (22):

$$\alpha_i = \sum_{k=1}^{n} a_{ki}\lambda_k, \qquad i = 1, 2, \ldots, n. \tag{23}$$

Now, the differential operators of the system (12) take the form

$$L_{ij} = \sum_{k_1+\ldots+k_n=r_j} A_{ij}^{k_1 \ldots k_n} \left(\sum_{l=1}^{n} a_{l1} \frac{\partial}{\partial \xi_l}\right)^{k_1} \ldots \left(\sum_{m=1}^{n} a_{mn} \frac{\partial}{\partial \xi_m}\right)^{k_n},$$

and the coefficients of the characteristic determinant after the transformation (23) may be written in the form

$$B_{ij} = \sum_{k_1+\ldots+k_n=r_j} A_{ij}^{k_1 \ldots k_n} \left(\sum_{l=1}^{n} a_{l1}\lambda_l\right)^{k_1} \ldots \left(\sum_{m=1}^{n} a_{mn}\lambda_m\right)^{k_n}.$$

This means that the law of transformation of the operators L_{ij} under the change of variables (22) is the same as the law of transformation of the elements of the characteristic determinant B_{ij} under the change of parameters (23). From this we draw the conclusion:

To obtain the characteristic determinant of the system of Eqs. (12) corresponding to the new independent variables (22), it is necessary and sufficient to introduce new parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ *in the characteristic determinant of the given system (12) by means of an affine transformation (23).*

This property of quasi-linear systems of equations is the basis on which the general classification of these equations is made.

If the characteristic determinant $\Delta(\alpha)$ of the system (12) can, by means of a nonsingular affine transformation of parameters, be reduced to a determinant or, as we say, to a form depending on fewer than n parameters, that is, fewer than the number of independent variables of the system, the system of equations is said to be *parabolic at the given point* and the form is said to be *degenerate*. When it is nondegenerate, the system (12) is said to be *elliptic at the given point* if all the roots of its characteristic equation are complex. (Here, one must not forget that $|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1$.) Also, in the nondegenerate case, the system (12) is said to be *hyperbolic at the given point* if, after a nonsingular affine transformation of the parameters of the characteristic determinant, the characteristic equation has $N = r_1 + r_2 + \ldots + r_n$ real roots (not necessarily distinct) for one of the parameters for arbitrary real values of all the others. Here, we should recall again that the characteristic equation includes the condition that $|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1$. A system of equations is said to be *elliptic*, *parabolic*, or *hyperbolic* in a region if it belongs to the corresponding type at all points of that region.

We note in the case of the general quasi-linear system of Eqs. (12) that the coefficients of the characteristic determinant $\Delta(\alpha)$

depend, as is clear from the preceding discussion, not only on the point in the space of the independent variables $x_1, x_2, \ldots, x_n$ but also on the choice of solution, which is being examined at the given point. The same applies to the characteristic directions, to the characteristics, and to the question of the type to which the system belongs; that is, at a given point $x_1^0, x_2^0, \ldots, x_n^0$, all this depends on the choice of solution of the system that we are considering. This situation does not take place in linear systems since then the characteristic determinant does not depend on the choice of solution. From the point of view of the general classification of equations, linear systems of first-order equations possess a number of special properties. Every system of this type can be written in the form

$$\sum_{j=1}^{p}\Big(\sum_{k=1}^{n} A_{ij}^{k}\frac{\partial}{\partial x_k}\Big)u_j + f_i = 0, \qquad i = 1, 2, \ldots, p, \tag{24}$$

where $f_i = \sum_{j=1}^{p} C_{ij}u_j + D_i$, A_{ij}^{k}, C_{ij} and the D_i are functions of the independent variables. Here, the characteristic determinant $\Delta(\alpha)$ may be written in the form

$$\Delta(\alpha) = \begin{vmatrix} \sum_{k=1}^{n} A_{11}^{k}\alpha_k & \ldots & \sum_{k=1}^{n} A_{1p}^{k}\alpha_k \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot & & \\ \sum_{k=1}^{n} A_{p1}^{k}\alpha_k & \ldots & \sum_{k=1}^{n} A_{pp}^{k}\alpha_k \end{vmatrix}. \tag{24'}$$

When we compare this characteristic determinant with the system of Eqs. (24), it is clear that, when we replace the equations in system (24) by linear combinations of them with coefficients depending on the independent variables, and also, when we replace the unknown functions by linear combinations of them with coefficients depending on the independent variables, the value of the characteristic determinant remains unchanged since such replacements affect the characteristic determinant only by replacing its rows with linear combinations of the rows or its columns with linear combinations of the columns. Thus, in the case of a first-order linear system, the characteristic determinant and, consequently, the characteristics and the property of belonging to one or the other of the types listed are invariant under replacement of the equations in the system and the unknown functions by linear combinations with coefficients depending only on the independent variables.

Let us give some simple examples that will clarify the basic concepts associated with the general classification of quasi-linear systems of equations.

For a first-order quasi-linear equation

$$\sum_{i=1}^{n} a_i p_i - a = 0, \quad p_i = \frac{\partial u}{\partial x_i} \tag{25}$$

the equation for the characteristic variables can be written in the form

$$\sum_{i=1}^{n} a_i \varphi_{x_i} = 0 \tag{26}$$

under the condition that $|\varphi_{x_1}|^2 + |\varphi_{x_2}|^2 + \ldots + |\varphi_{x_n}|^2 \neq 0$.

If we write the characteristic $\varphi(x_1, x_2, \ldots, x_n) = 0$ in parametric form $x_i = x_i(t_1, t_2, \ldots, t_{n-1})$, where the $t_1, t_2, \ldots, t_{n-1}$ are real variables, we see from (26) that the vector $(\varphi_{x_1}, \varphi_{x_2}, \ldots, \varphi_{x_n})$ giving the direction of the normal to the characteristic is orthogonal to the vector $(a_1, a_2, \ldots, a_n)$. This means that the last vector must lie in the tangent plane to the characteristic, that is, it must be a linear combination of the vectors $(\partial x_1/\partial t_i, \partial x_2/\partial t_i, \ldots, \partial x_n/\partial t_i)$ for $i = 1, 2, \ldots, n-1$. But for this, it is necessary and sufficient that

$$\delta = \begin{vmatrix} a_1 & a_2 & a_n \\ \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} & \dfrac{\partial x_n}{\partial t_1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial t_{n-1}} & \dfrac{\partial x_2}{\partial t_{n-1}} & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = 0. \tag{27}$$

We may regard Eq. (27) as the equation of the characteristics for the first-order quasi-linear equation (25) in vector form.

In the case of the three-dimensional Poisson equation, the characteristic equation is $\Delta(\alpha) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$ for $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$, and the equation of the characteristics is written in the form $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0$ under the conditions $|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_z|^2 \neq 0$ and $\varphi(x, y, z) = 0$. Here, there will be no real characteristic directions or real characteristics. In the case of the three-dimensional heat-flow equation, the characteristic equation takes the form $\Delta(\alpha) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$ with the condition $|\alpha_1|^2 = |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1$, and the equation of the characteristics is $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0$ under the conditions $|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_z|^2 + |\varphi_t|^2 \neq 0$ and $\varphi(x, y, z, t) = 0$. Here, the real characteristic directions will be the hyperplanes

$t - t^0 = 0$. These same hyperplanes will be the real characteristics. In the case of the three-dimensional wave equation, the characteristic equation is written in the form $\Delta(\alpha) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 = 0$ under the condition $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1$ and the equation of the characteristics is $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \varphi_t^2 = 0$ under the conditions $|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_z|^2 + |\varphi_t|^2 \neq 0$ and $\varphi(x, y, z, t) = 0$. Here, the real characteristic directions are the hyperplanes that form an angle equal to $\arccos \alpha_4 = \pi/4$ with the t-axis. The real characteristics will be these same hyperplanes and in addition the hypercones enveloping them with vertex angle equal to $\pi/2$.

For the system of equations

$$\begin{cases} au_x + bu_y - v_y = 0, \\ du_x + cu_y + v_x = 0, \end{cases}$$

where a, b, c, and d, are functions of x and y, the characteristic determinant can be written in the form

$$\Delta(a) = \begin{vmatrix} (a\alpha_1 + b\alpha_2) & -\alpha_2 \\ (d\alpha_1 + c\alpha_2) & \alpha_1 \end{vmatrix} = a\alpha_1^2 + (b + d)\alpha_1\alpha_2 + c\alpha_2^2.$$

A given system of equations is elliptic if $\delta = ac - (b + d)/2 > 0$; it is hyperbolic if $\delta < 0$; and it is parabolic if $\delta = 0$.

For the biharmonic equation

$$\Delta\Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

the characteristic determinant takes the form $\Delta(\alpha) = (\alpha_1^2 + \alpha_2^2)^2$. Thus, this equation is elliptic.

For the so-called equation of transverse vibrations of a plate

$$\Delta\Delta u - \frac{2h\rho}{D}\frac{\partial^2 u}{\partial t^2} = 0,$$

where h, ρ, and D are constants,* the characteristic equation can be written in the form $\Delta(\alpha) = (\alpha_1^2 + \alpha_3^2)^2 = 0$ with the condition $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$. Thus, this equation is parabolic. The planes $t = t^0$ are both the real characteristic directions and the real characteristics.

For the system of equations of a plane steady potential flow of gas

*For example, see S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, 1959, p. 334.

$$\left(1-\frac{v_x^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_y^2}{c_*^2}\right)\frac{\partial v_x}{\partial x}+$$

$$+\left(1-\frac{v_y^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_x^2}{c_*^2}\right)\frac{\partial v_y}{\partial y}-2v_xv_y\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)=0,$$

$$-\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}=0,$$

where v_x and v_y are the projection of the velocity vector onto the x- and y-axes and $\varkappa = c_p/c_v = c_*$ is a constant known as the critical velocity, the characteristic determinant can be written in the form

$$\Delta(\alpha)=\begin{vmatrix} A_1\alpha_1-2v_xv_y\alpha_2 & -2v_xv_y\alpha_1+A_2\alpha_2 \\ \alpha_2 & \alpha_1 \end{vmatrix},$$

where

$$A_1=1-\frac{v_x^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_y^2}{c_*^2},$$

$$A_2=1-\frac{v_y^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_x^2}{c_*^2},$$

or

$$\Delta(\alpha)=\left(1-\frac{v_x^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_y^2}{c_*^2}\right)\alpha_1^2+\left(1-\frac{v_y^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_x^2}{c_*^2}\right)\alpha_2^2.$$

Consequently, the given system of equations will be elliptic when

$$\delta=\left(1-\frac{v_x^2}{c_*^2}-\frac{\varkappa-1}{\varkappa+1}\frac{v_y^2}{c_*^2}\right)\left(1-\frac{v_y^2}{c_*^2}\frac{\varkappa-1}{\varkappa+1}\frac{v_y^2}{c_*^2}\right)>0$$

and hyperbolic when $\delta<0$. Thus, the dependence of the type of system on the choice of solution that we wish to consider shows up quite clearly.

2. Classification of second-order linear equations, their characteristics, and their reduction to canonical form.

A number of special features of the general classification of equations holds for a second-order linear equation. The general form of such an equation in n independent variables is

$$\sum_{i,j=1}^{n}A_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}+f=0, \tag{28}$$

where

$$f = \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Bu + C$$

and the A_{ij}, the B_i, B, and C are functions of the independent variables. The characteristic determinant $\Delta(\alpha)$ for this equation is a homogeneous quadratic form

$$\Delta(\alpha) = \sum_{i,j=1}^{n} A_{ij}\alpha_i\alpha_j. \tag{28'}$$

Consequently, as we know from algebra, there is a nonsingular affine transformation

$$\alpha_i = \sum_{k=1}^{n} a_{ki}\lambda_k, \quad i = 1, 2, \ldots, n \tag{23}$$

that reduces it to canonical form:

$$\Delta(\alpha) = \Delta(\lambda) = \sum_{i=1}^{n} \varepsilon_i \lambda_i^2, \tag{29}$$

where each of the ε_i is one or the other of the three constants +1, −1, and 0. The affine transformation of the independent variables associate with the transformation (23)

$$\xi_k = \sum_{i=1}^{n} a_{ki} x_i, \quad k = 1,2, \ldots, n \tag{22}$$

converts Eq. (28) to the form

$$\sum_{i,j=1}^{n} A_{ij}^{\bullet} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + f = 0, \tag{30}$$

where the $A_{ij}^{\bullet}$ are coefficients satisfying the condition

$$A_{ii}^{\bullet} = \varepsilon_i, \quad A_{ij}^{\bullet} = 0 \quad \text{for} \quad i \neq j \tag{31}$$

at the point $x_1^0, x_2^0, \ldots, x_n^0$.

Equation (30) is called *the canonical form of a second-order linear equation with n independent variables at a point* $x_1^0, x_2^0, \ldots, x^0$. In accordance with the general classification, this equation is

elliptic if all the ε_i are nonzero and have the same sign. It is hyperbolic if all the ε_i are nonzero and all but one of them have the same sign. It is parabolic if at least one of the ε_i is zero.

If conditions (31) hold not only at the given point $(x_1^0, x_2^0, \ldots, x_n^0)$ but throughout some region, Eq. (30) is called the *canonical form of the second-order linear equation in n independent variables in that region.* Typical examples of the canonical form of a second-order linear equation throughout the entire space of the independent variables are Poisson's equation $\Delta u = -F$ (elliptic case), the heat-flow equation $\Delta u - \partial u/\partial t = -F$ (parabolic case), and the wave equation $\Delta u - \partial^2 u/\partial t^2 = -F$ (hyperbolic case).

According to Eqs. (23) and (22), the question of reducing the linear second-order equation (28) to canonical form at a given point is completely solved by means of an affine transformation of the independent variables that is determined from the condition that this transformation must be associated with the transformation of parameters of the characteristic determinant (28′) that reduces it to canonical form.

The question arises as to whether it may not be possible by means of some, not necessarily affine, transformation of the independent variables to reduce a second-order linear equation in n independent variables to canonical form not just at a particular point but throughout some region. It turns out that in the general case, when the coefficients A_{ij} of the second derivatives are variable, this question does not have an affirmative answer.*

An affirmative answer to this question is known only for the case of two independent variables.

A second-order linear equation in two independent variables can be written in the general case in the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + f = 0, \tag{32}$$

where $f = D\partial u/\partial x + E\,\partial u/\partial y + Fu + G$, A, B, C, D, E, F, and G are functions of the independent variables x and y.** Here, all the quantities appearing in (32) are assumed to be real. The characteristic determinant in this case is $\Delta(\alpha) = A\alpha_1^2 + 2B\alpha_1\alpha_2 + C\alpha_2^2$. Denoting by δ the discriminant of the characteristic determinant

$$\delta = B^2 - AC, \tag{33}$$

we see that Eq. (32) will be hyperbolic if $\delta > 0$, parabolic if $\delta = 0$, and elliptic if $\delta < 0$.

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*If the coefficients of the highest derivatives are constants, this question obviously does not arise since it can be solved by the affine transformation of the independent variables indicated above.

**In all that follows, the linearity of the dependence of f on u, $\partial u/\partial x$, and $\partial u/\partial y$ is not essential. What is essential is that A, B, and C depend only on the independent variables.

First of all, let us note that if $A=C=0$, then Eq. (32) can be written in the form

$$\frac{\partial^2 u}{\partial x\,\partial y}+f_1=0, \tag{34}$$

where $f_1 = f/2B$. If we make the change of independent variables $\xi=x+y$, $\eta=x-y$, Eq. (34) is immediately reduced to the canonical form of the hyperbolic equation

$$\frac{\partial^2 u}{\partial \xi^2}-\frac{\partial^2 u}{\partial \eta^2}+f_1=0. \tag{34'}$$

Therefore, in all the following, we may without loss of generality take $A \neq 0$ in the region in question. This is evident since the rates of A and C can always be interchanged, and the case in which both these coefficients are zero has already been examined. The general transformation of independent variables

$$\xi=\xi(x, y),\ \eta=\eta(x, y), \tag{35}$$

where $\xi(x, y)$ and $\eta(x, y)$ are arbitrary twice continuously differentiable functions of x and y with nonvanishing Jacobian, reduces Eq. (32) to the form

$$A_1\frac{\partial^2 u}{\partial \xi^2}+2B_1\frac{\partial^2 u}{\partial \xi\,\partial \eta}+C_1\frac{\partial^2 u}{\partial \eta^2}+f'=0, \tag{32'}$$

where f_1 is a function that does not depend on the second derivatives of u and the coefficients A_1, B_1, and C_1 are determined by the equations

$$\begin{aligned} A_1&=A\xi_x^2+2B\xi_x\xi_y+C\xi_y^2,\\ C_1&=A\eta_x^2+2B\eta_x\eta_y+C\eta_y^2.\\ B_1&=A\xi_x\eta_x+B(\xi_x\eta_y+\xi_y\eta_x)+C\xi_y\eta_y. \end{aligned} \tag{36}$$

The equation of the characteristic variables that corresponds to Eq. (32) can be written in the form

$$A\varphi_x^2+2B\varphi_x\varphi_y+C\varphi_y^2=0 \tag{37}$$

with the condition $|\varphi_x|^2+|\varphi_y|^2\neq 0$. Equation (37) can be written in the form

$$\frac{\partial \varphi}{\partial x}=-\frac{B\pm\sqrt{\delta}}{A}\frac{\partial \varphi}{\partial y}. \tag{38}$$

For $\delta \geqslant 0$, that is, in the hyperbolic and parabolic cases, the coefficients in Eq. (38) are real. Therefore, the real characteristics $\varphi(x, y)=0$ are solutions of the ordinary differential equation

$$A\,dy^2 - 2B\,dx\,dy + C\,dx^2 = 0$$

or

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\delta}}{A} = \frac{C}{B \mp \sqrt{\delta}}. \tag{39}$$

When the coefficients A, B, and C in Eq. (39) are continuously differentiable, we obtain from it two families of real characteristics $\varphi(x, y)-\xi=0$ and $\psi(x, y)-\eta=0$ in the hyperbolic case and one family of real characteristics $\varphi(x, y)-\xi=0$ in the parabolic case. Here, ξ and η are arbitrary real parameters and $\varphi(x, y)$ and $\psi(x, y)$ are twice continuously differentiable functions of x and y. The Jacobian $\varphi_x\psi_y-\psi_x\varphi_y$ of the functions φ and ψ does not vanish since the slopes of the characteristics $\varphi(x, y)-\xi=0$ and $\psi(x, y)-\eta=0$, namely, $dy/dx = (B+\delta)/A$ and $dy/dx = (B-\delta)/A$, differ by a quantity which does not vanish. Therefore, in the hyperbolic case, we may take as our new variables the functions $\xi=\varphi(x, y)$ and $\eta=\psi(x, y)$. Here, comparison of Eqs. (36) with the equation of the characteristic variables (37) shows that $A_1=C_1=0$ and that the coefficient B_1 cannot vanish identically because the order of the equation is invariant under a change of independent variables. Thus, Eq. (32′) after division by $2B_1$ becomes Eq. (34), which, as we noted above, is easily reduced to the canonical form of a hyperbolic equation. In the parabolic case, we take as one of the independent variables the function $\xi=\varphi(x, y)$ and, as the other independent variable, we take an arbitrary function $\eta=\eta(x, y)$ whose Jacobian $\xi_x\eta_y-\xi_y\eta_x$ is nonzero. This last condition is always satisfied if we require that the trajectories $\xi=\text{const}$ and $\eta=\text{const}$ intersect at a nonzero angle. Now, just as in the preceding case, we have $A_1=0$, and we obtain

$$B_1 = A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\,\xi_y\eta_y = \eta_x(A\xi_x + B\xi_y) + \\ + \eta_y(B\xi_x + C\xi_y).$$

Remembering that $\delta=0$ by virtue of (39), we obtain $B_1=0$. If we divide Eq. (32′) by a coefficient C_1 (which, because of the invariance of the order of the equation under change of independent variables, is nonzero), this equation is reduced to the canonical form of a parabolic equation:

$$\frac{\partial^2 u}{\partial \eta^2} + f_1\left(\xi_1\,\eta,\ \frac{\partial u}{\partial \xi},\ \frac{\partial u}{\partial \eta}\right) = 0.$$

Let us now consider the elliptic case, that is, the case in which $\delta<0$. Here, since $|\varphi_x|^2+|\varphi_y|\neq 0$, the equation of the characteristic variables (37) or, what amounts to the same thing, (38) cannot be satisfied by any real function $\varphi(x, y)$. Therefore, it is natural to seek this function, the characteristic variable, in the form of a complex function $\varphi(x, y)=\xi+i\eta$, where $\xi=\xi(x, y)$ and $\eta=\eta(x, y)$ are real functions of x and y. For definiteness, let us take the negative sign in front of the radical $\sqrt{\delta}$ in Eq. (38). If we separate the real and imaginary parts of this equation in order to determine the functions ξ and η, we obtain the system

$$\begin{aligned}\xi_x &= -\frac{B}{A}\xi_y+\frac{\sqrt{|\delta|}}{A}\eta_y,\\ \eta_x &= -\frac{B}{A}\eta_y-\frac{\sqrt{|\delta|}}{A}\xi_y.\end{aligned} \tag{40}$$

The Jacobian of the functions $\xi(x, y)$ and $\eta(x, y)$ is then given by

$$\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\frac{\sqrt{|\delta|}}{A}(\xi_y^2+\eta_y^2). \tag{40'}$$

The existence of a solution of the system (40) when the determinant (40′) is not zero can be proven for arbitrary continuously differentiable coefficients A, B, and C.* At present, we shall prove that a solution to the system (40) exists under the assumption that the coefficients A, B, and C are analytic functions of x and y, using only the facts that we have already established. Equation (38) is a normal equation with respect to the variable x with unknown function φ. If we have analytic initial conditions $\varphi|_{x=x^0}=\varphi^0(y)$ in a neighborhood of the point (x^0, y^0) such that $\partial\varphi/\partial y\neq 0$ in a neighborhood of that point, we see that the conditions of Kovalevskaya's theorem are satisfied. (The hypotheses of that theorem permit the right-hand members of the system to be complex.) Therefore, there exists an analytic solution of Eq. (38) $\varphi(x, y)=\xi+i\eta$ and hence there exists a solution $\xi=\xi(x, y)$, $\eta=\eta(x, y)$ of the system (40). Furthermore, since $\partial\varphi/\partial y=\xi_y+i\eta_y$, we see on the basis of (40′) that the Jacobian $\xi_x\eta_y-\xi_y\eta_x$ is nonzero. Now, the functions $\xi=\xi(x, y)$ and $\eta=\eta(x, y)$ may be taken as the new independent variables. Recalling that the function $\varphi=\xi+i\eta$ must satisfy Eq. (37), when we separate the real and imaginary parts of that equation, we see that $A_1=C_1$ and $B_1=0$. If we now divide both parts of Eq. (32′) by the coefficients A_1 (which, because of the invariance of the order of the equation under transformation of the independent variables, is not zero), we obtain the canonical form of an elliptic equation

*See R. Courant and D. Hilbert, *Methods of Mathematical Physics*, New York, Interscience, 1962, pp. 159, 350. I. N. Vekua, *Doklady Akad. Nauk*, SSSR, *100*, No. 2, 1955.

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + f_1\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right) = 0.$$

This completely settles the question of reducing a second-order linear equation with two independent variables to canonical form in some region or other.

For example, consider the so-called Tricomi equation, which is often encountered in gas dynamics,

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

For $y>0$, by making the change of variables $\xi=x$, $\eta=2y^{3/2}/3$, we reduce this equation to the canonical form of an elliptic equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{3\eta}\frac{\partial u}{\partial \eta} = 0.$$

For $y<0$, by making the change of variables $\xi=x-2(-y)^{3/2}/3$, $\eta=x+2(-y)^{3/2}/3$, we reduce it to the canonical form of a hyperbolic equation

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} - \frac{1}{6(\xi-\eta)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0.$$

We note that the Tricomi equation is often called an *equation of mixed type* since it is elliptic for $y>0$, hyperbolic for $y<0$, and parabolic for $y=0$.*

3. Classification and characteristics of general nonlinear systems of equations. In conclusion, let us show how the concepts introduced for the case of quasi-linear systems of equations, and their classification, can be extended to the case of completely arbitrary systems of partial differential equations.

The most general form of a system of equations in p unknown functions $u_1, u_2, \ldots, u_p$, with the highest-order derivatives of these functions having orders $r_1, r_2, \ldots, r_p$, respectively, is

$$\Phi_i\left(x_1, x_2, \ldots, x_n, u_1, \ldots, u_p, \frac{\partial u_1}{\partial x_1}, \ldots, \frac{\partial^{r_p} u_p}{\partial x_n^{r_p}}\right) = 0$$
$$(i = 1, 2, \ldots, p),$$

where the Φ_i are given functions of the independent variables $x_1, x_2, \ldots, x_n$ of the unknown functions $u_1, u_2, \ldots, u_p$ and their derivatives

*For further reading, see F. Tricomi, "Sulle equazioni lineari alle derivate parziali di seconde ordine di tipo misto," *Rend.* reale accad. lincei, Ser. 5, *14*, 134-247 (1923); F. Tricomi, *Equazioni a derivate parziali*, Rome, Edizioni Cremonese, 1957, A. V. Bitsadze, *Equations of mixed type*, New York, Macmillan, 1964.

of order up to $r_1, r_2, \ldots, r_p$, respectively. For this system, we construct the characteristic determinant (16) at each point of the space of the independent variables for the chosen solution just as in the case of the quasi-linear system (12) except that, here, we get

$$A_{ij}^{k_1 k_2 \ldots k_n} = \frac{\partial}{\partial \left(\dfrac{\partial^{r_j} u_j}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}} \right)} \Phi_i.$$

Then, all the other concepts are introduced with the aid of this characteristic determinant just as for the case of quasi-linear systems. Thus, for example, in the case of a first-order equation

$$F(x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n) = 0, \quad p_i = \frac{\partial u}{\partial x_i} \tag{25'}$$

the characteristic determinant takes the form

$$\Delta(\alpha) = \sum_{i=1}^{n} F_{p_i} \alpha_i, \quad F_{p_i} = \frac{\partial F}{\partial p_i}, \tag{25'}$$

and the characteristic equation may be written in the form $\Delta(\alpha) = 0$ under the condition that $|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2 = 1$. The characteristic directions may be written in the form $\sum_{i=1}^{n} \alpha_i (x_i - x_i^0) = 0$, where the α_i are roots of the characteristic equation. The equation of the characteristic variables may be represented in the form

$$\sum_{i=1}^{n} F_{p_i} \varphi_{x_i} = 0$$

under the condition that $|\varphi_{x_1}|^2 + |\varphi_{x_2}|^2 + \ldots + |\varphi_{x_n}|^2 \neq 0$, and the equation of the characteristics is the same as the equation of the characteristic variables, except that it is subject to the extra condition that $\varphi(x_1, x_2, \ldots, x_n) = 0$. In particular, if the characteristics are represented in the vector form $x_i = x_i(t_1, t_2, \ldots, t_{n-1})$, where $t_1, t_2, \ldots, t_{n-1}$ are real variables, then their equation will be

$$\delta = \begin{vmatrix} F_{p_1} & F_{p_2} & \ldots & F_{p_n} \\ \dfrac{\partial x_1}{\partial t_1} & \dfrac{\partial x_2}{\partial t_1} & \ldots & \dfrac{\partial x_n}{\partial t_1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial x_1}{\partial t_{n-1}} & \dfrac{\partial x_2}{\partial t_{n-1}} & \ldots & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = 0. \tag{27'}$$

The classification for general systems of equations is accomplished in exactly the same way, that is, from the characteristic determinant of the given system as was indicated above.

In particular, for a second-order nonlinear equation

$$\Phi\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\right) = 0 \tag{41}$$

all the concepts associated with the classification are based on the equation

$$\sum_{i,j=1}^{n} A_{ij}(x_1, x_2, \ldots, x_n) \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{42}$$

where

$$A_{ij} = \frac{\partial}{\partial\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)} \Phi.$$

For example, Eq. (41) will be elliptic, hyperbolic, or parabolic at a given point for a particular solution, depending on which one of these types Eq. (42) belongs to at the given point.

We note that these concepts of hyperbolic, elliptic, and parabolic systems of equations include only the most important and the most frequently encountered systems of equations. Cases can be found in which the system does not belong to any one of these types. This, for example, is true of the equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{k=1}^{m} \frac{\partial^2 u}{\partial y_k^2} = 0 \quad (n, m \geqslant 2),$$

which is sometimes called *ultrahyperbolic.*

3. Solution of the Cauchy Problem for First-Order Equations. The Method of Characteristic Curves. Examples

Let $u = u(x_1, x_2, \ldots, x_n)$ be any solution of the first-order equation

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0, \; p_i = \frac{\partial u}{\partial x_i}, \; \sum_{i=1}^{n} F_{p_i}^2 > 0. \tag{25'}$$

Let us agree to interpret such a solution as an n-dimensional surface in the $(n+1)$-dimensional space of $x_1, x_2, \ldots, x_n, u$. We shall call this surface an *integral surface*. It turns out that Eq. (25′) does not uniquely determine the tangent plane of the integral surface at every fixed point $(x_1, x_2, \ldots, x_n\ u)$, instead, there is a set of possible tangent planes, which depends on $n-1$ parameters:

$$\sum_{i=1}^{n} p_i (X_i - x_i) - (U - u) = 0, \quad F(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0, \tag{43}$$

where X_i and U are the moving coordinates of the planes. The envelope of this set of possible tangent planes is some cone with rectilinear generators and with apex at the point $(x_1, x_2, \ldots, x_n, u)$. This cone is called the *characteristic cone* or the *Monge cone*. Let us find the equation of this cone and the equations of its rectilinear generators. Taking $p_2, p_3, \ldots, p_n$ as independent parameters, we see that any one of the planes (43) must intersect the planes

$$\left(p_1 + \frac{\partial p_1}{\partial p_k} dp_k\right)(X_1 - x_1) + \ldots + (p_k + dp_k)(X_k - x_k) + \ldots$$
$$\ldots + p_n (X_n - x_n) - (U - u) = 0, \quad k = 2, 3, \ldots, n$$

which are infinitesimally close to it. Consequently,

$$\frac{\partial p_1}{\partial p_2}(X_1 - x_1) + (X_2 - x_2) = 0, \ \ldots, \ \frac{\partial p_1}{\partial p_n}(X_1 - x_1) + (X_n - x_n) = 0.$$

On the other hand, on the basis of the second of Eqs. (43), we have

$$F_{p_1} \frac{\partial p_1}{\partial p_2} + F_2 = 0, \ \ldots, \ F_{p_1} \frac{\partial p_1}{\partial p_n} + F_{p_n} = 0$$

or

$$\frac{X_1 - x_1}{F_{p_1}} = \frac{X_2 - x_2}{F_{p_2}} = \ldots = \frac{X_n - x_n}{F_{p_n}}.$$

By using the first of Eqs. (43), we obtain

$$U - u = \frac{X_1 - x_1}{F_{p_1}} \left(p_1 F_{p_1} + p_2 F_{p_1} \frac{X_2 - x_2}{X_1 - x_1} + \ldots + p_n F_{p_1} \frac{X_n - x_n}{X_1 - x_1} \right) =$$
$$= \frac{X_1 - x_1}{F_{p_1}} \sum_{i=1}^{n} p_i F_{p_i}.$$

Thus, we obtain the desired equation of the characteristic cone in the form

$$\frac{X_1-x_1}{F_{p_1}}=\frac{X_2-x_2}{F_{p_2}}=\ldots=\frac{X_n-x_n}{F_{p_n}}=\frac{U-u}{\sum\limits_{i=1}^{n}p_iF_{p_i}},$$

$$F(x_1, \ldots, x_n, u, p_1, \ldots, p_n)=0. \tag{44}$$

For fixed values of the parameters $p_1, p_2, \ldots, p_n$, the equation of the cone (44) is the equation of its corresponding rectilinear generator.

In the case of a quasi-linear equation

$$\sum_{i=1}^{n}a_ip_i-a=0,\ p_i=\frac{\partial u}{\partial x_i} \tag{25}$$

the equation of the characteristic cone (44) takes the form

$$\frac{X_1-x_1}{a_1}=\frac{X_2-x_2}{a_2}=\ldots=\frac{X_n-x_n}{a_n}=\frac{U-u}{a}, \tag{45}$$

and, in this case, the characteristic cone itself degenerates into a ray passing through the point $(x_1, x_2, \ldots, x_n, u)$.

Thus, we conclude that at every point of the space $x_1, \ldots, x_n, u$ the first-order partial differential equation (25′) determines a characteristic cone, which is defined by the system of Eqs. (44) in the general case. The set of these characteristic cones play a role in the theory of first-order partial differential equations similar to that of the field of tangents to integral curves in the theory of ordinary differential equations.

The Cauchy problem for first-order equations may be formulated as follows: *Let S be an $(n-1)$-dimensional surface in the space of the independent variables $x_1, x_2, \ldots, x_n$, which, for definiteness, may be given in the vector form $x_i=x_i(t_1, t_2, \ldots, t_{n-1})$, where the $x_i(t_1, t_2, \ldots, t_{n-1})$ are continuously differentiable functions of real parameters $t_1, t_2, \ldots, t_{n-1}$. Suppose that the rank of the matrix $(\partial x_i/\partial t_\nu)$ is $n-1$. Find a solution $u=u(x_1, x_2, \ldots, x_n)$ of Eq. (25′) or, in particular, (25) such that*

$$u|_{x_i=x_i(t_1, \ldots, t_{n-1})}=u(t_1, t_2, \ldots, t_{n-1}),$$

where $u(t_1, t_2, \ldots, t_{n-1})$ is a given continuously differentiable function of its arguments. In other words, this means that we must find the integral surface of the given equation which passes through the $(n-1)$-dimensional surface S of the space $x_1, x_2, \ldots, x_n, u$ defined by the equations $x_i=x_i(t_1, t_2, \ldots, t_{n-1})$, $u=u(t_1, t_2, \ldots, t_{n-1})$.

Let us first consider this problem for the quasi-linear equation (25). Here, we assume that $\sum\limits_{i=1}^{n}a_i^2>0$ and that the coefficients

$a_1, a_2, \ldots, a_n$ and a in this equation are continuously differentiable with respect to all their arguments. We shall assume that the integral surface is continuously differentiable.

First of all, let us note that the envelopes of the characteristic cones—the rays (45) satisfy the system of differential equations

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \ldots = \frac{dx_n}{a_n} = \frac{du}{a} = ds, \tag{45'}$$

where s is a real variable. The system of equations (45′) is complete and may be treated independently of the way in which we obtain it. This system of equations is called the *characteristic system* corresponding to Eq. (25), and every solution of it is called a *characteristic curve*. From our knowledge of the properties of ordinary differential equations [the system (45′)], under the assumptions made regarding the coefficients in this system, must always have a unique continuously differentiable solution satisfying the stated initial conditions. If the initial conditions are continuously differentiable functions of certain parameters, so is the solution of the system.

Suppose now that $x_i = x_i(s, t_1, \ldots, t_{n-1})$, $u = u(s, t_1, \ldots, t_{n-1})$ is a solution of the system (45′) such that $x_i|_{s=0} = x_i(t_1, \ldots, t_{n-1})$ and $u|_{s=0} = u(t_1, \ldots, t_{n-1})$; that is, suppose that this solution is a characteristic curve passing through some point of the initial surface S^*. Let us see whether $x_i = x_i(s, t_1, \ldots, t_{n-1})$, $u = u(s, t_1, \ldots, t_{n-1})$ for variable values of the parameters $s, t_1, t_2, \ldots, t_{n-1}$ can be regarded as an n-dimensiomal surface and whether this surface will be an integral surface of Eq. (25).

From the characteristic system of Eq. (45′), we have

$$\begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_2}{\partial s} & \ldots & \frac{\partial x_n}{\partial s} \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \ldots & \frac{\partial x_n}{\partial t_1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x_1}{\partial t_{n-1}} & \frac{\partial x_2}{\partial t_{n-1}} & \ldots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \ldots & \frac{\partial x_n}{\partial t_1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x_1}{\partial t_{n-1}} & \frac{\partial x_2}{\partial t_{n-1}} & \ldots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = \delta.$$

From this it follows that if the surface S is not a characteristic surface, that is, if $\delta \neq 0$, we may take $x_1, x_2, \ldots, x_n$ as our independent variables instead of $s, t_1, t_2, \ldots, t_{n-1}$. Here, $s, t_1, \ldots, t_{n-1}$ will be continuously differentiable functions of $x_1, x_2, \ldots, x_n$ and $u = u(s, t_1, \ldots, t_{n-1})$ will be a continuously differentiable surface $u = u(x_1, x_2, \ldots, x_n)$ in the space $x_1, x_2, \ldots, x_n, u$. Furthermore, the last of the equations of the characteristic system $a^{-1}\,du = ds$ can be written in the form

$$a = \frac{du}{ds} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{dx_i}{ds} = \sum_{i=1}^{n} a_i p_i.$$

Consequently, $u = u(x_1, x_2, \ldots, x_n)$ will be an integral surface of Eq. (25). This integral surface will be unique. This is true because, if we were to find another integral surface passing through S^*, it must also be composed of the characteristic curves passing through S^*, and these characteristic curves are uniquely determined because of the uniqueness of the solutions of the characteristic system (45′). Thus, if the surface S on which the initial conditions are given is not a characteristic, the Cauchy problem will have a unique solution.

Suppose now that S is a characteristic. Let us assume that the Cauchy problem has a solution $u = u(x_1, x_2, \ldots, x_n)$. Then, on the original surface S^*, we have

$$\frac{\partial u}{\partial t_\nu} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_\nu}.$$

From the condition that $\delta = 0$, we obtain

$$a_i = \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial x_i}{\partial t_\nu},$$

where the λ_ν are parameters. Now, from the differential equation (25) itself, we obtain

$$a = \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial x_i}{\partial t_\nu} = \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial u}{\partial t_\nu}.$$

From this we see that the vector $(a_1, a_2, \ldots, a_n, a)$ determining the direction of the characteristic ray at a given point on S^* is a linear combination of the $n-1$ vector $(\partial x_1/\partial t_\nu, \ldots, \partial x_n/\partial t_\nu, \partial u/\partial t_\nu)$, where $\nu = 1, 2, \ldots, n-1$, and hence lies in the tangent plane to S^*. The surface S^* is composed of the envelopes of these rays, that is, of the characteristic curves. Thus, for the Cauchy problem to have a solution, it is necessary that the surface S^* be composed of the characteristic curves.

Let us suppose that this condition is satisfied. Let us construct an $(n-1)$-dimensional surface S_1^* intersecting S^* along some $(n-2)$-dimensional surface Q^* such that its projection S_1 onto the space of the independent variables $x_1, x_2, \ldots, x_n$ is not a characteristic. This can be done in infinitely many ways as can be seen especially clearly for $n=2$, when S^* coincides with the characteristic curve

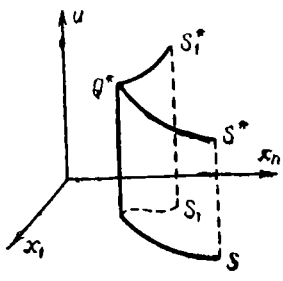

Fig. 3.

and Q^* is a point in three-dimensional space (see Fig. 3). Since S_1 is not a characteristic, there exists an integral surface passing through S_1^* and this integral surface must be composed of the characteristic curves passing through S_1^*. Those characteristic curves passing through Q^* together constitute the initial surface S^*. Thus, the integral surface passing through S_1^* will also pass through the initial surface S^*. But the surface S_1^* is not uniquely defined: it is possible to construct infinitely many such surfaces. Therefore, in the given case, the Cauchy problem has infinitely many solutions. This completely answers the question regarding the Cauchy problem for the quasi-linear equation (25).

Let us look now at the Cauchy problem for the general (not quasi-linear) equation (25′). Here, we shall assume that the function F is three times continuously differentiable with respect to all its arguments* and that the desired integral surface $u=u(x_1, x_2, \ldots, x_n)$ is twice continuously differentiable. In addition, we shall impose one other condition on the surface S on which the initial conditions are given and on the original surface S^*. Specifically, we shall require that the system of equations

$$u_{t_\nu} = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_\nu}, \quad \nu = 1, 2, \ldots, n-1, \tag{46}$$

$$F(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) = 0$$

be solvable uniquely for the parameters $p_1, p_2, \ldots, p_n$ on the surface S^* and that these parameters, as functions of $x_i=x_i(t_1, \ldots, t_{n-1})$, for $i=1, 2, \ldots, n$, be twice continuously differentiable with respect to $t_1, t_2, \ldots, t_{n-1}$. It is easy to see that this additional condition does not introduce any real restrictions since the first of Eqs. (46) can be solved for any $n-1$ of the parameters $p_1, p_2, \ldots, p_n$ (since the rank of the matrix $(\partial x_i / \partial t_\nu)$ is equal to $n-1$).

Let us assume first that Eq. (25′) has a twice continuously differentiable integral surface $u=u(x_1, x_2, \ldots, x_n)$. Through every point of this integral surface, there will pass a corresponding characteristic cone. One of the tangent planes of the characteristic cone will coincide with the tangent plane to the integral surface at the apex of the cone. One of the rectilinear generators of the characteristic cone, which we shall call the *characteristic ray*,

*As our subsequent calculations will show, this condition can be weakened. For example, we may replace it with the requirement that the functions $F_{p_1}, F_{p_2}, \ldots, F_{p_n}$ be twice continuously differentiable.

will lie in this tangent plane. The characteristic ray is uniquely determined by the coordinates of the point $(x_1, \ldots, x_n, u)$ through which it passes and by the quantities $p_1, p_2, \ldots, p_n, -1$ which represent the direction of the normal to the plane in which it lies. Let us set up the differential equations which the envelopes of the characteristic rays lying in the given integral surface must satisfy.

From (44), we have

$$\frac{dx_1}{F_{p_1}} = \frac{dx_2}{F_{p_2}} = \ldots = \frac{dx_n}{F_{p_n}} = \frac{du}{\sum_{i=1}^{n} p_i F_{p_i}} = ds, \tag{47}$$

where s is a real parameter. Furthermore, if we differentiate the last of Eqs. (44), that is, the differential equation (25′) itself, with respect to the independent variables, we obtain

$$F_{p_1}p_{1x_1} + F_{p_2}p_{2x_1} + \ldots + F_{p_n}p_{nx_1} + F_u p_1 + F_{x_1} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F_{p_1}p_{1x_n} + F_{p_2}p_{2x_n} + \ldots + F_{p_n}p_{nx_n} + F_u p_n + F_{x_n} = 0.$$

Since $p_i = \partial u/\partial x_i$, we obtain from this on the basis of (47)

$$\frac{dp_1}{ds} + F_u p_1 + F_{x_1} = 0, \ \ldots, \ \frac{dp_n}{ds} + F_u p_n + F_{x_n} = 0.$$

If we combine these equations with (47), we obtain differential equations which the envelopes of the characteristic rays must satisfy on the integral surface:

$$\frac{dx_1}{F_{p_1}} = \frac{dx_2}{F_{p_2}} = \ldots = \frac{dx_n}{F_{p_n}} = \frac{du}{\sum_{i=1}^{n} p_i F_{p_i}} =$$

$$= \frac{-dp_1}{p_1 F_u + F_{x_1}} = \frac{-dp_2}{p_2 F_u + F_{x_2}} = \ldots = \frac{-dp_n}{p_n F_u + F_{x_n}} = ds, \tag{48}$$

with the additional condition

$$F(x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n) = 0. \tag{48′}$$

The system of equations (48) is a complete system of ordinary differential equations with $2n+1$ unknown functions $x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n$ and it may be examined independently of the way in which we have obtained it.

This system is called the *characteristic system* corresponding to the given partial differential equation (25′).

On the basis of known properties of ordinary differential equations, we may say that, under the assumptions made above regarding the left-hand member of Eq. (25′), the characteristic system (48) with the given initial conditions has exactly one twice-continuously differentiable solution. Also, if the initial conditions are twice continuously differentiable functions of certain of the parameters, the solution of the characteristic system will possess this property also.

A particular property of the characteristic system (48) is that the left-hand member of the given partial differential equation (25′), that is, the function $F=F(x_1, \ldots, x_n, u, p_1, \ldots, p_n)$ is a first integral of it. By using (48), we have

$$\frac{dF}{ds} = \sum_{i=1}^{n} F_{x_i} \frac{dx_i}{ds} + F_u \frac{du}{ds} + \sum_{i=1}^{n} F_{p_i} \frac{dp_i}{ds} =$$

$$= \sum_{i=1}^{n} F_{x_i} F_{p_i} + F_u \sum_{i=1}^{n} p_i F_{p_i} + \sum_{i=1}^{n} - F_{p_i} (p_i F_u + F_{x_i}) \equiv 0.$$

But this means that the function F remains constant along the solutions of the characteristic system.

Every solution of the characteristic system (48)

$$x_1(s), \; x_2(s), \ldots, x_n(s), \; u(s), \; p_1(s), \ldots, p_n(s)$$

under the condition that

$$F(x_1, x_2, \ldots, x_n, u, p_1, p_2, \ldots, p_n) \equiv 0, \qquad (48')$$

along that solution, is called a *characteristic strip*, and the projection of this solution onto the space $x_1, x_2, \ldots, x_n, u$, that is, the curve $x_1(s), x_2(s), \ldots, x_n(s), u(s)$, which is the carrier of that strip, is called a *characteristic curve*.

Now, it will not be difficult to solve the Cauchy problem that has been posed. Suppose that

$$x_i = x_i(s, t_1, \ldots, t_{n-1}), \; u = u(s, t_1, \ldots, t_{n-1}), \; p_i = p_i(s, t_1, \ldots, t_{n-1})$$

is a solution of the characteristic system that satisfies the initial conditions

$$x_i|_{s=0} = x_i(t_1, \ldots, t_{n-1}), \quad u|_{s=0} = u(t_1, \ldots, t_{n-1}),$$

$$p_i|_{s=0} = p_i(t_1, \ldots, t_{n-1}),$$

where $p_i(t_1, \ldots, t_{n-1})$ are functions determined from the condition (46). Since F, is a first integral of the characteristic system, we have

$$F(x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n) = 0;$$

that is, the solution that we have constructed of the characteristic system is a characteristic strip.

By virtue of the characteristic system (48), we have

$$\begin{vmatrix} \frac{\partial x_1}{\partial s} & \dots & \frac{\partial x_n}{\partial s} \\ \frac{\partial x_1}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_1} \\ \frac{\partial x_1}{\partial t_{n-1}} & \dots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = \begin{vmatrix} F_{p_1} & \dots & F_{p_n} \\ \frac{\partial x_1}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_1} \\ \frac{\partial x_1}{\partial t_{n-1}} & \dots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} = \delta. \tag{49}$$

This means that if the surface S is not a characteristic, that is, if $\delta \neq 0$, we may take $x_1, x_2, \dots, x_n$ as our independent variables. Then, $s, t_1, \dots, t_{n-1}$ will, according to the implicit-function theorem, be twice continuously differentiable functions of $x_1, x_2, \dots, x_n$, and to the function $u = u(s, t_1, t_2, \dots, t_{n-1})$ will correspond, after a shift to the independent variables $x_1, x_2, \dots, x_n$, a completely determined twice continuously differentiable surface $u = u(x_1, x_2, \dots, x_n)$.

To prove that this surface is an integral surface, we need only show that $\partial u/\partial x_i = p_i$. For this, let us show that the quantities

$$u_s - \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial s} = V, \quad u_{t_\nu} - \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_\nu} = U_\nu, \tag{50}$$

$$\nu = 1, 2, \dots, n-1$$

are identically equal to 0 for arbitrary values of the variables $s, t_1, \dots, t_{n-1}$. From the characteristic system, we obtain directly

$$V = \sum_{i=1}^{n} p_i F_{p_i} - \sum_{i=1}^{n} p_i F_{p_i} \equiv 0.$$

Furthermore,

$$\frac{\partial U_\nu}{\partial s} = \frac{\partial U_\nu}{\partial s} - \frac{\partial V}{\partial t_\nu} = -\sum_{i=1}^{n} \left(\frac{\partial p_i}{\partial s} \frac{\partial x_i}{\partial t_\nu} - \frac{\partial p_i}{\partial t_\nu} \frac{\partial x_i}{\partial s} \right) =$$

$$= \sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial t_\nu} + \sum_{i=1}^{n} \frac{\partial x_i}{\partial t_\nu} (p_i F_u + F_{x_i}) =$$

$$= F_u \left(-u_{t_\nu} + \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_\nu} \right) + \left(\sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial t_\nu} + F_u u_{t_\nu} + \sum_{i=1}^{n} F_{x_i} \frac{\partial x_i}{\partial t_\nu} \right).$$

But the last term here vanishes since $F \equiv 0$. Therefore, we have

$$\frac{\partial U_\nu}{\partial s} = -F_u U_\nu, \quad U_\nu(s) = U(0)\, e^{-\int_0^s F_u ds}.$$

By virtue of the choice of initial conditions for p_i, specifically, by virtue of (46), we have $U_\nu(0) = 0$. Thus, $U_\nu = V \equiv 0$. Keeping this

as well as Eqs. (50) in mind, let us consider the following system of identities (which are a consequence of the rules for differentiation):

$$u_s - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial s} = 0, \quad u_{t_\nu} - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_\nu} = 0 \tag{51}$$

$$(\nu = 1, 2, \ldots, n-1).$$

This is a system of equations in the unknowns $\partial u/\partial x_i$. It coincides with the system of Eqs. (50) in the p_i. The determinant of this system is equal to $D(x_1, x_2, \ldots, x_n)/D(s, t_1, \ldots, t_{n-1})$ and consequently is nonzero. Since a system of linear algebraic equations with nonzero determinant has a unique solution, we conclude that $\partial u/\partial x_i = p_i$. This means that $u = u(x_1, x_2, \ldots, x_n)$ is indeed a twice continuously differentiable integral surface passing through the original surface S^*. It is easy to see that this surface is unique. This is true because any other integral surface passing through S^* must be composed of the characteristic curves passing through all possible points of the surface S^*. For these characteristic curves, the initial conditions for the p_i are determined from (46) and, consequently, such characteristic curves are uniquely determined and two distinct integral surfaces cannot be composed of them.

Suppose now that the surface S, that is, the surface on which the initial conditions are given, is a characteristic. Let us suppose that the desired solution of the Cauchy problem $u = u(x_1, x_2, \ldots, x_n)$ exists. Then, from the condition that $\delta = 0$, we have

$$F_{p_i} = \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial x}{\partial t_\nu} \qquad (i = 1, 2, \ldots, n),$$

where the $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are parameters. Therefore, since $p_i = \partial u_i/\partial x$ (because of the partial differential equation itself), we have

$$\sum_{i=1}^{n} p_i F_{p_i} = \sum_{i=1}^{n} u_{x_i} \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial x_i}{\partial t_\nu} = \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial u}{\partial t_\nu}.$$

If we differentiate the given equation (25′) with respect to x_k and replace $\partial p_i/\partial x_k$ with $\partial p_k/\partial x_i$, we obtain

$$-p_k F_u - F_{x_k} = \sum_{i=1}^{n} F_{p_i} \frac{\partial p_k}{\partial x_i} = \sum_{i=1}^{n} \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial x}{\partial t_\nu} \frac{\partial p_k}{\partial x_i} =$$

$$= \sum_{\nu=1}^{n-1} \lambda_\nu \sum_{i=1}^{n} \frac{\partial p_k}{\partial x_i} \frac{\partial x_i}{\partial t_\nu} = \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial p_k}{\partial t_\nu}.$$

Thus, for a solution to exist in this case, it is necessary that the vector

$$\vec{F} = \left(F_{p_1}, \ldots, F_{p_n}, \sum_{i=1}^{n} p_i F_{p_i}, -p_1 F_u - F_{x_1}, \ldots, -p_n F_u - F_{x_n}\right)$$

on the initial surface S^* be a linear combination of the $n-1$ vectors

$$\vec{X}_\nu = \left(\frac{\partial x_1}{\partial t_\nu}, \ldots, \frac{\partial x_n}{\partial t_\nu}, \frac{\partial u}{\partial t_\nu}, \frac{\partial p_1}{\partial t_\nu}, \ldots, \frac{\partial p_n}{\partial t_\nu}\right), \quad \nu = 1, 2, \ldots, n-1.$$

This means that the vector $\vec{F}$ must lie in the tangent plane to the $(n-1)$-dimensional surface S^{**} defined in the $(2n+1)$-dimensional space $x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n$ by the coordinates of the initial surface $x_i(t_1, t_2, \ldots, t_{n-1}), u(t_1, t_2, \ldots, t_{n-1})$, where $(i=1, 2, \ldots, n)$, and by the quantities $p_i(t_1, t_2, \ldots, t_{n-1})$, for $i=1, 2, \ldots, n$, which determine the direction of the normal to the tangent plane. Let us suppose now that this condition is satisfied. Let us construct in the space $x_1, x_2, \ldots, x_n, u$ a new initial surface S_1^* that intersects S^* along some $(n-2)$-dimensional surface Q^* in such a way that the projection S_1 of the surface S_1^* onto the space $x_1, x_2, \ldots, x_n$ is not a characteristic. This is always possible and in fact can be done in infinitely many ways, as can be shown especially clearly for $n=2$ when S^* is a characteristic curve and Q^* is a point (see Fig. 3). From what we have said above, there exists an integral surface passing through S_1^*, and this surface is composed of characteristic curves. Those characteristic curves passing through Q^* along with the quantities $p_1, p_2, \ldots, p_n$ that are assigned to them will constitute an $(n-1)$-dimensional surface R^{**} in the space $x_1, x_2, \ldots, x_n, u, p_1, \ldots, p_n$. By hypothesis, the vector $\vec{F}$ lies in the tangent plane to that surface and also in the tangent plane to S^{**}. Therefore, the surfaces R^{**} and S^{**} coincide because of the uniqueness of the solution of the characteristic system, and the projection R^{**} onto the space $x_1, \ldots, x_n, u$ coincides with the initial surface S^*. This means that the surface S^* is composed of characteristic curves and that the integral surface passing through S_1^* passes through S^*. Thus, we conclude that *if the surface S on which the initial conditions are given in a characteristic, then, for the Cauchy problem to have a solution, it is necessary that the initial surface S^* be composed of characteristic curves, and if this condition is satisfied, then the Cauchy problem has infinitely many solutions.*

This completely answers the question of the Cauchy problem for the general (not quasi-linear) first-order equation. This method of solving the Cauchy problem is known as the *method of characteristics.**

*The concept of a characteristic curve that we have just introduced is known in the literature on partial differential equations as a *characteristic of a first-order equation*, and the condition for a unique solution to the Cauchy problem is given apart from any connection with the general concept of a characteristic. In this literature, it is not pointed out that the condition $\delta = 0$ is an equation for the characteristic in the sense of the general classification presented in Sect. 2.

In conclusion, we shall present one method that leads very quickly to the solution of the Cauchy problem for a linear equation of the following form:

$$X(u) \equiv \frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_1, \ldots, x_n) \frac{\partial u}{\partial x_k} = F(t, x_1, x_2, \ldots, x_n), \tag{52}$$

$$u|_{t=t^0} = f(t_0, x_1, \ldots, x_n), \tag{52'}$$

where the a_k and f are given continuously differentiable functions x_1, x_2, ..., x_n and F is a continuously differentiable function of t, x_1, ..., x_n.

The characteristic system of equations corresponding to the homogeneous equation

$$X(u) \equiv \frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_l) \frac{\partial u}{\partial x_k} = 0, \tag{53}$$

is of the form

$$\frac{dx_1}{a_1(x_l)} = \ldots = \frac{dx_n}{a_n(x_l)} = \frac{dt}{1}. \tag{53'}$$

This system of equations admits the system of integrals

$$x_i = \varphi_i(t, C_1, \ldots, C_n) \qquad (i = 1, 2, \ldots, n), \tag{54}$$

which depend on the arbitrary constants C_1, ..., C_n and satisfy the initial conditions

$$x_i|_{t=t_0} = \varphi_i(t, C_1, \ldots, C_n)|_{t=t_0} = C_i, \quad i = 1, 2, \ldots, n. \tag{54'}$$

If we solve Eqs. (54) for the constants C_1, C_2, ..., C_n, we obtain n independent first integrals of the characteristic system (53')

$$\Phi_i(t, x_1, x_2, \ldots, x_n) = C_i \quad (i = 1, 2, \ldots, n). \tag{54''}$$

These first integrals are characterized by the fact that, on the basis of (54') they satisfy the initial conditions

$$\Phi_i(t, x_1, x_2, \ldots, x_n)|_{t=t_0} = x_i|_{t=t_0} \quad (i = 1, 2, \ldots, n). \tag{54'''}$$

On the other hand, an arbitrary function $\psi(\Phi_1, \Phi_2, \ldots, \Phi_n)$ is a solution of the homogeneous equation (53) since

$$X(\psi) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial \Phi_i} \left(\sum_{k=0}^{n} a_k \frac{\partial \Phi_i}{\partial x_k} \right) \quad (a_0 = 1;\ x_0 = t),$$

and, on the basis of the characteristic system (53′),

$$\sum_{k=0}^{n} a_k \frac{\partial \Phi_i}{\partial x_k} = \sum_{k=0}^{n} \frac{\partial \Phi_i}{\partial x_k} \frac{dx_k}{dt} = \frac{d\Phi_i}{dt} = 0.$$

In view of what has been said, the solution of the Cauchy problem (52)–(52′) can be written in the form

$$u(t, x_1, x_2, \ldots, x_n) = f[\Phi_i(t, x_1, \ldots, x_n)] + \\ + \int_{t_0}^{t} F[\tau, \Phi_i(t + t_0 - \tau; x_1, x_2, \ldots, x_n)]\, d\tau. \tag{55}$$

Specifically, we have

$$u(t, x_i)|_{t=t_0} = f(x_i),$$

$$X(u) = X[f(\Phi_i)] + \int_{t_0}^{t} X[F(\tau, \Phi_i)]\, d\tau + F(t, x_i).$$

This means that the function (55) satisfies Eq. (52) and the initial condition (52′).*

Example. If the coefficients a_i in Eq. (52) are constant, the first integrals (54″) satisfying the initial conditions (54″) will be of the form

$$\Phi_i(t, x_1, x_2, \ldots, x_n) = x_i - a_i(t - t_0) \quad (i = 1, 2, \ldots, n),$$

and the solution of the Cauchy problem (52)–(52′) may be written in explicit form:

$$u(t, x_1, x_2, \ldots, x_n) = f[x_i - a_i(t - t_0)] + \int_{t_0}^{t} F[\tau, x_i - a_i(t - \tau)]\, d\tau.$$

*Other methods of solving the Cauchy problem for special classes of first-order equations are taken up in detail in the book by N. M. Gyunter, *Integrirovaniye uravneniy pervogo poryadka v chastnykh proizvodnykh* (Integration of first-order partial differential equations), GTTI, 1934.

CHAPTER 3

Elliptic Equations. General Properties of Harmonic Functions. Potential Theory. Solution of Boundary-Value Problems

This chapter is devoted to elliptic differential equations. The fundamental boundary-value problems are formulated and are then investigated and solved by a variety of methods which are based primarily upon formulas for transforming volume integrals into surface integrals and the use of unit source and dipole functions. The general properties of harmonic functions are presented. Potential theory and the method of integral equations are developed in accordance with Lyapunov's works on potential theory. Formulas for the solution of boundary-value problems for canonical regions are presented.

1. The Formulation of Boundary-Value Problems and Their Physical Meaning

Let G denote a domain in a bounded portion of the xyz-space; let S be a piecewise-smooth surface bounding G; let $\vec{n}$ denote the inner normal to S;* and let P be a point with coordinates x, y, z.

For Laplace's equation in three independent variables

$$\Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

the simplest physical problems lead to the following boundary-value problems.

Find a solution $u=u(P)$ of Eq. (1) which has continuous second derivatives in the domain G and satisfies one of the following boundary conditions:

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*This means that the normal $\vec{n}$ is inwardly directed with respect to whichever of the regions bounded by the surface S we are considering in a particular case. If we were considering the region G^* lying outside S, the inner normal to S would be the normal directed into the region G^*.

(a) The solution $u=u(P)$ is continuous in the closed region $G \cup S$ and

$$u|_S = \Phi(P); \tag{1'}$$

(b) At every point on the surface S, the solution $u=u(P)$ has a limiting value of the normal derivative* and

$$\frac{\partial u}{\partial n}\Big|_S = \Phi(P); \tag{1''}$$

(c) The solution $u=u(P)$ is continuous in the closed region $G \cup S$; at every point of the surface S, it has a limiting value of the normal derivative and

$$\left(\frac{\partial u}{\partial n} - \alpha u\right)_S = \Phi(P), \quad \alpha > 0, \tag{1'''}$$

where $\Phi(P)$ and α are given continuous functions defined on S.

Each of the boundary-value problems corresponding to the cases (a), (b), and (c) is a *three-dimensional problem*, and, further, an *interior boundary-value problem*. The first of these is called the *Dirichlet* or the *first boundary-value problem*; the second is called the *Neumann* or *second boundary-value problem*; the third is called the *third boundary-value problem*.

If in each of these cases we seek a solution of Eq. (1) under the previously specified conditions, not in the bounded region G but in an unbounded region G^*, which is exterior to the surface S, under an additional condition, the so-called *condition of regularity at infinity*, we still speak of the three-dimensional first, second, and third boundary-value problems but, to distinguish them from the preceding cases, we call them *exterior problems* as opposed to the *interior* ones. The condition of regularity at infinity referred to above consists in the requirement that the solution $u=u(P)$ should tend to 0 uniformly as the point P approaches infinity.

Consider the two-dimensional form of Laplace's equation

$$\Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2}$$

We introduce the following notations: D denotes a bounded domain in the xy-plane; C denotes a piecewise-smooth contour bounding D; $\vec{n}$ denotes the inner normal to C; z denotes a point with coordinate x, y.** Then, we may pose the basic boundary-value problems as follows:

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*If $\vec{n}$ is the inner normal to S at the point P, the expression "limiting value of the normal derivative of the function u at the point P" refers to the limit of the derivative $\partial u/\partial n$ at the point $P_1 \in G$ as the point P_1 approaches the point P.

**We denote a point with coordinates x, y by z, having in mind certain cases in which we shall find it convenient to interpret it as a point in the complex plane $z=x+iy$.

Find the solution $u=u(z)$ of Eq. (2), which has continuous second derivatives in the domain D and which satisfies one of the following boundary conditions:

(a′) *The solution $u=u(z)$ is continuous in the closed region $D \cup C$ and*

$$u|_C = \Phi(z); \tag{2′}$$

(b′) *The solution $u=u(z)$ on the contour C has a limiting value of the normal derivative at each point and*

$$\frac{\partial u}{\partial n}\Big|_C = \Phi(z); \tag{2″}$$

(c′) *The solution $u=u(z)$ is continuous in the closed region $D \cup C$, at every point of the curve C it has a limiting value of the normal derivative, and*

$$\left(\frac{\partial u}{\partial n} - \alpha u\right)_C = \Phi(z), \quad \alpha > 0, \tag{2‴}$$

where $\Phi(z)$ and α are prescribed functions of z which are defined and continuous on C.

In these cases we speak respectively of the first (Dirichlet), second (Neumann), and third boundary-value problem. To distinguish them from the three-dimensional problems, we speak of these as *plane* or *two-dimensional* boundary-value problems.

If we are seeking a solution in the unbounded region D^*, exterior with respect to the contour C, under identical conditions as specified above, but with the additional condition of regularity at infinity, we will speak of the plane first, second, and third boundary-value problems but we call them *exterior* rather than interior problems. The condition for regularity at infinity in this case is different from the corresponding condition of regularity at infinity in the three-dimensional case: here, it consists in the requirement that the solution $u=u(z)$ remain bounded as the point z approaches infinity.

These three boundary-value problems are the basic ones for Laplace's equation and, in general, for linear second-order elliptic equations. In the case of equations more general than Laplace's, the boundary-value problems are posed in the same way but we speak of the solution of the corresponding elliptic equation.

We note that in certain cases, the boundary-value problems for Laplace's equation are also called the boundary-value problems of potential theory.

As degenerate cases of the first, second, and third internal boundary-value problems, we sometimes need to consider these problems for regions whose boundaries contain the point at infinity.

As a generalization of the basic boundary-value problems, we may mention the mixed boundary-value problem. This problem is formulated in the same way as the first, second, and third boundary-value problems above, except that in this case the boundary conditions of the first boundary-value problem are satisfied on one portion of the boundary of the region in which we are seeking a solution, those of the second boundary-value problem are satisfied on another portion, and those of the third on yet another.

These boundary-value problems can be automatically extended to the case of nonsimply connected regions if, in the three-dimensional case, S stands for the union of several piecewise-smooth closed surfaces and, in the two-dimensional case, C stands for the union of several piecewise-smooth closed contours. Here, the exterior boundary-value problems are characterized by the fact that the corresponding regions contain the point at infinity.

We note that in the case of the Dirichlet problem, the restrictions imposed on the surface S and on the contour C, although they may not be completely removed,* can be considerably weakened. For example, in the two-dimensional case, we may assume that the contours bounding the region D are Jordan curves.

We note also that in the case of the two-dimensional interior Dirichlet problem, it is sometimes necessary to consider cases in which the boundary function $\Phi(z)$ has a discontinuity of the first kind at some point $z_0 = x_0 + iy_0 \subset C$:

$$\Phi(z_0 + 0) - \Phi(z_0 - 0) = a,$$

where a is a given number and $\Phi(z_0+0)$ and $\Phi(z_0 - 0)$ are the limiting values of $\Phi(z)$ as z approaches the point z_0 along C from different directions. Let us denote by α the interior angle between the directions $z_0 + 0$ and $z_0 - 0$ and let us define a function $u_1(z) = u(z) - (a/\alpha)\arg(z - z_0)$. This function will be a twice continuously differentiable solution of Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain D, since the function $\chi(z) = (a/\alpha)\arg(z - z_0) = (a/\alpha)\arctan(y - y_0/x - x_0)$ is a solution of Laplace's equation, as can be verified by differentiation. Since $\chi(z_0 + 0) - \chi(z_0 - 0) = a$, we see that the function $\Phi_1(z) = \Phi(z) - (a/\alpha)\arg(z - z_0)$ will be continuous on C. Therefore, if we determine $u_1(z)$ by solving the Dirichlet problem (as formulated above) with the continuous boundary function $\Phi_1(z)$ and set $u(z) = u_1(z) + (a/\alpha)\arg(z - z_0)$, we obtain the solution of the Dirichlet problem for a discontinuous boundary function $\Phi(z)$. Therefore, in all that follows, we may, without loss of generality, consider the Dirichlet problem in the sense in which it was posed above, that is, for a continuous boundary function, requiring that the desired solution be continuous in the closed region.

We shall now show how certain important problems in physics and mechanics lead to the basic boundary-value problems for linear second-order elliptic equations in three and two independent variables.

The steady flow of heat in a homogeneous medium is described by the equation

*An example of a three-dimensional region for which the Dirichlet problem does not have a continuous solution in a closed region can be found in the book by R. Courant and D. Hilbert, *Methods of Mathematical Physics*, New York, Interscience, 1962, Vol. II, pp. 303-305.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{f}{k}, \tag{3}$$

where $u=u(P)$ is the temperature, k is the thermal conductivity, f is the intensity of heat sources (see Chapter 1, Sect. 2). The following problem is then posed: Find the temperature at all points of the body if the temperature on the surface S bounding it is known and is given by a function $\Phi(P)$. In this case, we obtain the first exterior boundary-value problem for Poisson's equation (3). If the body occupies the portion of space exterior to the surface S and if the temperature on S is equal to $\Phi(P)$ and approaches 0 uniformly as P approaches infinity, then determination of the temperature at all points of the body obviously reduces to solving the exterior first boundary-value problem for Eq. (3). For example, in this case, if S is the sphere $x^2+y^2+z^2=R^2$, $f=0$, and $u|_S=A=\text{const}$, the solution of the external problem is

$$u(P) = \frac{AR}{\sqrt{x^2+y^2+z^2}},$$

as can be directly verified by differentiation.

Suppose now that under the same conditions, we are given not the temperature on the surface S but the amount of heat Q flowing out of the right in unit time through a unit area of the surface S: $Q=\Phi(P)$. This quantity of heat must be equal to the amount of heat that flows from within up to the unit area of the surface S in unit time, and is given by the experimentally verified law $Q=k\,\partial u/\partial n$. On the basis of this, we arrive at the boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_S = \frac{1}{k}\Phi(P).$$

Thus, in the case of the region interior to S, we arrive at the second interior boundary-value problem for Eq. (3). In the case of the exterior of S, if the temperature approaches 0 uniformly as the point in question approaches infinity, we arrive at the second exterior boundary-value problem for Eq. (3). Suppose now that a body radiates heat into space with the same conditions imposed on the surface S. Then, by an experimentally verified law of Newton, the equation $k\,\partial u/\partial n = h(u-u_0)$ must be satisfied on S. Here, u_0 is the given temperature of points of the surrounding space and h is a positive proportionality constant known as the exterior thermal conductivity. With these physical hypotheses, we have the third interior boundary-value problem for Eq. (3) in the case of the interior of S. In the case of the exterior of S, we have the exterior third boundary-value problem provided the temperature approaches 0 uniformly as we approach infinity.

Now, let us assume that the flow of heat is a plane-parallel flow, i.e., independent of the coordinate z. Then, instead of Eq. (3), we have the two-dimensional equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{f}{k}. \tag{3'}$$

In our reasoning regarding Eq. (3), we now make the following substitutions: the contour C in place of the surface S, a continuous function $\Phi(z)$ in place of $\Phi(P)$, plane regions in place of three-dimensional ones, and the requirement that the temperature be bounded at infinity in place of the requirement that it vanish at infinity. Then, we obtain essentially the same boundary-value problems for Eq. (3′) as we did above for Eq. (3) except that they are now plane boundary-value problems instead of three-dimensional ones. For example, if C is the circle $x^2+y^2=R^2$, if $f=0$, and if $u|_C = A = \text{const}$, then the solution of the first exterior (or interior) boundary-value problem for Eq. (3′) may be written in the form $u(z)=A$. This example together with the example given above of the solution of the first exterior boundary-value problem for Eq. (3) emphasizes an essential distinction between the exterior boundary-value problems for the two- and three-dimensional cases.

If the condition of regularity at infinity in the three-dimensional case were the same as in the plane case, the physical process in the exterior three-dimensional region would not be uniquely determined. For example, in addition to the solution of the three-dimensional problem that we gave above, there would also be the solution $u(P)=A$.

In the statement of all the boundary-value problems for linear second-order elliptic equations, the conditions of regularity at infinity are formulated in such a way as to ensure unique determination of the physical processes corresponding to them or, what amounts to the same thing, uniqueness of the solution of the boundary-value problems.

Let us give one more example, this time from hydrodynamics. The potential function u of the steady potential flow of an incompressible ideal fluid satisfies Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{4}$$

Let us suppose that a fluid is flowing around a surface S and that its velocity at infinity is of magnitude v_0 and is directed along the positive x-axis. Let $\vec{v} = \nabla u$ denote the velocity vector in the region exterior to S. Since the flow is around a stationary surface S, the normal component of the velocity vector $\vec{v}$ on the surface S must obviously be 0; i.e., $\partial U/\partial n|_S = 0$. If we now define an auxiliary

function $U=u-v_0x$, this function U will, like the original one, satisfy Eq. (4) and, on the surface S, it will satisfy the boundary conditions

$$\left.\frac{\partial U}{\partial n}\right|_S = -v_0 \left.\frac{\partial x}{\partial n}\right|_S, \tag{4'}$$

where the right-hand member is a known function of the point P of the surface S. If we now require that the function U approach 0 as its argument approaches infinity, we obtain for this auxiliary potential function U the second exterior boundary-value problem for Eq. (4) with boundary conditions (4'). If the flow of the fluid is assumed to be planar, i.e., as if it were flowing around a contour C in the xy-plane, then Laplace's equation (4) becomes two-dimensional and the boundary conditions (4') take the form

$$\left.\frac{\partial U}{\partial n}\right|_C = -v_0 \left.\frac{\partial x}{\partial n}\right|_C. \tag{4''}$$

If we now assume the function U to be bounded at infinity, we obtain for U the two-dimensional boundary-value problem corresponding to the three-dimensional one above.

Suppose that a particular boundary-value problem for some differential equation or system of equations has the following property: For any positive number ε, there exists a positive number δ such that if the boundary conditions are changed by an amount not exceeding the number δ in absolute value, the change in the solution at any point of the domain in question or its boundary will not exceed ε in absolute value. The solution of such a boundary-value problem is said to be *stable* (with respect to the boundary conditions) or to *depend continuously* on the boundary conditions.

A boundary-value problem for a particular differential equation or system of equations is said to be *correctly posed* in a given domain if a solution of that boundary-value problem in the domain in question exists, is unique, and, furthermore, is stable.

Stability of a solution, and correct formulation of a boundary-value problem, are very important concepts applying to all partial differential equations and systems of equations regardless of the classification to which they belong. This is explained by the fact that, for every equation or system of equations, correctly posed boundary-value problems are of prime scientific and practical interest. A boundary-value problem that has either no solution or more than one solution cannot describe any actual physical process. On the other hand if the solution of the boundary-value problem exists and is unique but is not stable, it gives no valid information for describing any physical process because any modifications in the boundary conditions, no matter now small, or any errors made in determining them produce large changes in the solution in the region under investigation.

Therefore, the question as to whether a particular boundary-value problem has been posed correctly or incorrectly is of profound significance. It turns out, for all practical purposes, that all the boundary-value problems for second-order elliptic equations (including the first, second and third boundary-value problems) which are formulated on the basis of physical considerations, are correctly posed.

Let us give an example of an incorrectly posed boundary-value problem. As Hadamard has shown, the Cauchy problem for Laplace's equation $u_{xx} + u_{yy} = 0$ in the strip $0 < y < 1$ with the initial conditions $u|_{y=0} = 0$, $\partial u/\partial y|_{y=0} = e^{-\sqrt{k}} \cos kx$, where $k = \text{const} \neq 0$, is incorrectly posed since the solution of this problem, namely, $u = k^{-1} e^{-\sqrt{k}} \cos kx \sinh ky$ will, for sufficiently large k, differ from 0 in the strip referred to by an arbitrarily large amount for an arbitrarily small change in the initial conditions (together with their derivatives of every order).

2. Unit Source and Dipole Functions. Green's Function and the Solution of the First Boundary-Value Problem for a Circle and for a Sphere

1. Unit source and dipole functions. We shall say that the function

$$\delta(P, P_0) = \frac{1}{4\pi r}, \tag{5}$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is the distance between the points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$, is a *function of a unit source* (or a fundamental solution) *at the point* P_0 *for Laplace's equation in three dimensions*

$$\Delta u = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1}$$

We shall say that the function

$$\mu(P, P_0) = \lim_{P_0' \to P_0} \frac{\delta(P, P_0') - \delta(P, P_0)}{\rho}, \tag{6}$$

where ρ is the distance between the points P_0' and P_0 and where the point P_0' approaches the point P_0 along a directed segment $\vec{n}_0$ originating at the point P_0, is a *function of a unit dipole at the point* P_0 *with axis* $\vec{n}_0$ *for Eq. (1).*

Analogously, in the plane case, we shall say that the function

$$\delta(z, z_0) = \frac{1}{2\pi} \ln \frac{1}{r}, \tag{7}$$

where $r = |z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, is *a function of a unit source at a point z_0 for Laplace's equation in two dimensions*

$$\Delta u = 0, \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2}$$

We shall say that the function

$$\mu(z, z_0) = \lim_{z_0' \to z_0} \frac{\delta(z, z_0') - \delta(z, z_0)}{\rho}, \tag{8}$$

where ρ is the distance between the points z_0' and z_0 and where the point z_0' approaches the point z_0 along a directed line segment $\vec{n}_0$ originating at the point z_0, is a *function of a unit dipole at the point z_0 with axis $\vec{n}_0$ for Eq. (2)*. By making use of the definitions of the functions $\mu(P, P_0)$ and $\mu(z, z_0)$ and differentiating we obtain the following equations:

$$\mu(P, P_0) = \frac{\partial}{\partial n_0} \delta(P, P_0) = \frac{\partial}{\partial n_0} \frac{1}{4\pi r} = \frac{\cos \varphi_0}{4\pi r} = \frac{d\Theta_0}{4\pi dS_0}, \tag{6'}$$

$$\mu(z, z_0) = \frac{\partial}{\partial n_0} \delta(z, z_0) = \frac{\partial}{\partial n_0} \frac{1}{2\pi} \ln \frac{1}{r} = \frac{\cos \varphi_0}{2\pi r} = \frac{d\theta_0}{2\pi\, ds_0}. \tag{8'}$$

Here, φ_0 is the angle between the vector $\vec{P_0P}$ (or $\vec{z_0z}$ in the two-dimensional case) and the vector $\vec{n}_0$ (see Fig. 4), dS_0 is an element of surface containing P_0 and orthogonal to $\vec{n}_0$, and $d\Theta_0$ is the solid angle subtended at the point P by the element of area dS_0. This angle will be positive if $0 \leqslant \varphi_0 < \pi/2$ and negative if $\pi/2 < \varphi_0 \leqslant \pi$. Analogously, ds_0 is an element of arc length containing the point z_0 and orthogonal to the vector $\vec{n}_0$, and $d\theta_0$ is the angle subtended by

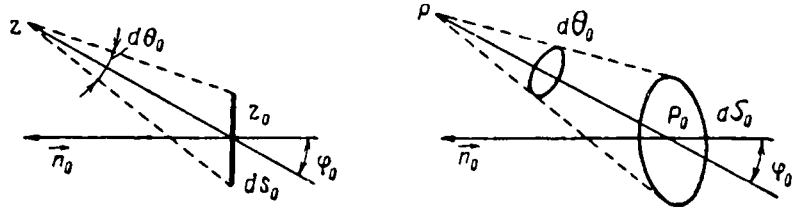

Fig. 4.

the arc element ds_0 at the point z. This angle is taken as positive if $0 \leqslant \varphi_0 < \pi/2$ and negative if $\pi/2 < \varphi_0 \leqslant \pi$ (see Fig. 4).

We call attention also to the following two formulas:

$$\iint_S \frac{\partial}{\partial n} \delta(P, P_0)\, dS = 1, \tag{5'}$$

$$\int_C \frac{\partial}{\partial n} \delta(z, z_0)\, ds = 1, \tag{7'}$$

where S is an arbitrary closed surface enclosing the point P_0 (or, in the two-dimensional case, C is a closed curve encircling the point z_0) and $\vec{n}$ is the inner normal to S or to C.

To prove these equations, note that the functions $\partial\delta(P, P_0)/\partial n = \partial(P_0, P)/\partial n$ and $\partial\delta(z, z_0)/\partial n = \partial\delta(z_0, z)/\partial n$ are functions of unit dipoles at the points P and z respectively and that the axes of these dipoles coincide with the normal $\vec{n}$. Consequently,

$$\frac{\partial}{\partial n} \delta(P_0, P) = \mu(P_0, P) = \frac{\cos\varphi}{4\pi r^2} = \frac{d\Theta}{4\pi\, dS}. \tag{9}$$

$$\frac{\partial}{\partial n} \delta(z_0, z) = \mu(z_0, z) = \frac{\cos\varphi}{2\pi r} = \frac{d\theta}{ds}, \tag{10}$$

where φ is the angle between the vectors $\vec{PP_0}$ and $\vec{n_0}$ in the three-dimensional case and between $\vec{zz_0}$ and $\vec{n}$ in the two-dimensional case, where $d\Theta$ is the solid angle subtended by the element of area dS at the point P_0, and where $d\theta$ is the angle subtended by the element of arc ds at the point z_0 (see Fig. 5). However, the entire surface S subtends a solid angle of 4π at the point P_0 and the curve C subtends an angle of 2π at the point z_0. This proves Eqs. (5') and (7').

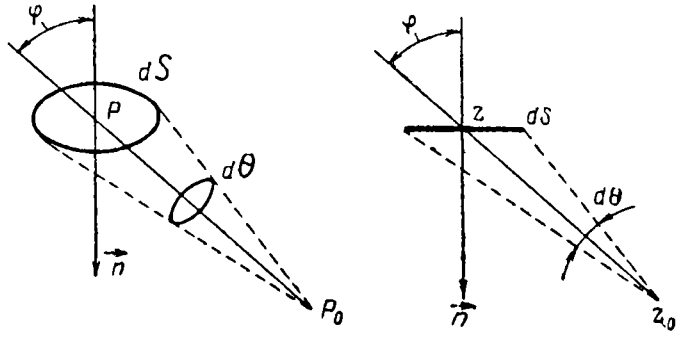

Fig. 5.

Differentiation immediately shows that the functions $\delta(P, P_0)$ and $\mu(P, P_0)$ on the one hand, and $\delta(z, z_0)$ and $\mu(z, z_0)$ on the other,

are twice continuously differentiable solutions of the three- and two-dimensional forms of Laplace's equation except at the points P_0 and z_0, respectively.

Unit source and unit dipole functions have a simple physical meaning. Suppose, for example, that a positive electric charge of magnitude $1/4\pi$ (in the electrostatic CGS units) is placed at the point P_0 in space. Then, by virtue of Coulomb's law (see Chapter 1, Sect. 2, subsection 5), the function $u(P) = \delta(P, P_0)$ will be the potential function of the electrostatic field due to that charge. Let us now place a negative charge of magnitude $1/4\pi\rho$ at the point P_0 and a positive charge of magnitude $1/4\pi\rho$ at the point P_0' where ρ is the distance between the points P_0' and P_0. These charges will create an electrostatic field with a potential function given by

$$u(P) = \frac{\delta(P, P_0') - \delta(P, P_0)}{\rho}. \tag{11}$$

Suppose now that the point P_0' is made to approach the point P_0 along a directed segment $\vec{\nu}$ originating at the point P_0. In the limit, we shall obtain an electrostatic field whose potential function coincides with the function of a unit dipole at the point P_0 with axis $\vec{\nu}$ for the three-dimensional Laplace equation.

To obtain a function of a unit source $\delta(z, z_0)$ for the two-dimensional Laplace equation as the potential function of an electrostatic field, we need only place positive electric charges of constant linear density $1/4\pi$ along a straight line passing through the point $z_0 = x_0 + iy_0$ and perpendicular to the xy-plane.

Indeed, for the potential function $u(x, y, z)$ corresponding to the electrostatic field caused by the charges placed along the line segment $-N \leqslant z \leqslant N$, where N is a positive number, we may write the equation

$$u(x, y, 0) = \frac{1}{4\pi}\int_{-N}^{N} \frac{ds}{\sqrt{s^2 + r^2}} - \frac{1}{2\pi}\ln(2N)$$

$$(r = \sqrt{(x - x_0)^2 + (y - y_0)^2}),$$

or

$$u(x, y, 0) = \frac{1}{2\pi}\int_{0}^{N} \frac{d\tau}{\sqrt{\tau^2 + r^2}} - \frac{1}{2\pi}\ln(2N) =$$

$$= \frac{1}{2\pi}\ln(\tau + \sqrt{\tau^2 + r^2})\Big|_0^N - \frac{1}{2\pi}\ln(2N).$$

Taking the limit as $N \to \infty$, we obtain

$$u(x, y, 0) = \frac{1}{2\pi}\ln\frac{1}{r} = \delta(z, z_0).$$

Suppose that we put a negative charge of linear density $1/4\pi\rho$ on the straight line referred to above and a positive charge of the same density on another straight line parallel to it, where ρ is the distance between the two lines. If we let the second line approach the first one, we obtain in the limit the potential function of the electrostatic field which will coincide with the function of a unit dipole at the point z_0 in the case of the two-dimensional form of Laplace's equation. The axis of this dipole will be a vector originating at the point z_0 with direction tangential to the trajectory described by the second line in the xy-plane.

We can give a completely analogous physical interpretation of unit source functions or unit dipole functions by looking at examples of potential flow of an incompressible ideal fluid, or of steady heat flow.

2. The representation of twice continuously differentiable functions as a sum of potentials. Let u and v be arbitrary twice continuously differentiable functions in a three-dimensional closed region $G \cup S$ or in a closed two-dimensional region $D \cup C$ in the xy-plane. Then, we have the formulas

$$-\iint_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS = \iiint_G (v\Delta u - u\Delta v)\, d\tau, \tag{12}$$

$$-\int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds = \iint_D (v\Delta u - u\Delta v)\, d\sigma, \tag{13}$$

where Δ is the Laplace operator (three-dimensional in the first case and two-dimensional in the second), where the surface S satisfies the conditions under which Ostrogradskiy's formula is applicable, and where the contour C satisfies the conditions under which Green's formula is applicable (see Chapter 1, Sect. 2). The first of these formulas is obtained from Ostrogradskiy's formula

$$-\iint_S v_n\, dS = \iiint_G \operatorname{div}\vec{v}\, d\tau,$$

if we set $\vec{v} = (vu_x - uv_x,\ vu_y - uv_y,\ vu_z - uv_z)$; the second is obtained from Green's formula

$$-\int_C v_n\, dS = \iint_D \operatorname{div}\vec{v}\, d\sigma,$$

if we set $\vec{v} = (vu_x - uv_x,\ vu_y - uv_y)$.

We shall call formula (12) Ostrogradskiy's formula for the Laplace operator and we shall call formula (13) Green's formula for the Laplace operator.*

*More commonly, both (12) and (13) are called Green's formulas. (Ed.).

Let us set

$$v = v(P_0, P) = \delta(P_0, P) + g(P_0, P) \tag{12'}$$

$$v = v(z_0, z) = \delta(z_0, z) + g(z_0, z), \tag{13'}$$

where $g(P_0, P)$ is a twice continuously differentiable function of the point P and $g(z_0, z)$ is a twice continuously differentiable function of the point z. Let us now apply formula (12) to the domain G minus a sufficiently small sphere e with center at the point P_0, and let us apply formula (13) to the domain D minus a sufficiently small circle e with center at the point z_0. Thus, we obtain the two equations

$$-\iint_{\gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = -\iint_{S}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS +$$

$$+\iiint_{G-e}(v\Delta u - u\Delta g)\,d\tau,$$

$$-\int_{\gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = -\int_{C}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds +$$

$$+\iint_{D-e}(v\Delta u - u\Delta g)\,d\sigma,$$

where γ is the boundary of the sphere e in the first case and of the circle e in the second. Note that, on γ,

$$\frac{\partial v(P_0, P)}{\partial n} = \frac{\partial v(P_0, P)}{\partial r} = -\frac{1}{4\pi r^2} + O(1)$$

$$\frac{\partial v(z_0, z)}{\partial n} = \frac{\partial v(z_0, z)}{\partial r} = -\frac{1}{2\pi r} + O(1),$$

where r is the radius of the sphere or circle e and $O(1)$ is a quantity that remains bounded as $r \to 0$. If we take the limit as $r \to 0$, we obtain

$$u(P_0) = \iint_{S}\left(u\frac{\partial v(P_0, P)}{\partial n} - v(P_0, P)\frac{\partial u}{\partial n}\right)dS -$$

$$-\iiint_{G}\left(v(P_0, P)\,\Delta u - u\Delta g\right)d\tau, \tag{12''}$$

$$u(z_0) = \int_{C}\left(u\frac{\partial v(z_0, z)}{\partial n} - v(z_0, z)\frac{\partial u}{\partial n}\right)ds -$$

$$-\iint_{D}\left(v(z_0, z)\,\Delta u - u\Delta g\right)d\sigma. \tag{13''}$$

In particular, if we set $g(P_0, P) \equiv 0$ and $g(z_0, z) \equiv 0$, these equations take the form

$$u(P_0) = \iint_S \left(u \frac{\partial \delta(P_0, P)}{\partial n} - \delta(P_0, P) \frac{\partial u}{\partial n} \right) dS - \iiint_G \delta(P_0, P) \Delta u \, d\tau, \tag{14}$$

$$u(z_0) = \int_C \left(u \frac{\partial \delta(z_0, z)}{\partial n} - \delta(z_0, z) \frac{\partial u}{\partial n} \right) ds - \iint_D \delta(z_0, z) \Delta u \, d\sigma. \tag{15}$$

Equation (14) means that every function $u(P)$ that is twice continuously differentiable in the closed region $G \cup S$ can be represented at every point P_0 of the domain G in the form of a sum of:

(a) a sourcelike function due to sources at points P of the surface S, the intensity of these sources being $\sigma(P) = -\partial u(P)/\partial n$,

$$U(P_0) = \iint_S \sigma(P) \delta(P_0, P) \, dS, \tag{14'}$$

(b) a sourcelike function due to dipoles at points P of the surface S of intensity $\vartheta(P) = u(P)$, the axes of these dipoles coinciding with the inner normal $\vec{n}$ to the surface S,

$$V(P_0) = \iint_S \vartheta(P) \mu(P_0, P) \, dS, \tag{14''}$$

(c) a sourcelike function due to sources at points of the domain D with intensity $\rho(P) = -\Delta u|_P$

$$I(P_0) = \iiint_G \rho(P) \delta(P_0, P) \, d\tau. \tag{14'''}$$

Analogously, in the plane case, formula (15) means that every function $u(z)$ that is twice continuously differentiable in a closed region $D \cup C$ can be represented at each point z_0 of the domain D as a sum of:

(a) a sourcelike function due to sources at points z of the contour C with intensity $\sigma(z) = -\partial u(z)/\partial n$

$$U(z_0) = \int_C \sigma(z) \delta(z_0, z) \, ds, \tag{15'}$$

(b) a sourcelike function due to dipoles at points z of the contour C with intensity $\vartheta(z) = u(z)$, the axes of these dipoles coinciding with the inner normal $\vec{n}$ to the curve C

$$V(z_0) = \int_C \vartheta(z) \mu(z_0, z) \, ds, \tag{15''}$$

(c) a sourcelike function due to sources at points of the domain D with intensity $\rho(z) = -\Delta u|_z$

$$I(z_0) = \iint_D \rho(z)\,\delta(z_0, z)\,d\sigma. \qquad (15''')$$

The integrals $U(P_0)$, $V(P_0)$, and $I(P_0)$ are usually called the *potential of a single layer*, the *potential of a double layer*, and the *potential of a volume distribution*, respectively, and the integrals $U(z_0)$, $V(z_0)$, and $I(z_0)$ are called, respectively, the *logarithmic potential of a single layer, of a double layer, and of a plane distribution.*

It is possible to give all these integrals a simple physical interpretation using the fact that they are obtained as a superposition of potentials of sources and dipoles.

3. Green's function and integral representations of solutions of boundary-value problems. Examples of Green's functions for the simplest regions. Suppose the function $g(P_0, P)$ in Eq. (12′) is such, that for every fixed point P_0 belonging to the domain G, the function $v(P_0, P)$ is a twice continuously differentiable function of P everywhere in G except at the point $P = P_0$ and otherwise satisfies all the conditions of one of the three-dimensional boundary-value problems posed above for Laplace's equation for the domain G with zero boundary conditions. Then, the function $v(P_0, P)$ is called a *Green's function* corresponding to the boundary-value problem for the domain G. In what follows, we shall denote such a Green's function by $G(P_0, P)$.

Analogously, in the plane case, if the function $g(z_0, z)$ in Eq. (13′) is such that, for every fixed point z_0 belonging to the domain D, the function $v(z_0, z)$ is a twice continuously differentiable function of z in the domain D and $-\{z_0\}$ satisfies all the conditions of any one of the basic two-dimensional boundary-value problems for Laplace's equation for the region D with zero boundary conditions, then the function $v(z_0, z)$ is called a Green's function corresponding to the boundary-value problem for the domain D. We shall denote such a function by $G(z_0, z)$.

For example, in the case of the first boundary-value problem for the three-dimensional domain G, Green's function $G(P_0, P)$ must be a twice continuously differentiable solution of the three-dimensional Laplace equation at all points of the domain G except at $P = P_0$, must be continuous on the surface S, and must satisfy $G(P_0, P)|_{P \in S} = 0$.

Green's function $G(z_0, z)$ for the second boundary-value problem for the half-plane $y > 0$ must be a twice continuously differentiable solution of the two-dimensional Laplace's equation for all $z \neq z_0$ in this half-plane, and the limiting values of the normal derivative on the x-axis must be equal to 0.

Formula (12″) shows that if the Green's function of any of the three-dimensional boundary-value problems for the domain G

exists and satisfies the conditions under which Ostrogradskiy's formula (12) is applicable, the solution of the corresponding boundary-value problem for Poisson's equation $\Delta u = -F$ (if we assume that it too satisfies the conditions of applicability of Ostrogradskiy's formula) can be written in explicit form. Indeed, for our three boundary-value problems, we obtain from (12″)

$$u(P_0) = \iint_S \Phi(P) \frac{\partial G(P_0, P)}{\partial n} dS + \iiint_G G(P_0, P) F(P) d\tau, \tag{16}$$

$$u(P_0) = \iint_S -G(P_0, P) \Phi(P) dS + \iiint_G G(P_0, P) F(P) d\tau, \tag{17}$$

$$u(P_0) = \iint_S -G(P_0, P) \Phi(P) dS + \iiint_G G(P_0, P) F(P) d\tau, \tag{18}$$

where $G(P_0, P)$ is the Green's function of the corresponding boundary-value problem for the domain G.

In just the same way, formula (13″) shows that if, in the plane case, Green's function corresponding to any of the basic boundary-value problems for the region D exists and satisfies on the contour C the conditions under which Green's formula (13) is applicable, then the solution of that boundary-value problem for Poisson's equation $\Delta u = -F(z)$ can be written in explicit form if we assume that it too satisfies the conditions of applicability of Green's formula. For the first, second, and third boundary-value problems, we obtain from (13″), respectively,

$$u(z_0) = \int_C \Phi(z) \frac{\partial G(z_0, z)}{\partial n} ds + \iint_D G(z_0, z) F(z) d\sigma, \tag{19}$$

$$u(z_0) = \int_C -G(z_0, z) \Phi(z) ds + \iint_D G(z_0, z) F(z) d\sigma, \tag{20}$$

$$u(z_0) = \int_C -G(z_0, z) \Phi(z) ds + \iint_D G(z_0, z) F(z) d\sigma, \tag{21}$$

where $G(z_0, z)$ is the Green's function of the corresponding boundary-value problem for the domain D.

We must make the following important observation concerning Green's function of the second boundary-value problem and concerning solutions given by Eqs. (17) and (20) of the second boundary-value problem.

For bounded regions, the second boundary-value problem does not have a Green's function in the sense described above. Such a Green's function can exist only for domains not contained in any bounded region in space or in the plane. (Soon we will convince ourselves of this by means of a particular example.) Therefore, for formulas (17) and (20) to be meaningful for bounded regions, it is necessary to generalize somewhat the concept of a Green's

function for the second boundary-value problem. By a *generalized Green's function* $G(P_0, P)$ of the second boundary-value problem for a bounded three-dimensional domain G, we shall mean the function (12′) if it satisfies zero boundary conditions for the second boundary-value problem on the surface S bounding G, if the function $g(P_0, P)$ is a twice continuously differentiable solution of Poisson's equation $\Delta u = 1/\omega$ in the domain G, where ω is the volume of the domain G, and if, in addition, the normalization condition

$$\iiint_G G(P_0, P)\, d\tau = 0$$

is satisfied.

We define analogously a generalized Green's function $G(z_0, z)$ for the second boundary-value problem for a bounded plane domain D. Here, the only difference is that instead of the three-dimensional Poisson equation, we have the two-dimensional equation $\Delta u = 1/\omega$, where ω is the area of the domain D and the normalization condition is written in the form

$$\iint_D G(z_0, z)\, d\sigma = 0.$$

Obviously, it is always possible to satisfy these normalization conditions since the boundary conditions of the second boundary-value problem determine the solution up to an arbitrary additive constant.

In what follows, we shall say that solutions of the second boundary-value problem that satisfy these normalization conditions are *normalized solutions* of that problem.

Let us show that replacement of Laplace's equation by Poisson's equation in introducing the generalized Green's function of the second boundary-value problem is necessary to ensure the existence of this generalized Green's function. We set $\Delta G = C = \text{const}$ and make use of

$$\left.\frac{\partial G(P_0, P)}{\partial n}\right|_{P \in S} = 0, \quad \left.\frac{\partial g(P_0, P)}{\partial n}\right|_{P \in S} = -\left.\frac{\partial \delta(P_0, P)}{\partial n}\right|_{P \in S}.$$

Then, from Ostrogradskiy's formula

$$\iiint_G \Delta g\, d\tau = -\iint_S \frac{\partial g(P_0, P)}{\partial n}\, dS = \iint_S \frac{\partial \delta(P_0, P)}{\partial n}\, dS = 1$$

we obtain $C = 1/\omega$, where ω is the volume of the region G.*

*$G(P_0, P)$ may be interpreted as the temperature of the body bounded by the surface S with thermal conductivity $k = 1$. Since $\partial G/\partial n|_S = 0$, it follows that, in order for all the heat liberated by a unit source at point P_0 to be absorbed, it is necessary to distribute the sources continuously at points of the body with density $-1/\omega$. This means that $\Delta G = 1/\omega$ (see Sect. 1).

Let us turn now to Eqs. (12″) and (13″). If we understand $u(P)$ and $u(z)$ in these equations to mean normalized solutions of the second boundary-value problem for a domain G in the three-dimensional case or a domain D in the two-dimensional case and if we understand $G(P_0, P)$ and $G(z_0, z)$ to mean the corresponding generalized Green's functions of the second boundary-value problem, these equations become formulas (17) and (20) since $\Delta g = 1/\omega = \text{const}$.

The meaning and significance of formulas (17) and (20) as applied to normalized solutions of the second boundary-value problem remain the same as that of formulas (16) and (19) as applied to solutions of the first boundary-value problem.

In general, however, it should be pointed out in connection with formulas (16)-(21) that even when we have a Green's function of some boundary-value problem or other in explicit form, we should still treat the right-hand members of these formulas as the expected result of solving the corresponding boundary-value problem which is still in need of verification. The reason for this is that formulas (16)-(21) were obtained under the assumption that Green's function for the boundary-value problem in question and the solution of that problem satisfy the conditions under which Ostrogradskiy's formula is applicable in the three-dimensional case and Green's formula is applicable in the two-dimensional case. However, this is not known to us beforehand since these conditions were not used in defining what is meant by a solution of a boundary-value problem, or in defining a Green's function. Thus, in any particular case, to show that the right-hand member of any of formulas (16)-(21) yields a solution of the corresponding boundary-value problem, an additional investigation is necessary, which, as a rule, yields affirmative results.

It follows from the definition of Green's function for a boundary-value problem that finding this function in explicit form reduces to solving the very boundary-value problem to which it corresponds. Thus, for example, to find the Green's function $G(P_0, P)$ of the first boundary-value problem for the domain G, it is necessary to solve this same boundary-value problem for the domain G for the function $g(P_0, P)$ under the boundary conditions

$$g(P_0, P)|_{P \in S} = -\delta(P_0, P)|_{P \in S}.$$

As a rule, therefore, we will not be able to find an explicit representation for a Green's function corresponding to some boundary-value problem in a given domain. However, in certain cases, this can be done. Every such case is very important and valuable.

Let us give a few known cases in which it is possible to express Green's function in explicit form.

(a) *Green's function of the first boundary-value problem for a sphere.* Let R denote the radius of a sphere with center at the

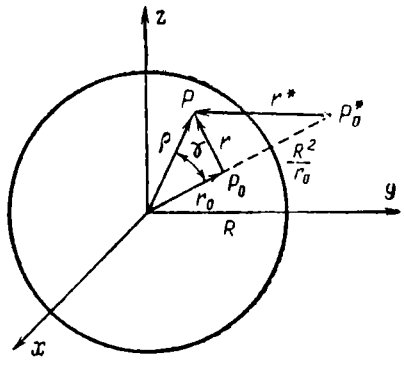

Fig. 6.

coordinate origin O, P_0 (x_0, y_0, z_0)—a point at which a unit source is located within the sphere, and r_0—the distance between this point and the coordinate origin. Let $P(x, y, z)$ denote a variable point, ρ—its distance from the coordinate origin. Let r—the distance between the points P_0 and P, and let γ denote the angle between the vectors $\vec{OP}_0$ and $\vec{OP}$ determining the position of the points P_0 and P (see Fig. 6). A unit source at the point P_0 induces a potential on the surface S of the sphere. By using the so-called method of images, we can obtain Green's function for the first boundary-value problem for the sphere by compensating for this potential on the sphere S by placing a suitably chosen source outside the sphere S. Specifically, let us place a source of intensity k (where k is some positive number) at a point P_0^* which is symmetric to the point P_0 with respect to the sphere S. If we denote by r^* the distance between the points P_0 and P_0^*, we obtain

$$r^* = \sqrt{\left(x - x_0\frac{R^2}{r_0^2}\right)^2 + \left(y - y_0\frac{R^2}{r_0^2}\right)^2 + \left(z - z_0\frac{R^2}{r_0^2}\right)^2} =$$

$$= \sqrt{\rho^2 + \frac{R^4}{r_0^2} - 2\frac{R^2}{r_0^2}\rho r_0\cos\gamma} = \frac{R}{r_0}\sqrt{R^2 + \rho^2\frac{r_0^2}{R^2} - 2\rho r_0\cos\gamma}\,.$$

Consequently, the potential due to the sources of the points P_0 and P_0^* is

$$\frac{1}{4\pi r} - \frac{k}{4\pi r^*} = \frac{1}{4\pi r} - \frac{kr_0}{4\pi R}\cdot\frac{1}{\sqrt{R^2 + \rho^2\dfrac{r_0^2}{R^2} - 2\rho r_0\cos\gamma}}\,.$$

From this it is clear that if we set $k = R/r_0$, this potential will vanish on the sphere S. This means that the function

$$G(P_0, P) = \frac{1}{4\pi r} - \frac{R}{r_0}\frac{1}{4\pi r^*} = \frac{1}{4\pi\sqrt{\rho^2 + r_0^2 - 2\rho r_0\cos\gamma}} -$$

$$- \frac{1}{4\pi\sqrt{R^2 + \rho^2\dfrac{r_0^2}{R^2} - 2\rho r_0\cos\gamma}} \qquad (22)$$

is the desired Green's function for the sphere.

In just the same way, we can find Green's function for the first and second boundary-value problems for the half-space $z>0$ in the following forms, respectively:

$$G(P_0, P) = \frac{1}{4\pi}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{4\pi}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}, \tag{23}$$

$$G(P_0, P) = \frac{1}{4\pi}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{4\pi}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}. \tag{23'}$$

The Green's function for the second boundary-value problem for a dihedral angle of magnitude $\pi/2$ is determined by the equation

$$G(P_0, P) = \frac{1}{4\pi}\left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right), \tag{24}$$

where r, r_1, r_2 and r_3 are the distances from the point P to the points P_0, P_1, P_2 and P_3 respectively. Here, P_1 is symmetric to P_0 about the xz-plane while, P_2 and P_3 are symmetric to P_1 and P_0 about the xy-plane (see Fig. 7).

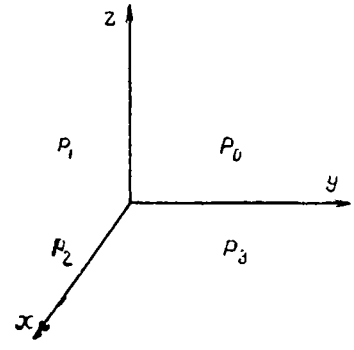

Fig. 7.

We suggest that the reader himself verify formulas (23), (23′), and (24). The method of images, which was used above, enables one to find the Green's function for the first and second boundary-value problems for a number of three-dimensional domains possessing particular symmetry properties. For example, it is possible to find explicit expressions for the Green's function for the first and second boundary-value problems for a dihedral angle of magnitude π/n (where n is an integer). Similarly, it is possible to find the Green's functions of the first boundary-value problem for a hemisphere, for a three-dimensional layer lying between two parallel planes, for a parallelepiped, etc. However, in the case of the second boundary-value problem, we cannot use the method of images to find the Green's function for a sphere, for example, or for any bounded region for that matter, since in this case the Green's function satisfies not Laplace's but Poisson's equation.

(b) *Green's function of the first boundary-value problem for a circle and for certain plane regions.* Let $z_0 = x_0 + iy_0$ be a point in the disk $|z| < R$. To compensate for the potential that exists on the circle $|z| = R$ as the result of a unit source at the point z_0, let us place a source of intensity -1 at the point $z_0^* = R^2/\bar{z}_0$ which is symmetric to the point z_0 with respect to the circle $|z| = R$. The potential function of these two sources will be of the form

$$\frac{1}{2\pi}\left(\ln\frac{1}{r}-\ln\frac{1}{r^*}\right), \quad r=|z-z_0|, \quad r^*=|z-z_0^*|.$$

On the circle $|z|=R$, this function will be equal to

$$\frac{1}{2\pi}\left(-\ln|Re^{i\varphi}-z_0|+\ln\left|Re^{i\varphi}-\frac{R^2}{\bar{z}_0}\right|\right)=\frac{1}{2\pi}\ln\frac{R}{r_0},$$

where $r_0=|z_0|$. Consequently, we may write the desired Green's function for the first boundary-value problem for the circle in the form

$$G(z_0, z)=\frac{1}{2\pi}\ln\frac{r^*r_0}{rR}=\frac{1}{2\pi}\ln\frac{r_0|z-z_0^*|}{R|z-z_0|}. \tag{25}$$

In just this way, we can obtain the Green's functions for the first boundary-value problem for a number of regions. For example, the Green's function for the half-plane $\operatorname{Im} z>0$ is

$$G(z_0, z)=\frac{1}{2\pi}\ln\frac{|z-\bar{z}_0|}{|z-z_0|}. \tag{26}$$

The Green's function for the semicircle $|z|<R$, $\operatorname{Im} z>0$ is

$$G(z_0, z)=\frac{1}{2\pi}\ln\frac{|z-z_0^*|\,|z-\bar{z}_0|}{|z-z_0|\,|z-\bar{z}_0^*|}. \tag{27}$$

The Green's function for the strip $0<\operatorname{Im} z<\pi$ is

$$G(z_0, z)=\frac{1}{2\pi}\ln\left|\frac{e^{z-\bar{z}_0}-1}{e^{z-z_0}-1}\right|. \tag{28}$$

The Green's function for the half-strip $0<\operatorname{Im} z<\pi$, $\operatorname{Re} z>0$ is

$$G(z_0, z)=\frac{1}{2\pi}\ln\frac{|e^{z-\bar{z}_0}-1|\,|e^{z+\bar{z}_0}-1|}{|e^{z-z_0}-1|\,|e^{z+z_0}-1|}. \tag{29}$$

The Green's function for the first boundary-value problem for the circle (25) can also be written in the following form:

$$G(z_0, z)=\frac{1}{2\pi}\ln\left|\frac{R^2-\bar{z}_0z}{R(z-z_0)}\right|, \tag{25'}$$

since the function $\zeta=R(z-z_0)/R^2-\bar{z}_0z$ is an analytic function of the complex variable $z=x+iy$ and maps the open disk $|z|<R$ into the unit open disk $|\zeta|<1$.

Now, let D denote an arbitrary simply connected domain not containing the point at infinity, let C denote a Jordan curve bounding

it, and let $w=w(z)$ be a function mapping the domain D conformally onto the unit disk $|w|<1$. Then, the function

$$\frac{w(z)-w(z_0)}{1-\overline{w(z_0)}\,w(z)}$$

is analytic in the domain D and has a zero of the first order at the point z_0 of that region. The contour C is mapped by w onto the unit circle. Consequently, the function

$$G(z_0, z)=\frac{1}{2\pi}\ln\left|\frac{1-\overline{w(z_0)}\,w(z)}{w(z)-w(z_0)}\right| \tag{30}$$

has a singularity at $z=z_0$ of the same type as a unit source. Everywhere else in D, this function is a twice continuously differentiable solution of the two-dimensional Laplace's equation. By virtue of the theorem on the correspondence of boundaries in the theory of conformal mappings, the function G is continuous and zero on the boundary C of the region D. Thus, *the function defined by Eq. (30) is a Green's function for the first boundary-value problem for the region D.* If the mapping of the domain D onto the unit disk is given explicitly, then (30) gives an explicit representation for the Green's function.

We know from the theory of functions of a complex variable that any simply connected domain can be mapped conformally onto the unit disk in such a way that any given point of the region will be mapped onto the coordinate origin and any given direction from that point will be mapped onto the direction of the positive real axis. This mapping is unique even when the boundary of the region is a Jordan curve, and it is a continuous one-to-one mapping in any closed region which maps into a closed disk. Therefore, Eq. (30) means that the Green's function of the first boundary-value problem exists for every simply connected domain bounded by a Jordan curve that does not enclose the point at infinity.*

It should be noted that the method of images can be used to obtain the Green's function of the second boundary-value problem for a number of two- as well as three-dimensional regions. For example, the Green's function of the second boundary-value problem for the upper half-plane can be found in the form

$$G(z_0, z)=\frac{1}{2\pi}\ln\frac{1}{|z-z_0|\,|z-\bar{z}_0|}. \tag{31}$$

The Green's function of the second boundary-value problem for the first quadrant of the plane, i.e., for an angle of magnitude $\pi/2$, may be written in the form

*We note that the solution of the first boundary-value problem for a plane region containing the point at infinity can, by the fractional linear transformation $\zeta = 1/z$, be reduced to the same problem in the ζ-plane for a region not containing the point at infinity.

$$G(z_0, z) = \frac{1}{2\pi} \ln \frac{1}{|(z^2 - z_0^2)(z^2 - \overline{z}_0^2)|}. \tag{32}$$

However, it should be noted that this method does not always lead to an easy construction of the Green's function of the second boundary-value problem even for certain very simple regions. This method does not provide the desired result even in the case of a circle.*

In conclusion, let us point out a general property of Green's functions for the basic boundary-value problems, namely, the property of symmetry.

Let $G(P_0, P)$ be the Green's function of the first, second, or third three-dimensional boundary-value problem for a domain G lying in a bounded portion of space. Let S denote its boundary, which is assumed to be piecewise smooth. (In the case of the second boundary-value problem, the Green's function is to be understood as a generalized Green's function.) If this function satisfies the conditions under which Ostrogradskiy's formula (12) for the Laplace operator can be applied, then, for arbitrary points P and P_0 of the region G,

$$G(P_0, P) = G(P, P_0). \tag{33}$$

Proof: In Eq. (12), we set

$$v(P) = G(P_0, P), \quad u(P) = G(P_0^*, P).$$

Then, we obtain

$$\iint_{\gamma} \left(G(P_0, P) \frac{\partial G(P_0^*, P)}{\partial n} - G(P_0^*, P) \frac{\partial G(P_0, P)}{\partial n} \right) dS +$$

$$+ \iint_{\gamma^*} \left(G(P_0, P) \frac{\partial G(P_0^*, P)}{\partial n} - G(P_0^*, P) \frac{\partial G(P_0, P)}{\partial n} \right) dS +$$

$$+ \iint_{S} \left(G(P_0, P) \frac{\partial G(P_0^*, P)}{\partial n} - G(P_0^*, P) \frac{\partial G(P_0, P)}{\partial n} \right) dS = 0, \tag{34}$$

where γ and γ^* are sufficiently small spheres with centers at the points P_0 and P_0^*.

Since the Green's function satisfies one of the homogeneous boundary conditions, (1′), (2′), or (3′), on the boundary, the last integral on the left-hand side of Eq. (34) vanishes. Therefore, when the radii on the spheres γ and γ^* approach 0, we obtain in the limit

*The solution of the second boundary-value problem for a circle will be given in Sect. 3, though without use of Green's function.

$$-G(P^{0*}, P_0)+G(P_0, P_0^*)=0.$$

This means that Eq. (33) holds.

The plane-case analog is as follows: *If the Green's function* $G(z_0, z)$ *for the first, second, or third boundary-value problem for a domain* D *lying in a bounded portion of the plane bounded by a piecewise-smooth contour* C *satisfies the conditions under which Green's formula (13) for the Laplace operator is applicable, then*

$$G(z_0, z) = G(z, z_0), \tag{35}$$

where z *and* z_0 *are arbitrary points in the domain* D. The proof of this is the same as the proof of Eq. (33) except that we need to use formula (13).

4. Solution of the first boundary-value problem of potential theory for a sphere and a circle. The above cases in which we can actually construct the Green's function enable us to write the solutions of the corresponding boundary-value problems in explicit form and, in particular, to establish the existence of these solutions. Let us show how this can be done with some important problems.

Suppose R is the radius of a sphere with center at the coordinate origin. As is clear from Eq. (22), in this case the Green's function $G(P_0, P)$ satisfies the conditions of applicability of Ostrogradskiy's formula. If we now assume that the solution of the three-dimensional Laplace's equation $\Delta u=0$ also satisfies these conditions, we obtain from formula (16)

$$u(P_0) = \iint_S \Phi(P) \frac{\partial G(P_0, P)}{\partial n} dS. \tag{36}$$

Simple calculation yields

$$\left.\frac{\partial G(P_0, P)}{\partial n}\right|_{\rho=R} = -\left.\frac{\partial G(P_0, P)}{\partial \rho}\right|_{\rho=R} =$$

$$= \frac{R - r_0\cos\gamma}{4\pi(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}} - \frac{\frac{r_0^2}{R} - r_0\cos\gamma}{4\pi(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}}$$

and Eq. (36) takes the form

$$u(P_0) = \frac{1}{4\pi R}\iint_S \Phi(P) \frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}} dS, \tag{37}$$

where $r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ is the distance from the point P_0 to the coordinate origin O and γ is the angle between the vectors $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$ (see Fig. 6).*

In particular, from (37) we have the identity

$$1 = \frac{1}{4\pi R} \int\int_S \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS. \tag{38}$$

Suppose that $\Phi(P)$ is an arbitrary continuous function of the point P lying on the sphere S. Then, the function $u(P_0)$ defined by Eq. (37) will, inside the sphere S, be a twice continuously differentiable solution of Laplace's equation since the Green's function has this property and differentiation under the integral sign is permissible. Now, suppose the point P_0 moves along the radius to a point P_0 lying on the sphere S. In view of the identity (38), we have

$$u(P_0) - u(P_0') =$$

$$= \frac{1}{4\pi R} \int\int_e (\Phi(P) - \Phi(P_0')) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS +$$

$$+ \frac{1}{4\pi R} \int\int_{S-e} (\Phi(P) - \Phi(P_0')) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS,$$

where e is a sufficiently small portion of the sphere S containing the point P_0'. On the basis of the identity (38), the first of the integrals on the right side of this last equation can be made arbitrarily small since the parenthesized expression immediately following the integral signs represents the change in a continuous function. The second of the integrals will become arbitrarily small when r_0 is sufficiently close to R. Thus, the function $u(P_0)$ approaches $\Phi(P_0')$ as P_0 approaches the point P_0' along the radius.

---

*In spherical coordinates, $z = \rho\cos\theta$, $x = \rho\sin\theta\cos\varphi$, $y = \rho\sin\theta\sin\varphi$. Then, $dS = R^2\sin\theta\, d\theta\, d\varphi$ and $\cos\gamma = \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0)$ since the direction cosines of the vectors $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$ are equal respectively to

$$(\sin\theta_0\cos\varphi_0,\ \sin\theta_0\sin\varphi_0,\ \cos\theta_0)$$

and

$$(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),$$

and since $\cos\gamma$ is equal to the scalar product of the unit vectors in the directions of $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$.

This convergence is uniform. It follows from this last fact that the function $u(P_0)$, if defined at points on the sphere to be equal to its limiting values, will be continuous in the closed region bounded by the sphere S. Thus, we conclude that *the solution of the first boundary-value problem of potential theory for a sphere with arbitrary continuous boundary conditions* $u|_S = \Phi(P)$ *exists and is given by Eq. (37).* Equation (37) is known as *Poisson's formula* or *Poisson's integral* for a sphere.

Suppose now that R is the radius of a disk, $|z| < R$. It is immediately clear from Eq. (25) that the Green's function $G(z_0, z)$ satisfies the conditions of applicability of Green's formula in this case. If we now assume that the solution of the two-dimensional Laplace's equation $\Delta u = 0$ also satisfies these conditions under the boundary conditions $u|_C = \Phi(z) = \Phi(Re^{i\theta})$, where θ is the argument of the point z lying in the circle C, then, from formula (19), we obtain

$$u(z_0) = \int_C \Phi(z) \frac{\partial G(z_0, z)}{\partial n} ds. \tag{39}$$

A simple calculation yields

$$\left.\frac{\partial G(z_0, z)}{\partial n}\right|_C = -\left.\frac{\partial G(z_0, z)}{\partial \rho}\right|_{\rho=R} = \operatorname{Re}\left\{\frac{e^{i\theta}}{2\pi} \frac{d}{dz} \ln \frac{R(z - z_0)}{R^2 - \bar{z}_0 z}\right\}_{|z|=R} =$$

$$= \operatorname{Re}\left\{\frac{e^{i\theta}}{2\pi}\left(\frac{1}{z - z_0} + \frac{\bar{z}_0}{R^2 - \bar{z}_0 z}\right)\right\}_{|z|=R} = \frac{R^2 - r_0^2}{2\pi R(R^2 + r_0^2 - 2Rr_0 \cos(\theta_0 - \theta)},$$

and Eq. (39) takes the form

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(Re^{i\theta}) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} d\theta, \tag{40}$$

where $r_0 = |z_0|$, $\theta_0 = \arg z_0$.

In particular, the identity

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} d\theta \tag{40'}$$

follows from (40).

Suppose that $\Phi(Re^{i\theta})$ is an arbitrary continuous function of the point z on the circle C. Then, the function $u = u(z_0)$ defined by Eq. (40) in the disk $|z_0| < R$ will be a twice continuously differentiable solution of the two-dimensional Laplace's equation because the

Green's function $G(z_0, z)$ has this property and differentiation under the integral sign is permissible. Let the point $z_0 = r_0 e^{i\theta_0}$ approach a point $z_0 = Re^{i\theta_0}$ lying on the circle C along a radial line from the origin. On the basis of identity (40′), we may write

$$u(z_0) - \Phi(Re^{i\theta_0}) = \frac{1}{2\pi} \int_e [\Phi(Re^{i\theta}) -$$

$$- \Phi(Re^{i\theta_0})] \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} d\theta + \frac{1}{2\pi} \int_{C-e} [\Phi(Re^{i\theta}) -$$

$$- \Phi(Re^{i\theta_0})] \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} d\theta,$$

where e is a sufficiently small arc of the circle C containing the point $Re^{i\theta_0}$. On the basis of the identity (40′), the first integral in the right-hand member of the last equation will become arbitrarily small in absolute value since the bracketed expression is the increment of a continuous function. The second integral in the right-hand member of this equation will be arbitrarily small when r_0 is sufficiently close to R because, in this case, the quantity $R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)$ is bounded below by a positive number. Thus, the function $u(z_0)$ approaches the function $\Phi(Re^{i\theta_0})$ uniformly as z_0 approaches points of the circle C along appropriate radii. Because the convergence is uniform, the function $u(z)$ must be continuous in the closed disk $|z| \leqslant R$ and on the boundary it assumes the given values $u|_C = \Phi(Re^{i\theta_0})$. This means that the solution of the first boundary-value problem of potential theory for a circle exists and is given by Eq. (40). This equation is known as Poisson's formula or integral for a circle.

Similarly, it is possible to obtain explicit formulas for the solutions of the first boundary-value problem of potential theory for a half-plane, a half-space, and other regions whose Green's function is known. It is suggested that the reader derive these formulas.

If D is an arbitrary simply connected plane domain bounded by a Jordan curve C which does not enclose $z = \infty$, and if $z = \omega(\zeta) = x + iy$ is a function mapping the region D conformally onto the disk $|z| < R$, then the first boundary-value problem of potential theory for the domain D can be reduced to the same type of boundary-value problem for the disk $|z| < R$.

Indeed, we immediately obtain from the Cauchy-Riemann equations the result that a twice continuously differentiable solution of Laplace's equation in the plane $\zeta = \xi + i\eta$ is also a twice continuously differentiable solution of Laplace's equation in the plane $z = x + iy$. On the basis of the theorem of correspondence between boundaries, the function $z = \omega(\zeta)$ represents a continuous one-to-one mapping in the closed region $D \cup C$. Consequently, the continuous boundary

conditions given on C transform into completely determined continuous boundary conditions on the circle $|z|=R$. After solving the first boundary-value problem of potential theory for this circle in accordance with Poisson's formula we shift from the variable z to the variable ζ, and obtain the desired solution for the region D. Thus, for the region D, the solution of the first boundary-value problem in potential theory exists and can be found by the use of Poisson's formula (40) together with a conformal mapping of the domain D onto a disk.

To obtain the formula for the desired solution, let us first write Poisson's formula (40) in the form

$$u(z_0) = \operatorname{Re} \frac{1}{2\pi i} \int_{|z|=R} \Phi(z) \frac{z+z_0}{z-z_0} \frac{dz}{z}, \tag{40''}$$

which is possible since

$$\frac{R^2-r_0^2}{R^2+r_0^2-2Rr_0\cos(\theta-\theta_0)} = \operatorname{Re} \frac{Re^{i\theta}+z_0}{Re^{i\theta}-z_0}, \quad z = Re^{i\theta}, \quad d\theta = \frac{dz}{iz}.$$

In formula (40''), let us make the substitutions: $z=\omega(\zeta)$, $z_0=\omega(\zeta_0)$, $u(z_0)=u(\omega(\zeta_0))=U(\zeta_0)$, $u(z)|_{|z|=R}=u(z(\zeta))|_{\zeta\subset C}=U(\zeta)|_C$. Then, we obtain the desired solution of the first boundary-value problem of potential theory for the domain D in the form

$$U(\zeta_0) = \operatorname{Re} \frac{1}{2\pi i} \int_C \Phi^*(\zeta) \frac{\omega(\zeta)+\omega(\zeta_0)}{\omega(\zeta)-\omega(\zeta_0)} \frac{d\omega(\zeta)}{d\zeta} \frac{d\zeta}{\omega(\zeta)}. \tag{40'''}$$

3. General Properties of Harmonic Functions. Applications to the Study of the Fundamental Boundary-Value Problems of Potential Theory

1. The Laplace operator and the divergence in orthogonal curvilinear coordinates. The definition of harmonic functions of three and two variables. Suppose that q_1, q_2, and q_3 are orthogonal curvilinear coordinates in xyz-space. If we set

$$x=\varphi_1(q_1, q_2, q_3), \quad y=\varphi_2(q_1, q_2, q_3), \quad z=\varphi_3(q_1, q_2, q_3),$$

then an element of arc length $ds=\sqrt{dx^2+dy^2+dz^2}$ in the coordinate system q_1, q_2, q_3 is given by the equation

$$ds^2 = H_1^2 dq_1^2 + H_2^2 dq_2^2 + H_3^2 dq_3^2, \tag{41}$$

where

$$H_1^2 = \left(\frac{\partial \varphi_1}{\partial q_1}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_1}\right)^2 + \left(\frac{\partial \varphi_3}{\partial q_1}\right)^2,$$

$$H_2^2 = \left(\frac{\partial \varphi_1}{\partial q_2}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_2}\right)^2 + \left(\frac{\partial \varphi_3}{\partial q_2}\right)^2,$$

$$H_3^2 = \left(\frac{\partial \varphi_1}{\partial q_3}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_3}\right)^2 + \left(\frac{\partial \varphi_3}{\partial q_3}\right)^2.$$

Analogously, in the plane case, if $x = \varphi_1(q_1, q_2)$ and $y = \varphi_2(q_1, q_2)$, where q_1 and q_2 are orthogonal curvilinear coordinates in the xy-plane, then the element of arc length $ds = \sqrt{dx^2 + dy^2}$ in the system of coordinates q_1, q_2 takes the form

$$ds^2 = H_1^2\, dq_1^2 + H_2^2\, dq_2^2, \tag{41'}$$

where

$$H_1^2 = \left(\frac{\partial \varphi_1}{\partial q_1}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_1}\right)^2, \quad H_2^2 = \left(\frac{\partial \varphi_1}{\partial q_2}\right)^2 + \left(\frac{\partial \varphi_2}{\partial q_2}\right)^2.$$

Suppose now that $\vec{A}$ is a three-dimensional vector in xyz-space. Then, from Ostrogradskiy's formula, we obtain

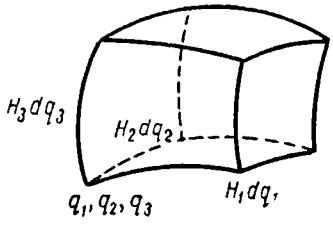

Fig. 8.

$$\operatorname{div} \vec{A} = \lim_{\tau \to 0} \frac{1}{\tau} \iint_S A_n\, dS, \tag{42}$$

where S is a closed surface and τ is the volume bounded by it. The integral

$$-\iint_S A_n\, dS$$

can be evaluated directly if, for the region bounded by S, we take the region bounded by the coordinate surfaces q_1, $q_1 + dq_1$, q_2, $q_2 + dq_2$, q_3, $q_3 + dq_3 = \text{const}$ (see Fig. 8).

We note that the element of arc length along the coordinate lines q_1, q_2, and q_3 is expressed by equations $ds_1 = H_1 dq_1$, $ds_2 = H_2 dq_2$, a and $ds_3 = H_3 dq_3$, respectively. We denote by A_1, A_2, and A_3 the projections of the vector $\vec{A}$ along directions tangent to the coordinate lines at the point (q_1, q_2, q_3).

Now, let us evaluate the above integral, interpreting the vector $\vec{A}$ as the velocity of a fluid.

Obviously, the rate of flow out of the volume τ through the faces $q_1 = \text{const}$ and $q_1 + dq_1 = \text{const}$ of the curvilinear parallelepiped is equal to

$$A_1\,ds_2\,ds_3\,|_{q_1+dq_1} - A_1\,ds_2\,ds_3\,|_{q_1} =$$

$$= \{H_2H_3A_1\,|_{q_1+dq_1} - H_2H_3A_1\,|_{q_1}\}\,dq_2\,dq_3 = \frac{\partial}{\partial q_1}(A_1H_2H_3)\,dq_1\,dq_2\,dq_3.$$

Analogously,

$$\frac{\partial}{\partial q_2}(A_2,\ H_1,\ H_3)\,dq_1dq_2dq_3,$$

is the rate of flow through the faces q_2 and $q_2+dq_2=\text{const}$, and

$$\frac{\partial}{\partial q_3}(A_3H_1H_2)\,dq_1dq_2dq_3$$

is the rate of flow through the faces q_3 and $q_3+dq_3=\text{const}$. Thus, noting that $\tau=H_1H_2H_3dq_1dq_2dq_3$, we obtain the general expression for the divergence in orthogonal curvilinear coordinates:

$$\operatorname{div}\vec{A} = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}(A_1H_2H_3) + \frac{\partial}{\partial q_2}(H_1A_2H_3) + \frac{\partial}{\partial q_3}(H_1H_2A_3)\right]. \quad (43)$$

Analogously, in the plane case, if we denote by B_1 and B_2 the projections of a plane vector $\vec{B}$ onto the coordinate axes q_1 and q_2, then from the formula

$$\operatorname{div}\vec{B} = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_C B_n\,ds, \quad (42')$$

where σ is the area of a domain bounded by a curve C, we obtain the formula

$$\operatorname{div}\vec{B} = \frac{1}{H_1H_2}\left[\frac{\partial}{\partial q_1}(B_1H_2) + \frac{\partial}{\partial q_2}(H_1B_2)\right]. \quad (43')$$

Let us now suppose that the vector field $\vec{A}$ has a scalar potential, $\vec{A}=\nabla u$. Then, since the projection of such a vector along any direction always coincides with the derivative of the potential function in that direction, we have

$$A_1 = \frac{\partial u}{\partial s_1} = \frac{1}{H_1}\frac{\partial u}{\partial q_1},\quad A_2 = \frac{\partial u}{\partial s_2} = \frac{1}{H_2}\frac{\partial u}{\partial q_2},$$

$$A_3 = \frac{\partial u}{\partial s_3} = \frac{1}{H_3}\frac{\partial u}{\partial q_3}.$$

If we substitute this into Eq. (43) and recall that $\operatorname{div}\nabla u=\Delta u$, we get an expression for the three-dimensional Laplace operator in orthogonal curvilinear coordinates

$$\Delta u = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2 H_3}{H_1}\frac{\partial u}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{H_3 H_1}{H_2}\frac{\partial u}{\partial q_2}\right)+\right.$$
$$\left.+\frac{\partial}{\partial q_3}\left(\frac{H_1 H_2}{H_3}\right)\frac{\partial u}{\partial q_3}\right]. \tag{44}$$

Analogously, in the plane case, if a vector field $\vec{B}$ has a scalar potential, $\vec{B} = \nabla u$, then, $B_1 = \frac{1}{H_1}\frac{\partial u}{\partial q_1}$, $B_2 = \frac{1}{H_2}\frac{\partial u}{\partial q_2}$, and we obtain from Eq. (43′) an expression for the two-dimensional Laplace operator in orthogonal curvilinear coordinates

$$\Delta u = \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_1}\frac{\partial u}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2}\frac{\partial u}{\partial q_2}\right)\right]. \tag{44′}$$

For example, in the case of an orthogonal transformation of coordinates x, y, and z, we have $q_1 = x_1$, $q_2 = y_1$, $q_3 = z_1$, $ds^2 = dx_1^2 + dy_1^2 + dz_1^2$, $H_1 = 1$, $H_2 = 1$, $H_3 = 1$ and, on the basis of (44) we have

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial z_1^2}.$$

In the case of spherical coordinates, where $q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$, we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2,$$
$$H_1 = 1,\ H_2 = r,\ H_3 = r\sin\theta$$

and the three-dimensional Laplace operator can be written in the form

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}. \tag{45}$$

In cylindrical coordinates, where $q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$, $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, and $z = z$, we have $H_1 = 1$, $H_2 = \rho$, $H_3 = 1$, and the three-dimensional Laplace operator can be written in the form

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}. \tag{46}$$

In particular, if u does not depend on z, we obtain from (46) the two-dimensional Laplace operator in polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$:

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}. \tag{46′}$$

If the function $\zeta=\zeta(z)=\xi+i\eta$ maps a domain in the plane $z=x+iy$ onto some domain in the ζ-plane conformally, then, on the basis of (44′), the two-dimensional Laplace operator in the variables ξ and η transforms into

$$\Delta u = \frac{1}{H_1^2}(u_{\xi\xi}+u_{\eta\eta}) = \left|\frac{d\zeta}{dz}\right|^2(u_{\xi\xi}+u_{\eta\eta}). \tag{47}$$

From this, it follows in particular that if a function $u(z)=u(x, y)$ is a twice continuously differentiable solution of Laplace's equation in two dimensions, $u_{xx}+u_{yy}=0$, it will also be a twice continuously differentiable solution of Laplace's equation $u_{\xi\xi}+u_{\eta\eta}=0$ as a function of the variables of the inversion mapping

$$\xi=\frac{x}{r^2},\quad \eta=\frac{y}{r^2}\qquad (r=\sqrt{x^2+y^2}). \tag{48}$$

Suppose that $u(P)=u(x, y, z)$ is a twice continuously differentiable solution of Laplace's equation in three dimensions, $u_{xx}+u_{yy}+u_{zz}=0$. We introduce the inversion transformation

$$\xi=\frac{x}{r^2},\quad \eta=\frac{y}{r^2},\quad \zeta=\frac{z}{r^2}\qquad (r=\sqrt{x^2+y^2+z^2}) \tag{48′}$$

and consider the function $v(\xi, \eta, \zeta)=ru(x, y, z)$ as a function of the variables ξ, η, ζ. On the basis of (45), the Laplace operator in the variables ξ, η, ζ in the spherical coordinate system $\xi=\rho\sin\theta\cos\varphi$, $\eta=\rho\sin\theta\sin\varphi$, $\zeta=\rho\cos\theta$ is written in the form

$$\Delta_{\xi\eta\zeta}=\frac{\partial^2}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+$$
$$+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

Since a change from ξ, η, ζ to x, y, z in a spherical coordinate system means only the change $\rho=1/r$, where $\rho=\sqrt{\xi^2+\eta^2+\zeta^2}$, we have

$$\Delta_{\xi\eta\zeta}=r^4\frac{\partial}{\partial r^2}+\frac{r^2}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{r^2}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

From this, we obtain

$$\Delta_{\xi\eta\zeta}v=r^4\left[\frac{\partial}{\partial r^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right](ru)=$$
$$=r^5\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]u=$$
$$=r^5\Delta_{xyz}u.$$

This means that the function $v(\xi, \eta, \zeta) = ru(x, y, z)$ as a function of ξ, η, ζ will be a twice continuously differentiable solution of Laplace's equation in three dimensions.

Now, we can give the general definition of harmonic functions of three or of two variables.

*A function $u(P) = u(x, y, z)$ is said to be a harmonic function of three variables in a bounded simply connected domain G in xyz-space if it is single-valued in this domain and is a twice continuously differentiable solution of the three-dimensional Laplace equation.** *A given function is said to be a harmonic function at a point in space if it is harmonic in some sufficiently small domain containing that point. The function $u(P) = u(x, y, z)$ is said to be harmonic at the point at infinity if the function $v(\xi, \eta, \zeta) = ru(x, y, z)$, considered as a function of the variables ξ, η, ζ, defined by the inversion transformation (48'), is a harmonic function at the coordinate origin $\xi = \eta = \zeta = 0$. If a function $u(P) = u(x, y, z)$ is harmonic at every point of a domain in the extended xyz-space, i.e., the space plus the point at infinity, we shall say that u is a harmonic function in that domain.*

Analogously, in the plane case, the definition of a harmonic function in a bounded simply connected domain and at a point in the plane remains the same except that solutions of the three-dimensional Laplace equation must be replaced with solutions of the two-dimensional equation. On the other hand, the definition of a function that is harmonic at infinity is quite different from the corresponding definition in the case of three variables. Specifically, *a function $u(z) = u(x, y)$, as a function of the variables x and y, is said to be harmonic at the point at infinity, $x = y = \infty$, if, as a function of the variables ξ and η defined by the inversion transformation (48), it is harmonic at the coordinate origin $\xi = \eta = 0$.* Furthermore, just as in the three-dimensional case, we shall say that *a function of two variables $u(z) = u(x, y)$ is harmonic in a domain of the extended xy-plane or the extended complex plane $z = x + iy$ if it is harmonic at every point of that region.*

It should be noted that, in accordance with the definition, harmonic functions in multiply connected domains need not be single-valued. As an example of a multi valued harmonic function, we can exhibit the harmonic function of two variables $u(x, y) = \arctan(y/x)$. As differentiation shows immediately, this function is a twice continuously differentiable solution of Laplace's equation, for example, in the annulus $1 < |z| < 2$. However, as we let z move around a closed contour encircling the coordinate origin, this function undergoes an increase equal to 2π.

We note that the reason for the quite different definitions of harmonic functions at infinity in the cases of two- and

*Here, as is usual, when we are speaking of solutions of Laplace's equation, these solutions are assumed to be real.

three- dimensions is that, in the two-dimensional case, the inverse transformation has the effect of multiplying the Laplace operator by r^{-4}:

$$u_{xx} + u_{yy} = (u_{\xi\xi} + u_{\eta\eta}) \frac{1}{r^4}, \qquad r = \sqrt{x^2 + y^2}, \tag{49}$$

whereas, in the three-dimensional case, we have a more complicated rule, expressed, in accordance with the calculations made above, by the equation

$$u_{xx} + u_{yy} + u_{zz} = [(ru)_{\xi\xi} + (ru)_{\eta\eta} + (ru)_{\zeta\zeta})] \cdot \frac{1}{r^5}, \quad r = \sqrt{x^2 + y^2 + z^2}. \tag{50}$$

All the simplest general properties of harmonic functions of three dimensions can be obtained from Ostrogradskiy's formula for the Laplace operator (12), from the representation of functions in the form of the sum of potentials (14), and from Poisson's formula for a sphere (37):

$$-\iint_S \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS = \iiint_G (v\Delta u - u\Delta v)\, d\tau, \tag{12}$$

$$u(P_0) = \iint_S \left(u \frac{\partial \delta(P_0, P)}{\partial n} - \delta(P_0, P) \frac{\partial u}{\partial n}\right) dS -$$

$$- \iiint_G \delta(P_0, P)\, \Delta u\, d\tau, \tag{14}$$

$$u(P_0) = \frac{1}{4\pi R} \iint_S \Phi(P) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS. \tag{37}$$

In just the same way, the general properties of harmonic functions of two variables can be obtained from Green's formula for the Laplace operator (13), from the representation of functions as sums of logarithmic potentials (15), and from Poisson's formula for a circle (40):

$$-\int_C \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) ds = \iint_D (v\Delta u - u\Delta v)\, d\sigma, \tag{13}$$

$$u(z_0) = \int_C \left(u \frac{\partial \delta(z_0, z)}{\partial n} - \delta(z_0, z) \frac{\partial u}{\partial n}\right) ds - \iint_D \delta(z_0, z)\, \Delta u\, d\sigma, \tag{15}$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(Re^{i\theta}) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0\cos(\theta - \theta_0)}\, d\theta. \tag{40}$$

Let us look at the general properties of harmonic functions in their most natural order.

2. A necessary condition for existence and uniqueness of solutions of the second interior boundary-value problem of potential theory. If we set $v \equiv 1$ in Eqs. (12) and (13) and take u in (12) to be a harmonic function, $u(P)$ of three variables and u in (13) to be a harmonic function $u(z)$ of two variables, we obtain

$$\iint_S \frac{\partial u}{\partial n}\, dS = 0, \quad \int_C \frac{\partial u}{\partial n}\, ds = 0. \tag{51}$$

In the three dimensional-case, this means that the second interior boundary-value problem of potential theory for domain G cannot have a solution which satisfies the conditions of applicability of Ostrogradskiy's formula (12), unless the boundary conditions $\partial u/\partial n\,|_S = \Phi(P)$ satisfy the first of Eqs. (51).

Analogously, in the plane case, for a solution of the second interior boundary-value problem for a domain D to exist and satisfy the conditions of applicability of Green's formula (13), it is necessary that the boundary conditions $\partial u/\partial n\,|_C = \Phi(z)$ satisfy the second of Eqs. (51).

If we interpret a harmonic function as a potential function corresponding to a steady flow of an incompressible fluid, then condition (51) means that the amount of fluid flowing into the region under consideration is zero.

If we set $v \equiv 1$ in Eqs. (12) and (13) and replace u with the square of a harmonic function $u(P)$ or, respectively, with the square of a harmonic function $u(z)$, we obtain, after some simple manipulations,

$$\begin{aligned} -\iint_S u\frac{\partial u}{\partial n}\, dS &= \iiint_G (u_x^2 + u_y^2 + u_z^2)\, d\tau, \\ -\int_C u\frac{\partial u}{\partial n}\, ds &= \iint_D (u_x^2 + u_y^2)\, d\sigma. \end{aligned} \tag{52}$$

It follows from these equations that if a solution of the second boundary-value problem for a domain G or a domain D exists and satisfies the conditions of applicability of Ostrogradskiy's formula in the three-dimensional case or Green's formula in the two-dimensional case, then this solution is unique up to an arbitrary additive constant. To see this, assume the theorem to be false and apply Eqs. (52) to the difference between two solutions with the same boundary conditions.

3. The mean-value theorem and the maximum principle for harmonic functions. Suppose the function u in Eqs. (14) and (15) is a harmonic function of three or two variables, respectively. Let S (in the first case) be a sphere of radius R with center at P_0 and let C (in the second case) be a circle of radius R with center at z_0. But since

$$\frac{\partial}{\partial n}\delta(P_0, P)\Big|_S = \frac{1}{4\pi R^2}, \quad \frac{\partial}{\partial n}\delta(z_0, z)\Big|_C = \frac{1}{2\pi R}$$

we obtain, by virtue of (51),

$$u(P_0) = \frac{1}{4\pi R^2}\iint_S u(P)\,dS, \quad u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\varphi})\,d\varphi. \tag{53}$$

The first of these is called *the mean-value theorem for harmonic functions of three variables*, the second *the mean-value theorem for harmonic functions of two variables*.

Theorem (the maximum principle for harmonic functions). *A nonconstant harmonic function of three or two variables cannot assume either its maximum or its minimum value at an interior point of the region in which it is harmonic.**

Proof: Let $u(P)$ be a harmonic function of three variables in a bounded domain G, and suppose it assumes its maximum value M at P_0, an interior point of G. From the mean-value theorem, we have

$$u(P_0) = \frac{1}{4\pi R^2}\iint_S u(P)\,dS = \begin{cases} = M, & \text{if } u(P) \equiv M \\ < M, & \text{if } u(P) \not\equiv M. \end{cases}$$

However, $u(P_0) = M$ by hypothesis. Consequently, the second alternative in the last inequality cannot hold. This means that $u(P) \equiv M$ in any sphere with center P_0 which is contained in G. If P_1 is any other point contained in one of these spheres, we can repeat the argument and conclude that $u(P) \equiv M$ in any sphere with center P_1 which is contained in G. By making use of the Heine-Borel theorem, we can cover any closed region contained in G by a finite number of such spheres, and this implies $u(P) \equiv M$ at every point in G. Thus, the maximum principle for the three-dimensional case has been proved. The statement about the minimum value can be reduced to this case merely by changing the sign of u. The same

*In the case of three variables, it is assumed that the region in question does not include the point at infinity. For example, the function $u = 1/r$ (where $r = \sqrt{x^2+y^2+z^2}$) is harmonic in the region $r > 1$ and, in this case $u|_{r=1} = 1$, $u|_{r=\infty} = 0$. In the case of two independent variables, the region in question can even include the point at infinity.

proof holds for the case of two variables, except that we must employ the mean-value theorem for harmonic functions of two variables.

4. The uniqueness and stability of solutions to the first boundary-value problem of potential-theory. *In the three- and two-dimensional cases, the solution of the first interior boundary-value problem of potential theory is unique and stable.*

Proof: If two harmonic functions of three variables are continuous in a closed region $G \cup S$ and if the absolute value of their difference on S does not exceed a positive number ε, then, because of the maximum principle, the absolute value of this difference does not exceed ε anywhere in the closed region $G \cup S$. Therefore, if the boundary conditions are identical for two solutions of the first boundary-value problem, these solutions are also identical, and if the boundary conditions differ by an amount less than ε in absolute value, this will also be true of the solutions in the corresponding region. Obviously, the same reasoning holds in the plane case.

It is important that we have not only established the stability of the first interior boundary-value problem but also given a measure of that stability.

5. Theorems on removable singularities of harmonic functions. *Theorem. Suppose a function $u(z)$ of two variables x and y is harmonic and single-valued in a neighborhood of a point $z_0 = x_0 + iy_0$. Suppose also that $u(z)/\ln|z - z_0|$ approaches zero uniformly as $z \to z_0$. Then, $u(z)$ has a limiting value as $z \to z_0$ and, if we define $u(z_0)$ to be equal to this value, then $u(z)$ will be harmonic at z_0.*

Proof: Let $v(z)$ be a solution of the first boundary-value problem of potential theory in a sufficiently small circle of radius R with center at the point z_0, such that $v|_C = u|_C$ on the boundary C. Such a solution exists and is given by Poisson's integral (40) for a circle. The function $U(z) = u(z) - v(z)$ will be continuous in closed annulus $\rho \leqslant r \leqslant R$, where $r = |z - z_0|$ and ρ is an arbitrarily small positive number, and it will be harmonic in the interior of this annulus. By hypothesis, we can choose ρ in such a way that, on the circle $|z - z_0| = \rho$, the function U will satisfy the inequality $|U| < \varepsilon(\rho) \ln(1/r)$, where $\varepsilon(\rho)$ is an arbitrarily small number and $\varepsilon(\rho) \to 0$ as $\rho \to 0$. We now see that the harmonic function $\varepsilon(\rho) \ln(1/r)$ majorizes the function U in the annulus $\rho \leqslant r \leqslant R$ since the latter function vanishes on the circle C and satisfies the inequality $|U| < \varepsilon(\rho) \ln(1/\rho)$ on the circle $r = \rho$. If we now assume that the function U does not vanish at some point $z \neq z_0$ and if we let ρ approach 0, we immediately arrive at a contradiction since at the given point $\varepsilon(\rho) \ln(1/r) \to 0$ as $\rho \to 0$. This means that $u(z) \equiv v(z)$ for $z \neq z_0$. If we set $u(z_0) = v(z_0)$, we conclude that the function $u(z)$ is harmonic at the point z_0 since Poisson's integral (40) possesses this property.

Theorem. If a function $u(P)$ of three variables x, y, and z is harmonic and single-valued in a neighborhood of a point P_0 and if u is bounded or if it grows in such a way that $ru(P)$, where r is the

distance between points P and P_0, approaches 0 uniformly as $P \to P_0$, then u approaches a limit as P approaches P_0. If we assign this value to $u(P_0)$, then $u(P)$ will be harmonic at P_0.

Proof: Let $v(P)$ be a solution of the first boundary-value problem in potential theory in a sufficiently small sphere of radius R with center at the point P_0. Suppose also that on the boundary S of this sphere we have $v|_S = u|_S$. Such a solution exists and is given by Poisson's integral for a sphere (37).

The function $U = u(P) - v(P)$ is harmonic in a region $\rho < r < R$, where ρ is a sufficiently small positive number, and it is continuous on the boundary of that region. Furthermore,

$$U|_S = 0, \quad |U|_{r=\rho} < \varepsilon(\rho) \frac{1}{r}\Big|_{r=\rho},$$

where $\varepsilon(\rho)$ is a quantity that approaches 0 as $\rho \to 0$.

We conclude on the basis of the maximum principle that the harmonic function $\varepsilon(\rho)/r$ majorizes the function U in the closed region $\rho \leqslant r \leqslant R$. If we now suppose that $U \not\equiv 0$ at some point other than the point P_0, we immediately arrive at a contradiction since the function $\varepsilon(\rho)/r$ can be made arbitrarily small at that point with sufficiently small ρ. Consequently, if we set $u(P_0) = v(P_0)$, we see that the functions $u(P)$ and $v(P)$ coincide everywhere. This means that the function $u(P)$, being Poisson's integral for the sphere, is harmonic at the point P_0.

We note that the requirement that the harmonic function be single-valued in a neighborhood of the point in question is essential in these theorems. For example, the function $u(z) = \arctan(y/x)$, defined in a neighborhood of the coordinate origin, satisfies all the requirements of the theorem on removable singularities of harmonic functions of two variables except the requirement of single-valuedness and, as one can see, this function does not have a limit as z approaches the origin.

6. Harmonicity of solutions of the exterior boundary-value problem of potential theory at infinity and the behavior of harmonic functions in a neighborhood of infinity. *The solutions of the exterior boundary-value problem of potential theory in the three- and two-dimensional cases are harmonic functions at the point at infinity.*

Both of these solutions are, by hypothesis, harmonic functions in a neighborhood of infinity and they satisfy the regularity condition at infinity. In the plane case, it follows from the theorem on removable singularities and from the regularity condition at infinity that a solution of the problem $u(z)$, regarded as a function of the inversion variables (48), is harmonic at the point $\xi = \eta = 0$. But, by definition, this means that the function $u(z)$ is a harmonic function of the variables x and y at the point $x = y = \infty$. In the three-dimensional case, it follows from the condition of regularity at

infinity and from the property of the three-dimensional Laplace operator (50) that the function $v(\xi, \eta, \zeta) = ru(P)$, where $r = \sqrt{x^2+y^2+z^2}$, regarded as a function of the inversion variables (48′), satisfies the conditions of the theorem on removable singularities of harmonic functions of three variables in a neighborhood of the point $\xi=\eta=\zeta=0$. Consequently, the function $v(\xi, \eta, \zeta)$ is a harmonic function of ξ, η, and ζ at the coordinate origin. By definition, this means that the function $u(P)$ is harmonic at the point $x=y=z=\infty$.

As a consequence of the proven proposition about the harmonic character of solutions of exterior boundary-value problems, we see that, by means of an inversion in the plane case, the first, second, and third exterior boundary-value problems reduce, respectively, to the first, second, and third interior boundary-value problems. In the three-dimensional case, the first exterior problem is again reduced to the first interior problem by means of the inversion transformation, but the second and the third exterior problems may be reduced either to the second interior problem or to the third interior problem. The validity of the assertion regarding the plane case is immediately obvious from the boundary conditions (2′)-(2″)-(2‴); in the three-dimensional case, it is clear from the boundary conditions (1′)-(1″)-(1‴) and from consideration of the function $v(\xi, \eta, \zeta) = ru(P)$ as a function of the variables ξ, η, ζ.

If $u(P)$ is a harmonic function at the point at infinity of xyz-space, then in a neighborhood of the point at infinity,

$$|u| < \frac{A}{r}, \quad \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial u}{\partial y}\right|, \left|\frac{\partial u}{\partial z}\right| < \frac{A}{r^2}, \tag{54}$$

where $r = \sqrt{x^2+y^2+z^2}$ and A is a constant.

Proof: The first of the inequalities (54) follows from the fact that the function $v(\xi, \eta, \zeta) = ru(P)$ is harmonic as a function of the inversion variables ξ, η, ζ at the point $\xi=\eta=\zeta=0$. Also,

$$\frac{\partial u}{\partial x} = v\frac{\partial}{\partial x}\frac{1}{r} + \left(\frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)\frac{1}{r}.$$

Therefore, it follows immediately from (48′) that the second of inequalities (54) is valid. In the same way, we can see that inequalities (54) are valid in the general case.

Similarly, if $u(z)$ is harmonic at the point at infinity of the xy-plane, then in a neighborhood of infinity,

$$|u| < A, \quad \left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial u}{\partial y}\right| < \frac{A}{r^2}, \tag{54′}$$

where $r = |z|$ and A is constant.

The validity of these inequalities follows immediately from the definition of a harmonic function at the point at infinity in the plane.

7. The first theorem on the convergence of harmonic functions. *Theorem. If a sequence of functions of three or two variables which are harmonic in some domain and continuous in the closure of that domain converges uniformly on the boundary of that domain, then it converges uniformly in the closed domain and the function representing the limit is harmonic in that domain and continuous in its closure.*

Proof: For definiteness, let $u_1(P), \ldots, u_n(P), \ldots$ denote a sequence of functions that are harmonic in a domain G and continuous in its closure $G \cup S$. It follows from the uniform convergence of the sequence on the surface S that, for an arbitrary given number ε, the inequality $|u_m(P) - u_n(P)| < \varepsilon$ will hold everywhere on the surface whenever $m, n > N$, where N is a sufficiently large number. But we already know that this last inequality will hold throughout the entire region $G \cup S$ since the expression between the absolute-value bars is a function which is harmonic in G and continuous in $G \cup S$. Therefore, the sequence $\{u_n(P)\}$ converges uniformly in the closed region $G \cup S$ to some limit function $u(P)$. This function $u(P)$ is continuous in the closed region $G \cup S$ since it is the limit of a uniformly convergent sequence of continuous functions. Now, let P_0 denote a point in the domain G and let R denote the radius of a sufficiently small sphere S_0 which has its center at the point P_0 and is contained in G. From Poisson's formula for a sphere, we have

$$u_n(P_0) = \frac{1}{4\pi R} \iint_{S_0} u_n(P) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS.$$

Taking the limit under the integral sign, we see that the limit function $u(P_0)$ is represented by Poisson's integral in the interior of the sphere. Consequently, it is a harmonic function at the point P_0. Since the point P_0 is an arbitrary point in the domain G, the function $u(P)$ is harmonic in G. In the case of two variables, our reasoning is the same except that we use Poisson's formula for a circle instead of a sphere.

The theorem that we have just proven is known as *Harnack's first theorem.*

8. Inequalities for positive harmonic functions. Liouville's theorem. *Suppose $u(P)$ is a function of three variables and is harmonic and positive in a sphere of radius R with center at the point P_0. Then, for an arbitrary point P lying at a distance $r<R$ from the point P_0,*

$$\frac{R(R-r)}{(R+r)^2} u(P_0) \leqslant u(P) \leqslant \frac{R(R+r)}{(R-r)^2} u(P_0). \tag{55}$$

Analogously, in the plane case, if $u(z)$ is a function that is harmonic and positive in a circle of radius R with center at a point z_0, then, for an arbitrary point z lying at a distance $r<R$ from z_0,

$$\frac{R-r}{R+r}u(z_0) \leqslant u_1^*(z) \leqslant \frac{R+r}{R-r}u(z_0). \qquad (55')$$

Proof: If we replace cos γ with +1 in Poisson's formula for a sphere (37) and use the mean-value theorem (53), we obtain the second inequality of (55). Similarly, if we replace cos γ in the same formula with -1, we obtain the first inequality of (55). Inequalities (55′) are obtained in exactly the same way by using Poisson's formula for a circle.

Theorem (Liouville). *If a function of three (resp. two) variables is harmonic at all points of a three (resp. two)-dimensional space (except the point at infinity), is bounded, or does not change sign, then it is identically equal to a constant.*

Proof: Without loss of generality, we may assume the given function to be positive since otherwise we could either multiply it by -1 or add a sufficiently large positive number to it. In inequalities (55) [res. (55′)], we may take R arbitrarily large. Taking the limit as $R \to \infty$, we obtain $u(P) = \text{const}$ (resp. $u(z) = \text{const}$).

9. The second theorem about the convergence of harmonic functions. *Theorem. If a sequence of functions of three or two variables that are harmonic in a domain increases monotonically, convergence of this sequence at any point of the domain implies its uniform convergence in every closed subdomain contained in the original domain in which the functions of the sequence are harmonic.*

Proof: Let $u_1(P) \leqslant u_2(P) \leqslant \ldots \leqslant u_n(P)$ be a monotically increasing sequence of harmonic functions of three variables in a region G and let P_0 be a point at which this sequence converges. We denote by R the radius of a sphere which has its center at the point P_0 and is contained in G. From inequalities (55), we obtain, for $m > n$,

$$0 \leqslant u_m(P) - u_n(P) \leqslant \frac{R(R+r)}{(R-r)^2}(u_m(P_0) - u_n(P_0)).$$

Consequently, in some sphere with center at the point P_0, having a radius $R/2$, for example, the sequence of functions converges uniformly. If we apply this reasoning to any other point of this sphere, we obtain a new sphere in which the sequence converges uniformly. According to the Heine-Borel theorem, we can cover any closed region contained in G with a finite number of such spheres. This means that, in the closure of the domain in question, the sequence of functions converges uniformly. In the case of two independent variables, the reasoning is the same except that we use inequalities (55′) instead of (55).

This theorem is known as *Harnack's second theorem.*

10. **Equicontinuity and compactness of sets of harmonic functions.** Theorem. *A set of harmonic functions of three or two variables which is uniformly bounded in a given domain is equicontinuous in every closed subdomain of the given domain.**

Proof: Let $u(P)$ be a function belonging to a set of functions of three variables which are harmonic in a domain G, and uniformly bounded in that domain, $|u(P)|<M$, $M=\text{const}$. For every pair of points P_0, P_0' in the interior of a sphere of radius R with center at the point P_1 and contained entirely in G, we have, on the basis of Poisson's formula,

$$u(P_0') - u(P_0) = \frac{1}{4\pi R}\iint_S u(P)\left\{\frac{R^2 - r_0'^2}{(R^2 + r_0'^2 - 2Rr_0'\cos\gamma')^{3/2}} - \right.$$

$$\left. - \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0\cos\gamma)^{3/2}}\right\} dS,$$

where r_0' and r_0 are the distances of the points P_0' and P_0 from the point P_1 and where γ' and γ are, respectively, the angles between the vectors $\overrightarrow{P_1P}$ and $\overrightarrow{P_1P_0'}$, on the one hand, and $\overrightarrow{P_1P}$ and $\overrightarrow{P_1P_0}$, on the other. From this it is obvious that, independently of the choice of function $u(P)$ belonging to the given set of functions, we have

$$|u(P_0') - u(P_0)| < \varepsilon,$$

provided the distance between the points P_0 and P_0' is sufficiently small. But, in a finite number of steps, we can cover every closed region contained in the region G. Consequently, the set of functions in the closed region in question is equicontinuous. In the case of two variables, the proof is the same except that then we would need to use Poisson's formula for a circle.

A set of harmonic functions of three (resp. two) variables which is uniformly bounded in a three- (resp. two-) dimensional domain is compact in that domain; i. e. every infinite sequence of functions belonging to that set, has a subsequence which converges uniformly in every closed subdomain contained in the interior of the given domain.

Proof: On the basis of a theorem of Arzelà, from every sequence of arbitrary functions which are equicontinuous in a closed region and uniformly bounded at one point of that region, one can select a subsequence which converges uniformly in the given closed region. In view of this, we see that the property of compactness of a set of harmonic functions follows from its equicontinuity.

11. **The property of analyticity of harmonic functions.** Theorem. *Harmonic functions of three or two variables are analytic functions of these variables.*

Proof: Without loss of generality, we can consider the case of a function which is harmonic in the interior and on the boundary of a unit sphere (circle, in the two dimensional case) with center at the origin, and examine it in some neighborhood of the origin.** Then, Poisson's formulas for a circle and for a sphere take the following forms:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi}\Phi\left(e^{i\theta}\right)\frac{1 - r_0^2}{1 + r_0^2 - 2r_0\cos(\theta - \theta_0)}\,d\theta,$$

$$u(P_0) = \frac{1}{4\pi}\iint_S \Phi(P)\,\frac{1 - r_0^2}{(1 + r_0^2 - 2r_0\cos\gamma)^{3/2}}\,dS.$$

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*We recall that a set of functions $\{u(P)\}$ defined in a closed region G is said to be equicontinuous in that closed region if, for an arbitrarily small positive number ε, there exists a positive number δ such that the inequality $|u(P_1) - u(P_2)| < \varepsilon$ holds for every function $u(P)$ of the given set whenever $|P_1 - P_2| < \delta$, where P_1 and P_2 belong to G.

**This condition can be satisfied by means of a similarity transformation and a translation of the coordinate origin.

For $r_0 < 1$, the function $1 - r_0^2 / 1 + r_0^2 - 2r_0 \cos\gamma$ can be expanded in a power series

$$\frac{1-r_0^2}{1+r_0^2-2r_0\cos\gamma} = -1 + \frac{1}{1-r_0e^{i\gamma}} + \frac{1}{1-r_0e^{-i\gamma}} = 1 + \\ + \sum_1^\infty r_0^n (e^{in\gamma} + e^{-in\gamma})$$

and, consequently, Poisson's integral (shown above) for a circle can be written in the form

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{i\theta}) \left\{1 + \sum_{n=1}^\infty r_0^n [(\cos(\theta-\theta_0) + i\sin(\theta-\theta_0))^n + \right. \\ \left. + (\cos(\theta-\theta_0) - i\sin(\theta-\theta_0))^n]\right\} d\theta. \tag{56}$$

If we replace $r_0\cos\theta$ and $r_0\sin\theta$ with x_0 and y_0 in this expression, we obtain an expansion of the function $u(z_0)$ in a series of powers of the variables x_0 and y_0.

For $|x| < 1$ and r_0 sufficiently small, the function $(1 + r_0^2 - 2r_0x)^{-\frac{1}{2}}$ can be expanded in a series of powers of r_0:

$$(1+r_0^2-2r_0x)^{-\frac{1}{2}} = 1 - \frac{1}{2}(r_0^2-2r_0x) + \frac{3}{8}(r_0^2-2r_0x)^2 + \ldots = \\ = 1 + r_0x + r_0^2\left(\frac{3}{2}x - \frac{1}{2}\right) + \ldots = \sum_{n=0}^\infty P_n(x) r_0^n,$$

where $P_n(x)$ denotes the coefficients of r_0^n. It is easy to see that the $P_n(x)$ (where $n = 0, 1\ldots$) are polynomials in x of degree n and that they satisfy the conditions

$$P_n(-x) = (-1)^n P_n(x), \quad n = 0, 1, 2, \ldots . \tag{57}$$

For example,

$$P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{2}(3x^2 - 1).$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \; P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

The polynomials $P_n(x)$ are among the simplest special functions. They are known as *Legendre polynomials*.

Let us differentiate the equation

$$\frac{1}{\sqrt{1+r_0^2-2r_0x}} = \sum_{n=0}^\infty P_n(x) r_0^n$$

with respect to r_0 and multiply the result by $2r_0$. If we add this product to the equation just shown, we obtain

$$\frac{1-r_0^2}{(1+r_0^2-2r_0x)^{3/2}} = P_0(x) + 3P_1(x) r_0 + \ldots + (2n+1) P_n(x) r_0^n + \ldots .$$

Therefore, Poisson's formula for a sphere, shown above, can be written in the form

$$u(P_0) = \frac{1}{4\pi} \iint_S \Phi(P) \left(\sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) r_0^n \right) dS.$$

Let us replace r_0^2 with $x_0^2 + y_0^2 + z_0^2$. Then, by virtue of (57) and the fact that

$$r_0 \cos\gamma_0 = r_0 [\cos\theta_0 \cos\theta + \sin\theta_0 \sin\theta \cos(\varphi - \varphi_0)] =$$

$$= z_0 \cos\theta + x_0 \sin\theta \cos\varphi + y_0 \sin\theta \sin\varphi$$

we obtain, after termwise integration of the series, an expansion of the function $u(P_0)$ in a series of powers of the independent variables x_0, y_0, and z_0.

12. Analytic continuation of harmonic functions. Theorem. (the symmetry principle). *Suppose that a function of three variables $u(x, y, z)$ is harmonic in a domain G, that it is continuous in the interior of G, and that it vanishes on a plane or spherical portion S_0 of the boundary G. Then, the function u can be analytically continued* beyond S_0. For this, it is sufficient to assign to the function at points symmetric about S_0 values equal in magnitude but opposite in sign.*

Proof: Without loss of generality, we may assume that S_0 is a plane portion of the boundary S of the region G. (If this is not the case, we need only use the inversion transformation (48′) to map one of the points of a spherical surface S_0 into the point at infinity and consider the function $v(\xi, \eta, \zeta) = ru(x, y, z)$ instead of the function $u(x, y, z)$ [see Sect. 3, subsection 1].) Therefore, let us assume that S_0 lies in the xy-plane and that G is contained in the upper half-space $z>0$. Then, about every point $P_0 \in S_0$, we can construct a sufficiently small sphere S_ε with center at the point P_0 and write Poisson's integral for this sphere in terms of a boundary function which is an odd function of z and that coincides with $u(x, y, z)$ on that portion of the sphere S_ε contained in G. In this case, Poisson's integral, as can be seen from the expression (37) for it, will be equal to zero when $z=0$. Consequently, for $z>0$, it coincides with the function $u(x, y, z)$. On the other hand, it is a harmonic function throughout the interior of the sphere S_ε. Consequently, it is an analytic continuation of the function $u(x, y, z)$ in a neighborhood of an arbitrary point P_0 on the surface S_0. This completes the proof of the theorem.

Obviously, this theorem and its proof remain unchanged for harmonic functions of two variables except that instead of a plane portion of the boundary of a three-dimensional region, we consider a rectilinear portion of the boundary of a plane region and, in the proof, we use Poisson's integral for a circle instead of a sphere.

For the case of harmonic functions of two variables, this theorem is easily generalized.

Theorem. *Suppose a function of two variables $u(x, y)$ is harmonic in a domain D. Suppose the value of u on an analytic arc** γ which constitutes a portion of the boundary D are expressible as an analytic function of the arc length. Then, this function can be analytically continued beyond the arc γ.*

Proof: Since harmonicity is preserved under conformal transformation of independent variables and since (by virtue of the fact that $dx(s)/ds + i\,dy(s)/ds \neq 0$) a neighborhood of the arc γ can always be mapped conformally and one-to-one into a neighborhood of a segment of the real axis in the complex plane $s+i\tau$, we can, without loss of generality, assume that γ is a segment of the real axis. Now, since the boundary values of the function $u(x, y)$ constitute an analytic function $u(x, 0) = f(x)$, we have $f(x+iy) = u_1(x, y) + iv_1(x, y)$, where $u_1(x, y)$ and $v_1(x, y)$ are harmonic conjugate functions and where $u_1(x, 0) = u(x, 0)$.

The function $U(x, y) = u(x, y) - u_1(x, y)$ satisfies the conditions of the preceding theorem. Consequently, it can be analytically continued beyond the arc γ. Since the function $u_1(x, y)$ possesses this same property, the function $u(x, y)$ also possesses it. This completes the proof of the theorem.

---

*This means that it is possible to construct a function that is harmonic in a domain containing the open portion of S_0 and which coincides with the given function $u(x, y, z)$ in the intersection of the new domain and the domain G.

**A curve $x = x(s)$, $y = y(s)$, where s is the arclength of a curve, is called an *analytic arc if* $x(s)$ and $y(s)$ are analytic functions of s and $dx(s)/ds + i[dy(s)/ds]$ does not vanish on the curve.

13. Harmonic conjugate functions of two variables and reduction of the second boundary-value problem to the first. In the case of two independent variables, it is convenient to consider, together with a given harmonic function $u(x, y)$, the harmonic function $v(x, y)$ related to it by the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{58}$$

The functions u and v are called harmonic *conjugates*. The combination of harmonic conjugates $f(z) = u + iv$ is an analytic function of the complex variable $z = x + iy$. If n, s is a rectangular coordinate system obtained from the xy-system by a rotation and translation of the coordinate origin, conditions (58) can be written in the form

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}, \quad \frac{\partial u}{\partial s} = -\frac{\partial v}{\partial n}. \tag{58'}$$

It follows from these equations that if we know the normal derivative $\partial v/\partial n$ on a closed contour C, we also know the tangential derivative of the function u.

This property of harmonic conjugate functions of two variables enables us to reduce the second boundary-value problem for a region D with the boundary conditions

$$\left.\frac{\partial v}{\partial n}\right|_C = \Phi(z) \tag{59}$$

to the first boundary-value problem for the same region

$$u|_C = \Phi^*(z), \quad \Phi^*(z) = \int_0^s \Phi(z)\, ds, \tag{59'}$$

where u is the harmonic conjugate of v. It should be noted that the function $\Phi^*(z)$ will be single-valued on the contour C; i.e., it will not acquire a nonzero increment on going around the contour C. This follows from the condition which is necessary for the existence of a solution to the second boundary-value problem

$$\int_C \Phi(z)\, ds = 0. \tag{51'}$$

Let us consider, for example, the case in which the region D is an open disk $|z| < R$. Here, the function $u(z)$, which satisfies the boundary conditions (59'), can be written in the form of Poisson's integral

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^*(z)\, \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)}\, d\theta. \tag{60}$$

The corresponding analytic function whose real part is the function u can be written in the form

$$f(z_0) = u(z_0) + iv(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \Phi^*(z)\,\frac{Re^{i\theta}+z_0}{Re^{i\theta}-z_0}\,d\theta, \tag{61}$$

since

$$\operatorname{Re}\frac{Re^{i\theta}+z_0}{Re^{i\theta}-z_0} = \frac{R^2-r_0^2}{R^2+r_0^2-2Rr_0\cos(\theta-\theta_0)}.$$

Consequently, the solution of the second boundary-value problem for the disk $|z|<R$ with boundary conditions (59) should be sought in the form of the imaginary part $v(z_0)$ of the function of the complex variable (61).

We have

$$f(z_0) = \frac{1}{2\pi i}\int_C \Phi^*(z)\,\frac{z+z_0}{z-z_0}\,\frac{dz}{z} = \frac{1}{2\pi i}\int_C \frac{2\Phi^*(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\int_C \Phi^*(z)\,\frac{dz}{z}.$$

A simple calculation yields

$$f(z_0) = \frac{1}{2\pi i}\left[\Phi^*(z)\ln(z-z_0)\right]_C - \frac{1}{\pi i}\int_C \frac{d\Phi^*(z)}{ds}\,e^{-i\left(\theta+\frac{\pi}{2}\right)}\ln(z-z_0)\,dz - \frac{1}{2\pi i}\int_C \Phi^*(z)\,\frac{dz}{z} =$$

$$= -\frac{1}{\pi i}\int_C \Phi(z)\ln(z-z_0)\,ds - \frac{1}{2\pi}\int_C \Phi^*(z)\,ds.$$

Taking only the imaginary part of this equation, we obtain the so-called formula of Dini

$$v(z_0) = -\frac{1}{\pi}\int_C \Phi(z)\ln\frac{1}{r}\,ds, \tag{62}$$

where $r=|z-z_0|$ and $|z|=R$. Formula (62) gives the solution of the second boundary-value problem in the potential theory for the disk $|z|<R$ in the case when the normal derivative of its right-hand member has limiting values on the circle $|z|=R$. The existence of these limiting values of the normal derivative will follow from

a study of potential theory, which we shall take up in the following section.

The methods that we have just presented can be used to obtain explicit formulas for the solution of the second boundary-value problem for a half-plane and for a number of other plane regions for which a solution of the first boundary-value problem is known.

4. Potential Theory. The Method of Integral Equations. Solution of the Basic Boundary-Value Problems for Various Regions

1. Properties of potentials at points outside the region of integration. The integrals

$$U(P_0) = \iint_S \sigma(P)\,\delta(P_0, P)\,dS, \tag{14'}$$

$$V(P_0) = \iint_S \vartheta(P)\,\mu(P_0, P)\,dS, \tag{14''}$$

$$I(P_0) = \iiint_G \rho(P)\,\delta(P_0, P)\,d\tau. \tag{14'''}$$

are called, respectively, the *potentials* of a *single layer*, of a *double layer*, and of a *volume distribution*.

In this section, unlike Sect. 2, S denotes an arbitrary smooth surface or set of such surfaces, G denotes a bounded three-dimensional domain or the union of several such domains, $\sigma(P)$ and $\vartheta(P)$ denote piecewise-continuous integrable functions defined on S, and $\rho(P)$ denotes a given piecewise-continuous integrable function defined in G. The other notations are used as in Sect. 2.

Analogously, *in the plane case, the integrals*

$$U(z_0) = \int_C \sigma(z)\,\delta(z_0, z)\,ds, \tag{15'}$$

$$V(z_0) = \int_C \vartheta(z)\,\mu(z_0, z)\,ds, \tag{15''}$$

$$I(z_0) = \iint_D \rho(z)\,\delta(z_0, z)\,d\sigma. \tag{15'''}$$

are called, respectively, the *logarithmic potentials* of a single layer, of a double layer, and of a *distribution in the plane*.

Again in contrast with Sect. 2, C here denotes an arbitrary smooth curve or the union of several such curves, D denotes a bounded plane domain or the union of several such domains, $\sigma(z)$ and $\vartheta(z)$ denote given piecewise-continuous integrable functions defined on C, and $\rho(z)$ denotes a given piecewise-continuous

integrable function defined in D. The other notations are used as in Sect. 2.

Each of the functions $\sigma(P)$, $\vartheta(P)$, $\rho(P)$, $\sigma(z)$, $\vartheta(z)$ and $\rho(z)$ is called the *density* of the corresponding potential.

If a point $P_0(x_0, y_0, z_0)$ lies entirely outside the region of integration of the potentials (14′) and (14″), that is, outside S, or if it lies outside the region of integration G of the potential (14‴), then, obviously, these potentials can be differentiated with respect to the variables x_0, y_0, and z_0 under the integral sign infinitely many times. Since functions of a unit source and of a unit dipole satisfy Laplace's equation, we conclude that the potentials of a simple layer, a double layer, or a volume outside the corresponding region of integration are harmonic functions of three variables.

It follows immediately from Eqs. (14′), (14″), and (14‴) that, as we let the point P_0 approach infinity, the following estimates hold:

$$U(P_0) - M\frac{1}{4\pi r_0} = O\left(\frac{1}{r_0^2}\right), \qquad M = \iint_S \sigma(P)\, dS,$$

$$I(P_0) - M\frac{1}{4\pi r_0} = O\left(\frac{1}{r_0^2}\right), \qquad M = \iiint_G \rho(P)\, d\tau,$$

$$V(P_0) = O\left(\frac{1}{r_0^2}\right), \qquad r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}.$$

On the basis of the theorem on removable singularities, we conclude that *at infinity the potentials of a single layer, a double layer, and a volume distribution are harmonic functions of three variables and that the potential of a double layer has a zero of at least second order.*

Similarly, in the plane case, since the functions $\delta(z_0, z)$ and $\mu(z_0, z)$ are harmonic functions of two variables for $z_0 \neq z$, we conclude that each of the logarithmic potentials of a single layer, a double layer, and a plane distribution are harmonic functions of two variables outside the corresponding region of integration. From Eqs. (15′), (15″), and (15‴), it follows that, when we let the point z_0 approach infinity, we obtain

$$U(z_0) - M\frac{1}{2\pi}\ln\frac{1}{r_0} \to 0, \qquad M = \int_C \sigma(z)\, ds.$$

$$I(z_0) - M\frac{1}{2\pi}\ln\frac{1}{r_0} \to 0, \qquad M = \iint_D \rho(z)\, d\sigma.$$

$$V(z_0) = O\left(\frac{1}{r_0}\right), \qquad r_0 = \sqrt{x_0^2 + y_0^2}.$$

From this, we draw the following conclusion:

The logarithmic potential of a double layer is a harmonic function of two variables at the point of infinity and possesses there a zero of at least first order. The logarithmic potential of a single layer and the logarithmic potential of a plane distribution have singularities at infinity such that the functions

$$U(z_0) - M\frac{1}{2\pi}\ln\frac{1}{r_0} \quad \left(M = \int_{\mathcal{L}} \sigma(z)\,ds\right)$$

and

$$I(z_0) - M\frac{1}{2\pi}\ln\frac{1}{r_0} \quad \left(M = \iint_{D} \rho(z)\,d\sigma\right)$$

are harmonic functions at infinity.

2. A criterion for uniform convergence of integrals and a theorem on the differentiation of uniformly convergent integrals. Suppose that $Q(x_1, \ldots, x_n)$ and $Q_0(x_1^0, \ldots, x_n^0)$ are points in n-dimensional space and that $D(Q_0, Q)$ is a function of the point Q in a domain Ω that it depends on the point Q_0 as a parameter. Then, the integral

$$I(Q_0) = \int_{\Omega} D(Q_0, Q)\,d\omega \tag{63}$$

is said to *converge uniformly* at a point Q_0' of this n-dimensional space if it can be represented in the form

$$I(Q_0) = \int_{\Omega - e} D(Q_0, Q)\,d\omega + \int_{e} D(Q_0, Q)\,d\omega, \tag{63'}$$

where e is a sufficiently small subdomain of Ω such that the second integral in the right-hand member of Eq. (63') is, in absolute value, less than any preassigned positive number ε for all points Q_0 in some neighborhood of the point Q_0', while the first of these integrals is a continuous function of Q_0 at the point Q_0'.

Let us agree to call the second integral in the right-hand member of Eq. (63') the *singular integral.*

It follows immediately from the definition of a uniformly convergent integral that it is a continuous function at points at which it converges uniformly. Let us give a criterion for uniform convergence of an integral.

Theorem. The integral (63) converges uniformly at every point Q_0' of a closed region $\Omega \cup T$ (where T is the boundary of the domain Ω) if the function $D(Q_0, Q)$, as a function of both points Q and Q_0, is continuous in the closed region whenever $Q \neq Q_0$ and satisfies the inequality

$$|D(Q_0, Q)| < \frac{C}{|Q_0 - Q|^{n-\alpha}}, \quad C, \ \alpha = \text{const}, \ \alpha > 0, \tag{63''}$$

where $|Q_0 - Q|$ *is the distance between the points* Q *and* Q_0.

Proof: Let e denote the neighborhood of Q_0' defined by the inequality $|Q - Q_0'| < \eta$. For the singular integral, under the condition that the point Q belongs to e, we have

$$\left| \int_e D(Q_0, Q)\, d\omega \right| < \int_e \frac{C\, d\omega}{|Q_0 - Q|^{n-\alpha}} \leqslant \int_{|Q_0 - Q| < 2\eta} \frac{C\, d\omega}{|Q_0 - Q|^{n-\alpha}}.$$

It is easy to see that, for sufficiently small η, the integral on the right can be made arbitrarily small since it is the singular integral of a convergent integral. Since $D(Q_0, Q)$ is a continuous function of the point Q_0 when $|Q_0 - Q| > \eta/2$ and since this continuity is uniform with respect to Q, we conclude that the integral (63) converges uniformly at the point Q_0'.

We shall use this test for uniform convergence of an integral later on.

Theorem (on the differentiation of uniformly convergent integrals). *Suppose the integral*

$$j(Q_0) = \int_\Omega D(Q_0, Q)\, d\omega \tag{64}$$

has continuous derivatives in the domain Ω. *Then, the integral*

$$I(Q_0) = \int_\Omega \rho(Q)\, D(Q_0, Q)\, d\omega \tag{64'}$$

has continuous derivatives in Ω *if the following conditions are satisfied:*

(a) *The function* $\rho(Q)$ *is Hölder-continuous* and bounded in the domain* Ω.

(b) *The function* $D(Q_0, Q)$ *and its derivatives* $D_{x_i^0}(Q_0, Q)$ *(for* $i = 1, 2, \ldots, n$*), as functions of both points* Q *and* Q_0, *are continuous and satisfy the inequalities*

$$|D(Q_0, Q)| < \frac{C}{|Q - Q_0|^{n-1}}, \quad |D_{x_i^0}(Q_0, Q)| < \frac{C}{|Q - Q_0|^n}, \ C = \text{const}. \tag{64''}$$

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*A function $\rho(Q)$ is said to be *Hölder-continuous* in a domain Ω if, for an arbitrary closed subdomain of Ω and for arbitrary points Q and Q' in that subdomain, there exist positive numbers K and α, with $\alpha \leqslant 1$, such that $|\rho(Q) - \rho(Q')| < K|Q - Q'|^\alpha$.

in the closed region $\Omega \cup T$ *when* $Q \neq Q_0$. *These derivatives of the integral* $I(Q_0)$ *are given by the equations**

$$I_{x_i^0}(Q_0) = \rho(Q_0)\, j_{x_i^0}(Q_0) + \int_\Omega [\rho(Q) - \rho(Q_0)]\, D_{x_i^0}(Q_0, Q)\, d\omega \qquad (64'')$$

$$(i = 1, 2, \ldots, n).$$

Proof: On the basis of our test for uniform convergence of an integral, the right-hand members of Eqs. (64‴) are continuous functions of the point Q_0 in the domain Ω. It remains to show that the right-hand members of Eqs. (64‴) are equal to the left-hand members. Since Q_0' is a point obtained from the point $Q_0(x_1^0, \ldots, x_n^0)$ by increasing one of the coordinates x_i^0 by an amount h. We have

$$\frac{I(Q_0') - I(Q_0)}{h} = \int_\Omega \rho(Q)\, \frac{D(Q_0'Q) - D(Q_0, Q)}{h}\, d\omega.$$

If in the right-hand member of this equation we add and subtract the expression

$$\rho(Q)\, j_{x_i^0}(Q_0),\ \rho(Q_0) \int_\Omega \frac{D(Q_0'Q) - D(Q_0, Q)}{h}\, d\omega,$$

we obtain

$$\frac{I(Q_0', Q) - I(Q_0, Q)}{h} = \rho(Q_0)\, j_{x_i^0}(Q_0) +$$

$$+ \int_\Omega [\rho(Q) - \rho(Q_0)]\, \frac{D(Q_0', Q) - D(Q_0, Q)}{h}\, d\omega + \rho(Q) \left\{ - j_{x_i^0}(Q_0) + \right.$$

$$\left. + \int_\Omega \frac{D(Q_0', Q) - D(Q_0, Q)}{h}\, d\omega \right\}.$$

The last term in this equation approaches zero as $h \to 0$. Therefore, we need only show that the quantity

$$\varphi_h(Q_0) = \int_\Omega [\rho(Q) - \rho(Q_0)]\, \frac{D(Q_0', Q) - D(Q_0, Q)}{h}\, d\omega$$

differs by an arbitrarily small amount from

$$\varphi_0(Q) = \int_\Omega [\rho(Q) - \rho(Q_0)]\, D_{x_i^0}(Q_0, Q)\, d\omega$$

for sufficiently small h. We have

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*An analogous theorem was proven and extensively used in the work by E. Hopf, "Über den funktionalen, insbesondere der analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung," *Math. Zeitschr., 34,* 1931, 194-233.

$$\varphi_h(Q_0) - \varphi_0(Q_0) = \int_{\Omega - e} [\rho(Q) - \rho(Q_0)] \left[\frac{D(Q_0', Q) - D(Q_0, Q)}{h} - \right.$$

$$\left. - D_{x_i^0}(Q_0, Q) \right] d\omega + \int_e [\rho(Q) - \rho(Q_0)] \frac{D(Q_0', Q) - D(Q_0, Q)}{h} d\omega -$$

$$- \int_e [\rho(Q) - \rho(Q_0)] D_{x_i^0}(Q_0, Q) d\omega,$$

where e is the neighborhood of the point Q_0 defined by the inequality $|Q - Q_0| < \eta$, where η in turn is a sufficiently small number. On the basis of our test for uniform convergence of an integral, the third integral on the right side of the last equation is arbitrarily small for η sufficiently small and the first integral will be arbitrarily small for sufficiently small h. Therefore, we need only show that the integral

$$E = \int_e [\rho(Q) - \rho(Q_0)] \frac{D(Q_0', Q) - D(Q_0, Q)}{h} d\omega$$

can be made arbitrarily small if η is sufficiently small. If we set

$$D(Q_0, Q) = \frac{\lambda(Q_0, Q)}{|Q - Q_0|^n},$$

we see that the function $\lambda(Q_0, Q)$ is continuous with respect to both Q and Q_0 in the closed region $\Omega \cup T$ and will have continuous bounded derivatives $\lambda_{x_i^0}(Q_0, Q)$ in that closed region when $Q_0 \neq Q$. Therefore, if we use the mean-value theorem and if we set $|Q - Q_0| = r$ and $|Q - Q_0| = r_1$, we have

$$\left| \frac{D(Q_0', Q) - D(Q_0, Q)}{h} \right| = \frac{1}{|h|} \left| \frac{\lambda(Q_0', Q)}{r_1^n} - \frac{\lambda(Q_0, Q)}{r^n} \right| \leqslant$$

$$\leqslant \frac{1}{|h|} \left| \lambda(Q_0', Q) \left(\frac{1}{r_1^n} - \frac{1}{r^n} \right) \right| + \frac{1}{r^n} \left| \frac{\lambda(Q_0', Q) - \lambda(Q_0, Q)}{h} \right| =$$

$$= \frac{1}{|h|} \left| \frac{\lambda(Q_0', Q)}{r_1^n} \right| \frac{|r^n - r_1^n|}{r^n} + \frac{1}{r^n} \left| \frac{\lambda(Q_0', Q) - \lambda(Q_0, Q)}{h} \right| \leqslant$$

$$\leqslant M \left(\frac{r^{n-1} + r^{n-2} r_1 + \ldots + r_1^{n-1}}{r_1^{n-1} r^n} + \frac{1}{r^n} \right), \qquad M = \text{const.}$$

Therefore, recalling that the function $\rho(Q)$ is Hölder-continuous, we obtain

$$|E| \leqslant MK \int_e r^\alpha \frac{r^{n-1} + r^{n-2} r_1 + \ldots + r_1^{n-1}}{r_1^{n-1} r^n} \, d\omega + MK \int_e \frac{1}{r^{n-\alpha}} d\omega, \quad (65')$$

$K = \text{const.}$

The second integral on the right will, by virtue of the test for uniform convergence of an integral, be arbitrarily small for sufficiently small η. To get a bound for the first integral, we partition the region of integration e into two subregions e' and e'' such that $r < r_1$ in e' and $r_1 < r$ in e''. Then, this integral will be majorized by the sum of the integrals

$$MKn \int_{e'} \frac{d\omega}{r^{n-\alpha}} + MKn \int_{e''} \frac{d\omega}{r_1^{n-\alpha}},$$

each of which is arbitrarily small for sufficiently small η, which completes the proof.

This theorem is a convenient tool for investigating potentials. Note that the restrictions imposed on their densities are comparatively weak.

3. The first and second derivatives of a potential due to a volume distribution and of a logarithmic potential due to a plane distribution. Let us establish the basic properties of the potential of a volume distribution and of the logarithmic potential of a plane distribution at points within their regions of integration.

Theorem 1 (on the derivatives of the potential of a volume distribution and of the logarithmic potential of a plane distribution). *If a function $\rho(P)$ is piecewise continuous and bounded in a region G,* then the volume potential (14‴) has continuous derivatives everywhere in the space. Analogously, if a function $\rho(z)$ is piecewise continuous and bounded in a region D, then the logarithmic potential (15‴) has continuous derivatives everywhere in the plane.***

It is sufficient to consider the derivatives with respect to the variable x_0 of the integrals $I(P_0)$ and $I(z_0)$:

$$\frac{\partial I(P_0)}{\partial x_0} = \iiint_G \rho(P) \frac{\partial}{\partial x_0} \frac{1}{4\pi r} \, d\tau, \quad (66)$$

*In the present subsection, the letters G and D denote domains of the same kind as in subsection 1.

**Here, as everywhere, when we speak of all space or of the entire plane, we mean all points of the space or, respectively, all points of the plane except the point at infinity.

$$\frac{\partial I(z_0)}{\partial x_0} = \iint_D \rho(z) \frac{\partial}{\partial x_0}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right) d\sigma. \qquad (66')$$

By virtue of the test for the uniform convergence of integrals, the right-hand members of these equations are continuous functions of the point P_0 throughout all space and of the point z_0 throughout the entire plane, respectively. It remains to show that the right-hand members of these equations are equal to their left-hand members. In the three-dimensional case, if we set

$$r_1 = \sqrt{(x - x_0 - h)^2 + (y - y_0)^2 + (z - z_0)^2},$$

we have

$$\left|\iiint_G \rho(P)\,\frac{\frac{\partial}{4\pi r_1} - \frac{1}{4\pi r}}{h}\,d\tau - \iiint_G \rho(P)\,\frac{\partial}{\partial x_0}\,\frac{1}{4\pi r}\,d\tau\right| \leqslant$$

$$\leqslant \left|\iiint_e \rho(P)\,\frac{\partial}{\partial x_0}\,\frac{1}{4\pi r}\,d\tau\right| + \left|\iiint_{G-e} \rho(P)\left[\frac{\frac{1}{4\pi r_1} - \frac{1}{4\pi r}}{h} - \frac{\partial}{\partial x_0}\,\frac{1}{4\pi r}\right] d\tau\right| +$$

$$+ \left|\iiint_e \rho(P)\,\frac{\frac{1}{4\pi r_1} - \frac{1}{4\pi r}}{h}\,d\tau\right|, \qquad (66'')$$

where e is a sphere of sufficiently small radius η with center at the point P_0 (see Fig. 9). The first two integrals in the right-hand member of the last inequality will be arbitrarily small for sufficiently small η and h. If we partition the region of integration e into two subregions e' and e'' in such a way that $r<r_1$ in e' and $r_1<r$ in e'', we obtain, for the last integral in equality (66″):

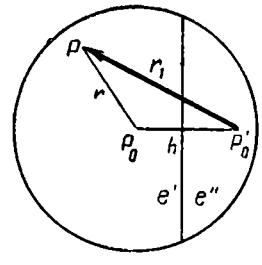

Fig. 9.

$$\left|\iiint_e \rho(P)\,\frac{\frac{1}{4\pi r_1} - \frac{1}{4\pi r}}{h}\,d\tau\right| \leqslant \frac{1}{4\pi}\iiint_{e'} |\rho(P)|\,\frac{1}{r^2}\,d\tau +$$

$$+ \frac{1}{4\pi}\iiint_{e''} |\rho(P)|\,\frac{1}{r_1^2}\,d\tau.$$

If we now replace integration over e' with integration over the sphere $|P - P_0| < 2\eta$ and integration over e'' with integration over the sphere $|P_0' - P| < 2\eta$ (see Fig. 9) and then shift to spherical coordinates, we see that the integral of the left-hand member of the

last inequality will be arbitrarily small for sufficiently small η, which proves Eq. (66).

Analogously, the proof of Eq. (66′) is reduced to proving that the integral

$$A = \iint_e \rho(z) \frac{\ln r_1 - \ln r}{2\pi h} d\sigma, \quad r = |z - z_0|, \quad r_1 = |z - z_0'|,$$

where e is the circle of radius η with center at the point z_0, will be arbitrarily small in absolute value if η is sufficiently small. If we partition e into two subregions e' and e'' such that $r < r_1$ in e' and $r_1 < r$ in e'' and use the inequality $\ln(1+\delta) < \delta$ for $\delta > 0$, we obtain

$$0 < \ln \frac{r_1}{r} < -1 + \frac{r_1}{r}$$

in e' and

$$0 < \ln \frac{r}{r_1} < -1 + \frac{r}{r_1}$$

in e''. From this, we have

$$|A| < \iint_{e'} |\rho(z)| \frac{1}{2\pi r} d\sigma + \iint_{e''} |\rho(z)| \frac{1}{2\pi r_1} d\sigma$$

If we shift to polar coordinates with origin at the point z_0 in the first case or with origin at the point z_0' in the second case, we immediately see that the integral in which we are interested will be arbitrarily small in absolute value for η sufficiently small. This completes the proof of Theorem 1.

Theorem 2 (on the second derivatives of potential of a volume distribution and of the logarithmic potential of a plane distribution). *If a function $\rho(P)$ is Hölder-continuous and bounded in a domain G, then the volume potential (14‴) has continuous second derivatives at every point of G and satisfies the three-dimensional Poisson equation $\Delta I(P_0) = -\rho(P_0)$ in that domain. Analogously, if a function $\rho(z)$ is Hölder-continuous and bounded in a D, the logarithmic potential (15‴) has continuous second derivatives at every point of D and satisfies the two-dimensional Poisson equation $\Delta I(z_0) = -\rho(z_0)$.*

Proof: Let P_0' be a fixed point belonging to the domain G. Consider the integral

$$I^*(P_0) = \iiint_e \rho(P) \frac{1}{4\pi r} d\tau,$$

where e is a sphere of sufficiently small radius with center at the point P_0'. The difference $I(P_0) - I^*(P_0)$ is a harmonic function at the point P_0'. Therefore, to prove the assertions of the theorem at the point P_0' it will be sufficient to prove them for the integral $I^*(P_0)$.

If we apply Ostrogradskiy's formula to the derivatives of the integral

$$j^*(P_0) = \iiint_e \frac{1}{4\pi r}\, d\tau,$$

we obtain

$$\frac{\partial j^*(P_0)}{\partial x_0} = \iiint_e \frac{\partial}{\partial x_0}\frac{1}{4\pi r}\, d\tau = -\iiint_e \frac{\partial}{\partial x}\frac{1}{4\pi r}\, d\tau = \iint_\gamma \frac{1}{4\pi r}\cos nx\, dS,$$
$$\frac{\partial j^*(P_0)}{\partial y_0} = \iint_\gamma \frac{1}{4\pi r}\cos ny\, dS, \quad \frac{\partial j^*(P_0)}{\partial z_0} = \iint_\gamma \frac{1}{4\pi r}\cos nz\, dS, \tag{67}$$

where γ is the surface of a sphere e.* It follows from Eqs. (67) that the integrals

$$\frac{\partial j^*(P_0)}{\partial x_0}, \quad \frac{\partial j^*(P_0)}{\partial y_0}, \quad \frac{\partial j^*(P_0)}{\partial z_0},$$

and the integrals

$$\frac{\partial I^*(P_0)}{\partial x_0}, \quad \frac{\partial I^*(P_0)}{\partial y_0}, \quad \frac{\partial I^*(P_0)}{\partial z_0},$$

satisfy the conditions of the theorem on differentiation of uniformly convergent integrals. Therefore, the integral $I^*(P_0)$ has continuous second derivatives in the interior of the sphere e. If we use formulas (64‴), we obtain

$$\Delta I^*(P_0)|_{P_0'} = \rho(P_0)\,\Delta j^*(P_0)|_{P_0'} + \iiint_e [\rho(P) - \rho(P_0)] \times$$
$$\times \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{4\pi r}\, d\tau.$$

From Eqs. (67), we obtain by differentiating under the integral sign

$$\Delta j^*(P_0') = -\iint_\gamma \frac{1}{4\pi r^2}(\cos^2 nx + \cos^2 ny + \cos^2 nz)\, dS = -1.$$

Consequently, $\Delta I^*(P_0') = -\rho(P_0')$. This completes the proof for the three-dimensional case. The proof for the plane case is the same except that instead of Ostrogradskiy's formula we use Green's formula.

Corollary. The interior three- and two-dimensional boundary-value problems for Poisson's equation reduce respectively to the corresponding boundary-value problems for Laplace's equation if the right-hand member of Poisson's equation is Hölder-continuous and bounded in the domain in question.

*Here, application of Ostrogradskiy's formula is permissible since the integrand is of order $O(1/r)$ as the point P_0 approaches the point P_0'.

Proof: In the case of the three-dimensional Poisson equation $\Delta u(P_0) = -\rho(P_0)$, if we set

$$u(P_0) = v(P_0) + \iiint_G \rho(P) \frac{1}{4\pi r} d\tau,$$

we see on the basis of Theorem 2 that the function $v(P_0)$ is harmonic in the region in question. On the basis of Theorem 1, the continuous boundary conditions (1′), (1″), and (1‴) for the function u are transformed into continuous boundary conditions of the same respective types for the function v. In the case of the two-dimensional Poisson equation $\Delta u(z_0) = -\rho(z_0)$, the reasoning is the same except that we shall introduce the function $v(z_0)$ defined by the equation

$$u(z_0) = v(z_0) + \iint_D \rho(z) \frac{1}{2\pi} \ln \frac{1}{r} d\sigma.$$

We note that if the requirement of Hölder continuity of the functions $\rho(P)$ and $\rho(z)$ is replaced with simple continuity in the regions G and D, the assertions of Theorem 2 do not remain valid.*

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*This can be seen for an example given by N. M. Gunther, *La théorie du potential et ses applications aux problèmes fondamentaux de la physique mathématique*, Paris, Gauthiers-Villars, 1934. In spherical coordinates ρ, θ, φ, the potential of a sphere $\rho < R < 1$ with density $(3z^2/\rho^2 - 1)/\ln \rho$ at a point $P_0(a, 0, 0)$ can be written in the form

$$I(a) = \frac{1}{2} \int_0^R \frac{\rho^2}{\ln \rho} \left\{ \int_0^\pi \frac{\sin\theta (3\cos^2\theta - 1)\, d\theta}{\sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}} \right\} d\rho.$$

If we set $t = \sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}$, we obtain

$$\int_0^\pi \frac{\sin\theta (3\cos^2\theta - 1)\, d\theta}{\sqrt{\rho^2 + a^2 - 2a\rho\cos\theta}} = \frac{1}{a\rho} \int_{|a-\rho|}^{a+\rho} \left[3\left(\frac{\rho^2 + a^2 - t^2}{2a\rho} \right)^2 - 1 \right] dt =$$

$$= \begin{cases} \frac{4}{5} \frac{\rho^2}{a^3}, & \text{if } \rho < a \\ \frac{4}{5} \frac{a^2}{\rho^3}, & \text{if } \rho > a. \end{cases}$$

Therefore,

$$I(a) = \frac{2}{5} \left[\frac{1}{a^3} \int_0^\pi \frac{\rho^4}{\ln \rho} d\rho + a^2 \int_a^R \frac{d\rho}{\rho \ln \rho} \right],$$

$$I'(a) = \frac{2}{5} \left[-\frac{3}{a^4} \int_0^a \frac{\rho^4\, d\rho}{\ln \rho} + 2a \int_a^R \frac{d\rho}{\rho \ln \rho} \right], \qquad I'(0) = 0.$$

4. The continuity of the potential and of the logarithmic potential of a single layer. In the present subsection, unlike subsection 1, we shall let S denote a smooth surface possessing the following property: There exists a sphere of fixed radius such that, if we were to place the center of the sphere at any point of the surface S, the sphere would contain a portion of the surface S that touches every straight line parallel to the normal to the surface S at the center of the sphere no more than one point. Similarly, the letter C will be used to denote a smooth curve such that a circle of fixed radius with center at any point of the curve will contain a portion of the curve that will touch every straight line parallel to the normal of the curve at the center of the circle at no more than one point.

Theorem 3 (on the continuity of the potential and of the logarithmic potential of a single layer). *If a function $\sigma(P)$ is piecewise continuous and bounded on a surface S, the potential of a single layer (14‴) is a continuous function everywhere in space.*

Analogously, if a function $\sigma(z)$ is piecewise continuous and bounded on a curve C, the logarithmic potential of a single layer (15′) is a continuous function throughout the entire plane.

Proof: The continuity of the potential of a single layer (14′) at points not on S is obvious. Let P_0 denote a point on the surface S. We choose a coordinate system x, y, z so that the z-axis coincides with the direction of the normal to S at the point P_0. Let e denote that portion of the surface S intercepted by the cylinder $x^2+y^2=\eta^2=$ const. For the singular integral of the single layer potential (14′), we have

$$\iint_e \sigma(P)\frac{1}{4\pi r}\,dS \ll \iint_e |\sigma(P)|\frac{1}{4\pi r}\,dS \leqslant M \iint_{\sqrt{x^2+y^2}<\eta} \frac{dx\,dy}{r_1} <$$

$$< M \iint_{r_1<2\eta} \frac{dx\,dy}{r_1}, \qquad r_1 = \sqrt{(x-x_0)^2+(y-y_0)^2}, \quad M = \text{const}.$$

Consequently, the integral (14′) converges uniformly at the point P_0 and is a continuous function of the point P_0 at that point.

If we let a approach 0 in the right-hand side of the equation

$$\frac{I'(a)-I'(0)}{a} = \frac{2}{5}\left[-\frac{3}{a^5}\int_0^a \frac{\rho^4\,d\rho}{\ln\rho} + 2\int_a^R \frac{d\rho}{\rho\ln\rho}\right],$$

the first term approaches 0 and the second term approaches infinity. This means that the potential in question does not have a second derivative with respect to the variable $x_0=a$ at the coordinate origin. Obviously, this is due to the fact that the function $(3z^2/\rho^2)-1/\ln\rho$, although continuous, is not Hölder continuous in a neighborhood of the coordinate origin.

In the plane case, if the y-axis coincides with the normal to the curve at some point z_0' on the curve, then, if we denote by e that portion of the curve C intercepted by the straight lines $|x|=\eta$, the singular integral of the logarithmic potential of a single layer (15′) is given by

$$\int_e \sigma(z)\frac{1}{2\pi}\ln\frac{1}{r}\,ds \ll \int_e \left|\sigma(z)\frac{1}{2\pi}\ln\frac{1}{r}\right| ds < M\int_{|x|<\eta} |\ln r_1|\,dx <$$

$$< M\int_{|x-x_0|<2\eta} |\ln r_1|\,dx, \quad r_1 = |x-x_0|, \quad M = \text{const}.$$

Therefore, the integral (15′) converges uniformly at the point z_0' and is a continuous function of the point z_0. This completes the proof of the theorem.

Remark: Obviously, the assertions of this theorem remain valid if, in the integral (14′), S denotes several surfaces satisfying the conditions stated at the beginning of this subsection. The same is true if, in the integral (15′), C denotes several curves satisfying the conditions stated. This follows from the fact that the sum of continuous functions is a continuous function.

5. Potentials of a double layer and normal derivatives of potentials of a single layer as functions of points of the regions of integration. Lyapunov surfaces and curves. In contrast with the preceding sections, here, the letters S and C will denote, respectively, a so-called Lyapunov surface or a Lyapunov curve.

A surface S is called a *Lyapunov surface* if the following conditions are satisfied:

1. The surface S has a tangent plane at every point.

2. There exists a number $\eta>0$ such that a sphere of radius η with center at an arbitrary point of the surface intercepts a portion of the surface which touches any straight line parallel to the normal to the surface at the center of the sphere at no more than one point.

3. The angle between neighboring normals* $\omega(P_0', P_0)$ satisfies the condition

$$\omega(P_0', P_0) < Ar^{\lambda} \qquad A, \lambda = \text{const}, \quad 0 < \lambda \leqslant 1, \tag{68}$$

where r is the distance between the points P_0' and P_0.

The definition of a Lyapunov curve is the same except that instead of having a sphere and a tangent plane to the surface, we have a circle of fixed radius with center at an arbitrary point of the curve and a tangent line.

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*By *angle between neighboring normals* we mean the angle between the normals at two neighboring points P'_0 and P_0 of the surface or curve.

To compare with the surfaces and curves of the preceding subsection, we note that Lyapunov surfaces and curves must satisfy the additional constraint (68), which is the condition that the angle between neighboring normals, as a function of the distance between points on a closed surface or a closed curve, should be Hölder-continuous.

Theorem 4. If a function $\vartheta(P)$ is piecewise continuous and bounded on a closed Lyapunov surface S, the double-layer potential (14″) is a continuous function of P.

Analogously, if a function $\vartheta(z)$ is piecewise continuous and bounded on a closed Lyapunov curve C, the logarithmic potential of a double layer (15″) is a continuous function of z.

Proof: Let P_0 be a fixed point of the surface S. Set up an xyz-coordinate system such that the z-axis coincides with the normal to S at the point P_0 and the coordinate origin coincides with P_0 (see Fig. 10). Let e denote that portion of the surface cut out by the cylinder $\sqrt{x^2+y^2}=\delta$, where δ is a sufficiently small number. Let P_0 be a point on e and let ξ, η, ζ be a coordinate system such that the ζ-axis coincides with the normal $\vec{n}_0$ to the surface S at the point P_0 and the coordinate origin coincides with that point. The equation of the surface S in a neighborhood of the point P_0 in the $\xi\eta\zeta$-coordinate system can be written in the form

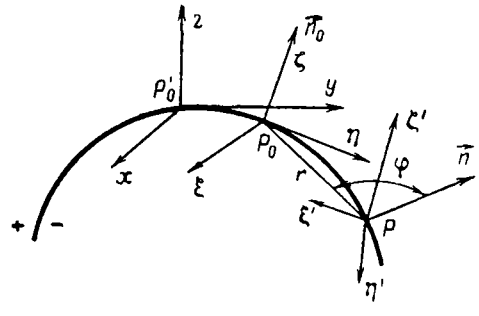

Fig. 10.

$$\zeta = \zeta(\xi, \eta), \quad \zeta(0, 0) = \zeta_\xi(0, 0) = \zeta_\eta(0, 0) = 0.$$

The direction cosines of the normal $\vec{n}$ to the surface S at a point $P \in e$ in the $\xi\eta\zeta$-coordinate system can be written in the form

$$\cos\alpha = \frac{\zeta_\xi}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}}, \quad \cos\beta = \frac{\zeta_\eta}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}},$$

$$\cos\gamma = \frac{-1}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}}.$$

Consequently,

$$\zeta_\xi = -\frac{\cos\alpha}{\cos\gamma}, \quad \zeta_\eta = -\frac{\cos\beta}{\cos\gamma}.$$

Let ξ', η', ζ' be a coordinate system with origin at the point P such that (1) the ξ'-axis coincides with the projection of the vector

$\vec{n}$ onto the $\xi\eta$-plane, (2) the ζ'-axis is parallel to the ζ-axis, and (3) the η'-axis is perpendicular to the ξ'- and ζ'-axes. For the unit vectors $\vec{n}$, $\vec{\xi}$, and $\vec{\eta}$ in the $\xi'\eta'\zeta'$ coordinate system, we have*

$$\vec{n} = \left[\cos\left(\frac{\pi}{2} - \omega\right),\ 0,\ \cos\omega\right],\quad \vec{\xi} = [\cos(\xi, \xi'),\ \cos(\xi, \eta'),\ 0],$$

$$\vec{\eta} = [\cos(\eta, \xi'),\ \cos(\eta, \eta'),\ 0]\ *.$$

If we now take into account the equations $\cos\alpha = (\vec{n}, \vec{\xi})$ and $\cos\beta = (\vec{n}, \vec{\eta})$, we obtain $\cos\alpha = \sin\omega\cos(\xi, \xi')$ and $\cos\beta = \sin\omega\cos(\eta, \xi')$. On the basis of (68), it follows from this that the following inequalities hold at the point P:

$$|\zeta_\xi|,\ |\zeta_\eta| < A'r^\lambda,\quad |\zeta| = |\xi\zeta_\xi + \eta\zeta_\eta| < A'r^{1+\lambda},\quad A' = \text{const}.$$

If we now recall that $\cos\varphi = \left(\frac{\vec{PP_0}}{r}, \vec{n}\right)$ (see Fig. 10), we obtain

$$\cos\varphi = \frac{\xi}{r}\cos\alpha + \frac{\eta}{r}\cos\beta + \frac{\zeta}{r}\cos\gamma \ll |\cos\alpha| + |\cos\beta| + \frac{|\zeta|}{r} \leqslant 3A'r^\lambda. \tag{69}$$

Now, for the singular integral of the double-layer potential (14″) corresponding to the point P_0, we have

$$E = \iint_e \vartheta(P)\frac{\cos\varphi}{4\pi r}\,dS \leqslant M\frac{3A'}{4\pi}\iint_e \frac{dS}{r^{2-\lambda}} \leqslant M' \iint\limits_{\sqrt{x^2+y^2}<\delta} \frac{dx\,dy}{r^{2-\lambda}},$$

$$(M,\ M' = \text{const}).$$

We now introduce the quantity $r_1 = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, which is the length of the projection of the vector r onto the xy-plane. We then obtain

$$E \leqslant M' \iint\limits_{\sqrt{x^2+y^2}<\delta} \frac{1}{r_1^{2-\lambda}}\,dx\,dy \leqslant M' \iint\limits_{r_1<2\delta} \frac{1}{r_1^{2-\lambda}}\,dx\,dy < \varepsilon,$$

where ε is an arbitrarily small positive number. This means that the integral (14″), as a function of the point P_0 on the surface S,

*Here, ω is the angle between the normals at the points P_0 and P.

converges uniformly at the point P_0' and is a continuous function of P_0 at P_0'.

The theorem is proven for the logarithmic potential of a double layer (15″) in the same way except that in this case we consider the singular integral over that portion of the curve cut out by the straight lines $|x| = \delta$.

Theorem 5. If a function $\sigma(P)$ is piecewise continuous and bounded, the normal derivative of the potential of a single layer

$$\frac{\partial U(P_0)}{\partial n_0} = \iint_S \sigma(P) \frac{\partial}{\partial n_0} \frac{1}{4\pi r} dS = \iint_S \sigma(P) \frac{\cos \varphi_0}{4\pi r^2} dS, \tag{70}$$

where $\vec{n}_0$ is the normal to S at the point P_0 and φ_0 is the angle between the vectors $\vec{P_0P}$ and $\vec{n}_0$, is a continuous function of the point P_0 of the closed Lyapunov surface S.

Analogously, if a function $\sigma(z)$ is piecewise continuous and bounded, the normal derivative of the logarithmic potential of a single layer

$$\frac{\partial U(z_0)}{\partial n_0} = \int_C \sigma(z) \frac{\partial}{\partial n_0} \left(\frac{1}{2\pi} \ln \frac{1}{r} \right) ds = \int_C \sigma(z) \frac{\cos \varphi_0}{2\pi r} ds, \tag{70'}$$

where $\vec{n}_0$ is the normal to C at the point z_0 and φ_0 is the angle between the vectors $\vec{z_0z}$ and $\vec{n}_0$, is a continuous function of the point z_0 on the closed Lyapunov curve C.

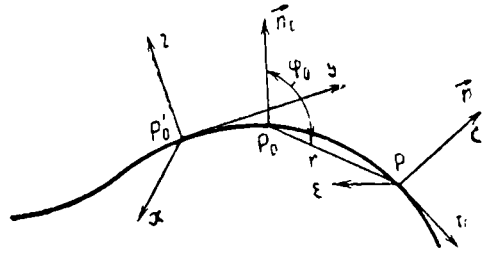

Fig. 11.

Proof: Let P_0' be a fixed point of the surface S. Let x, y, z be a coordinate system such that the z-axis coincides with the normal to S at the point P_0' and the coordinate origin coincides with P_0' (see Fig. 11). Let e denote that portion of the surface S cut out by the cylinder $\sqrt{x^2+y^2} = \delta$, where δ is a sufficiently small number. Let P_0 and P be points on e. Let ξ, η, ζ be a coordinate system such that the ζ-axis coincides with the normal $\vec{n}$ to the surface S at the point P and the coordinate origin coincides with P. The equation

of the surface S in a neighborhood of the point P may be written in the $\xi\,\eta\,\zeta$-coordinate system in the form

$$\zeta = \zeta(\xi, \eta), \quad \zeta(0, 0) = \zeta_\xi(0, 0) = \zeta_\eta(0, 0) = 0.$$

The direction cosines of the normal $\vec{n_0}$ to the surface S at the point P_0 in the $\xi\,\eta\,\zeta$-coordinate system may be written in the form

$$\cos\alpha = \frac{\zeta_\xi}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}}, \quad \cos\beta = \frac{\zeta_\eta}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}},$$

$$\cos\gamma = \frac{-1}{\sqrt{1+\zeta_\xi^2+\zeta_\eta^2}}$$

Consequently, ζ and the partial derivatives ζ_ξ, ζ_η as in the preceding theorem, will satisfy the conditions

$$|\zeta_\xi|, \ |\zeta_\eta| < A'r^\lambda, \ |\zeta| = |\xi\zeta_\xi + \eta\zeta_\eta| < A'r^{1+\lambda}, \ A' = \text{const}.$$

Therefore, on the basis of the equation $\cos\varphi_0 = \left(\frac{\vec{P_0P}}{r}, \vec{n_0}\right)$ (see Fig. 11), we have

$$\cos\varphi_0 = \frac{\xi}{r}\cos\alpha + \frac{\eta}{r}\cos\beta + \frac{\zeta}{r}\cos\gamma \ll 3A'r^\lambda. \qquad (69')$$

Therefore, the singular integral of the normal derivative of the single-layer potential (70) corresponding to the point P_0' is

$$E = \iint\limits_e \sigma(P)\frac{\cos\varphi_0}{4\pi r^2}dS \ll M\frac{3A'}{4\pi}\iint\limits_e \frac{dS}{r^{2-\lambda}} \leqslant M' \iint\limits_{\sqrt{x^2+y^2}<\delta} \frac{1}{r_1^{2-\lambda}}\,dx\,dy \leqslant$$

$$\leqslant M' \iint\limits_{r_1<2\delta} \frac{dx\,dy}{r_1^{2-\lambda}} < \varepsilon \quad M, M' = \text{const},$$

where $r_1 = \sqrt{(x-x_0)^2+(y-y_0)^2}$ and ε is an arbitrarily small positive number. This means that the integral (70) as a function of the point P_0 on S converges uniformly at the point P_0' and, consequently, is a continuous function of P_0 on the closed surface S. The theorem is proved in the same way for the case of the integral (70′) except that in this case we examine the singular integral over that portion of the curve intercepted by the straight lines $|x| = \delta$.

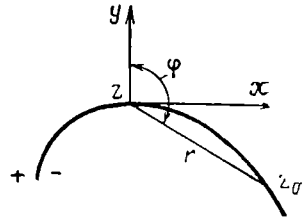

Fig. 12.

Let us note one property of Lyapunov curves which possess continuous curvature. Specifically, on these curves, the ratio $(\cos\varphi)r$ is a continuous function of the two points z and z_0 lying on these curves. To see this, we need only show that this function as a function of z_0 is continuous at $z_0 = z$.

Let x, y be a coordinate system such that the y-axis coincides with the normal $\vec{n}$ to the curve C at the point z and the coordinate origin coincides with z (see Fig. 12). We have

$$\frac{\cos\varphi}{r} = \frac{y}{x^2+y^2} = \frac{1}{1+\frac{y^2}{x^2}}\frac{y}{x^2} \to \frac{1}{2}\frac{d^2y}{dx^2}\Big|_{x=0}.$$

On the other hand, the curvature K of the curve at the point z is

$$K = \frac{d}{ds}\operatorname{arctg}\frac{dy}{dx}\Big|_{x=0} = \frac{1}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}\frac{d^2y}{dx^2}\Big|_{x=0} = \frac{d^2y}{dx^2}\Big|_{x=0}.$$

Consequently,

$$\lim_{z_0\to z}\frac{\cos\varphi}{r} = \frac{1}{2}K. \tag{71}$$

6. The limiting values of the potential and of the logarithmic potential of a double layer. *Theorem 6. If a function $\vartheta(P)$ is continuous on a closed Lyapunov surface S, then the potential of a double layer $V(P_0)$ converges uniformly to its continuous limiting values as P approaches points P_0' of that surface both from within and from without, and these limiting values are given, respectively, by the equations*

$$V^+(P_0') = \frac{\vartheta(P_0')}{2} + V(P_0'), \quad V^-(P_0') = -\frac{\vartheta(P_0')}{2} + V(P_0'),$$
$$V(P_0') = \iint_S \vartheta(P)\frac{\cos\varphi}{4\pi r_0^2}\,dS, \quad r_0 = |P - P_0'|. \tag{72}$$

Analogously, if the function $\vartheta(z)$ is continuous on a closed Lyapunov C, the logarithmic potential of a double layer $V(z_0)$ converges uniformly to its continuous limiting values as z_0 approaches points z_0' of that curve both from within and from without, and these limiting values are given, respectively, by the equations*

$$V^+(z_0') = \frac{\vartheta(z_0')}{2} + V(z_0'), \quad V^-(z_0') = -\frac{\vartheta(z_0')}{2} + V(z_0'),$$
$$V(z_0') = \int_C \vartheta(z)\frac{\cos\varphi}{2\pi r_0}\,ds, \quad r_0 = |z - z_0'|. \tag{72'}$$

*We call attention to the fact that, in view of (72), as the surface S is crossed, $V(P_0)$ undergoes a discontinuity:

$$V^+(P_0') - V^-(P_0') = \vartheta(P_0').$$

The same conclusion can be drawn from (72'). Therefore, Theorem 6 is often called the theorem on the *discontinuities of the potential of a double layer.*

To prove the assertions of the theorem regarding the potential of a double layer $V(P_0)$, we introduce the integral

$$W(P_0) = \int_S \vartheta(P_0') \frac{\cos\varphi}{4\pi r^2} dS = \frac{1}{4\pi} \int_S \int \vartheta(P_0') d\Theta,$$

where P_0' is a fixed point on the surface S.

Since $d\Theta$ is an element of solid angle subtended, we have

$$W(P_0) = \begin{cases} \vartheta(P_0'), & \text{if } P_0 \in G; \\ \frac{1}{2}\vartheta(P_0'), & \text{if } P_0 \in S; \\ 0, & \text{if } P_0 \notin G \cup S. \end{cases}$$

We define

$$\chi(P_0) = V(P_0) - W(P_0) = \int_S \int [\vartheta(P) - \vartheta(P_0')] \frac{\cos\varphi}{4\pi r^2} dS. \quad (73)$$

Suppose that, for every point $P_0' \in S$, there exists a sufficiently small open portion e of the surface S containing the point P_0' such that

$$E(P_0) = \int_e \int \left| \frac{\cos\varphi}{4\pi r^2} \right| dS < M \qquad (r = |P - P_0|),$$

where P_0 is a point lying on the normal to S at the point P_0' (or on its continuation) and M is a constant independent of the points P_0 and P_0'. In such a case, we shall say that the surface S satisfies condition α.

When condition α is satisfied for the singular integral corresponding to the integral (73), we have

$$\int_e \int [\vartheta(P) - \vartheta(P_0')] \frac{\cos\varphi}{4\pi r^2} dS \ll \varepsilon E(P_0) < \varepsilon M,$$

where ε is an arbitrarily small positive number. Therefore, the integral (73) converges uniformly at the point P_0' under the condition that the point P_0 lies on the normal to S at the point P_0' (or on its continuation) and is a continuous function of P_0 at $P_0 = P_0'$. This means that the function $\chi(P_0)$ approaches its limiting value at the point P_0' as the point P_0 approaches the point P_0' along the normal either from the interior or from the exterior side of the surface S

$$\chi(P_0') = V(P_0') - \frac{1}{2}\vartheta(P_0').$$

Because of the uniform continuity of the function $\vartheta(P)$ on the surface S, all the above inequalities are independent of the choice of the point P_0'. Therefore, the limiting values of $\chi(P_0')$ define a continuous function of the point P_0'. On the basis of the general properties of continuous functions, it follows from this that $\chi(P_0)$ is a continuous function of the point P_0 in a three-dimensional region containing the surface S. In particular,

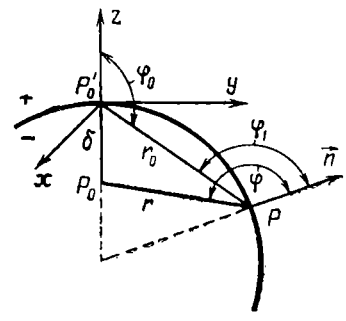

Fig. 13.

$$\chi^+(P_0') = \chi^-(P_0') = \chi(P_0'),$$

or

$$V^+(P_0') - \vartheta(P_0') = V^-(P_0') = V(P_0') - \frac{1}{2}\vartheta(P_0').$$

From this follow the validity of the relations (72) and also the uniform convergence of $V(P_0)$ to its limiting values from within and without.

Thus, it only remains now for us to show that every closed Lyapunov surface possesses property α.* We write the integral $E(P_0)$ in the form

$$E(P_0) = \iint_{e'} \left| \frac{\cos\varphi}{4\pi r^2} \right| dS + \iint_{e''} \left| \frac{\cos\varphi}{4\pi r^2} \right| dS, \tag{73'}$$

where e' is that portion of e lying within the sphere $|P - P_0'| = 2\delta$ (where in turn $\delta = |P_0 - P_0'|$ and $e'' = e - e'$ (see Fig. 13).

Since $r/\sin\varphi_0 = \delta/\sin(\varphi - \varphi_1)$ (see Fig. 13), for $P \in e'$ we can assume $r > \delta/2$ and $\cos nz \geqslant a > 0$, where $a = \text{const}$. Therefore, the first of the integrals in the left-hand side of Eq. (73'), when written in polar coordinates, is

$$\iint_{e'} \left| \frac{\cos\varphi}{4\pi r^2} \right| dS \leqslant \frac{1}{4\pi\left(\frac{\delta}{2}\right)^2} \int_0^{2\pi}\int_0^{2\delta} \frac{\rho\, d\rho\, d\varphi}{\cos nz} \leqslant$$

$$\leqslant \frac{1}{4\pi\left(\frac{\delta}{2}\right)^2 a} \int_0^{2\pi}\int_0^{2\delta} \rho\, d\rho\, d\varphi \leqslant M', \tag{73''}$$

where M' is a constant independent of P_0 and P_0'.

*For certain types of surfaces, for example, for convex surfaces, condition α is obviously satisfied. Therefore, for such surfaces, the proof of the theorem requires no further arguments.

On the basis of inequality (69), we have $|\cos\varphi_1| < 3A' r_0^{\lambda}$, where $A' = \text{const}$ (see Figs. 10 and 13). Therefore, to show that the second integral on the right-hand side of Eq. (73′) is bounded above by a constant independent of P_0 and P_0', it will be sufficient to show that the integral

$$A = \iint_{e''} \left| \frac{\cos\varphi}{4\pi r^2} - \frac{\cos\varphi_1}{4\pi r_0^2} \right| dS$$

possesses this property.

By means of projections onto the normal $\vec{n}$ (see Fig. 13), we find

$$-r\cos\varphi = -r_0\cos\varphi_1 + \delta\cos nz.$$

Simple calculation yields

$$\frac{\cos\varphi}{r^2} - \frac{\cos\varphi_1}{r_0^2} = \frac{r\cos\varphi - r_0\cos\varphi_1}{r^3} + r_0\cos\varphi_1\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right) =$$

$$= \frac{-\delta\cos nz}{r^3} + r_0\cos\varphi_1\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right).$$

We have

$$\iint_{e''} \left| \frac{-\delta\cos nz}{4\pi r^3} \right| dS \leqslant \int_0^{2\pi}\int_{2\delta}^{d} \frac{\delta\rho\, d\rho\, d\varphi}{4\pi\rho^3} = \frac{2\pi}{4\pi}\,\delta\,\frac{-1}{\rho}\Big|_{2\delta}^{d} \leqslant M'', \qquad (73''')$$

where d is a fixed number (greater than 2δ) and M'' is a constant independent of P_0 and P'_0. Therefore, it remains only to show that the integral

$$B = \iint_{e''} \left| \frac{r_0\cos\varphi_1}{4\pi}\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right) \right| dS$$

is bounded.

From the triangle $P_0P_0'P$ (Fig. 13), we have $-\delta < r - r_0 < \delta$ or $1 - \delta/r_0 < r/r_0 < 1 + \delta/r$. Since $r_0 > 2\delta$ for $P \in e''$, we have $1/2 < r/r_0 < 3/2$. Therefore,

$$r_0\left|\frac{1}{r^3} - \frac{1}{r_0^3}\right| = \frac{|r_0^3 - r^3|}{r^3 r_0^3} \leqslant \frac{\delta(r_0^2 + r_0 r + r^2)}{r^3 r_0^3} \leqslant \delta\,\frac{r_0^2 + \frac{3}{2}r_0^2 + \frac{9}{4}r_0^2}{\left(\frac{1}{2}\right)^3 r_0^5} \leqslant$$

$$\leqslant C\frac{\delta}{r_0^3}, \qquad (C = \text{const}).$$

In view of the inequality $|\cos\varphi_1| \leqslant 3A'r_0^{\lambda}$, we obtain

$$B \leqslant \frac{3A'}{4\pi} C\delta \iint_{e''} \frac{1}{r_0^{3-\lambda}}\, dS = \frac{3A'}{4\pi} C\delta \int_0^{2\pi}\int_{2\delta}^{d} \frac{\rho\, d\rho\, d\varphi}{\rho^{3-\lambda}\cos nz} \leqslant$$

$$\leqslant C'\delta \frac{-1}{\rho^{1-\lambda}}\Big|_{2\delta}^{d} \leqslant M''', \qquad (C' = \text{const}),$$

where M''' is a constant independent of P_0 and P'_0. Combining this last inequality with inequalities (73″) and (73‴), we conclude that the integral (73′) is bounded by a constant independent of P_0 and P'_0. This completes the proof for the general case of a Lyapunov surface. The assertions of the theorem having to do with the logarithmic potential of a double layer are proved analogously.

7. The limiting values of the normal derivative of the potential and of the logarithmic potential of a single layer.

Theorem 7. If a function $\sigma(P)$ is continuous on a closed Lyapunov surface S, then the normal derivative of the potential of a single layer $\partial U(P_0)/\partial n_0$ approaches its continuous limiting values uniformly as P_0 approaches points P_0 of that surface along the normal from within and from without, and these limiting values are given respectively by the equations

$$\frac{\partial U^+(P'_0)}{\partial n_0} = -\frac{\sigma(P'_0)}{2} + \frac{\partial U(P'_0)}{\partial n_0}, \quad \frac{\partial U^-(P'_0)}{\partial n_0} = \frac{\sigma(P'_0)}{2} + \frac{\partial U(P'_0)}{\partial n_0}$$

$$\frac{\partial U(P'_0)}{\partial n_0} = \iint_S \sigma(P)\frac{\cos\varphi_0}{4\pi r_0^2}\, dS, \quad r_0 = |P - P'_0|, \tag{75}$$

where φ_0 is the angle formed by the vector $\overrightarrow{P'_0P}$ and the normal $\vec{n}_0$ to the surface S at the point P'_0 (see Fig. 14).

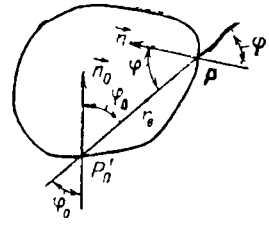

Fig. 14.

If a function $\sigma(z)$ is continuous on a closed Lyapunov curve C, then the normal derivative of the logarithmic potential of a single layer $\partial U(z_0)/\partial n_0$ approaches its continuous limiting values uniformly as z_0 approaches points z_0' of that curve along the normal from within and from without, and these limiting values are given respectively by the equations

$$\frac{\partial U^+(z'_0)}{\partial n_0} = -\frac{\sigma(z'_0)}{2} + \frac{\partial U(z'_0)}{\partial n_0}, \quad \frac{\partial U^-(z'_0)}{\partial n_0} = \frac{\sigma(z'_0)}{2} + \frac{\partial U(z'_0)}{\partial n_0},$$

$$\frac{\partial U(z'_0)}{\partial n_0} = \int_C \sigma(z)\frac{\cos\varphi_0}{2\pi r_0}\, ds, \quad r_0 = |z - z_0|, \tag{75′}$$

*where φ_0 is the angle between the vector $\overrightarrow{z'_0 z}$ and the normal $\vec{n}_0$ to the curve C at the point z'_0.**

Proof: The derivative of the logarithmic potential $U(z_0)$ of a simple layer in the direction of the normal $\vec{n}_0$ to the curve C at the point z'_0 can be obtained by differentiating under the integral sign and taking the limit (the point z_0 lies on the normal $\vec{n}_0$):

$$\frac{\partial U(z_0)}{\partial n_0} = \int_C \sigma(z) \frac{\partial}{\partial n_0} \frac{1}{2\pi} \ln \frac{1}{r} ds = \int_C \sigma(z) \frac{\cos \alpha}{2\pi r} ds,$$

where $r = |z - z_0|$ and α is the angle between the vectors $\overrightarrow{z_0 z}$ and $\vec{n}_0$ (see Fig. 15). We introduce the logarithmic potential of a double layer

$$V(z_0) = \int_C \sigma(z) \frac{\cos \varphi}{2\pi r} ds$$

and the function

$$\chi(z_0) = \frac{\partial U(z_0)}{\partial n_0} + V(z_0) = \int_{C-e} \sigma(z) \frac{\cos \varphi + \cos \alpha}{2\pi r} ds +$$

$$+ \int_e \sigma(z) \frac{\cos \varphi + \cos \alpha}{2\pi r} ds,$$

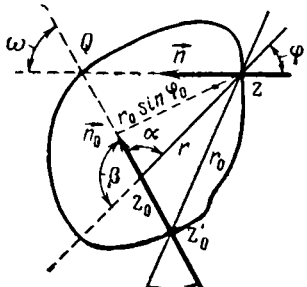

Fig. 15.

where e is a sufficiently small open arc of the curve C containing the point z'_0. Let us show that the last integral converges uniformly at the point z'_0 under the condition that the point z_0 lies on the normal $\vec{n}_0$ (or on its continuation). From the triangle $z_0 z Q$ (see Fig. 15), we have $\alpha + \varphi + \omega = \pi$, where ω is the angle between the normals to C at the points z and z_0. Since $\alpha = \pi - \beta$ (see Fig. 15), we have $\beta - \varphi = \omega$. Therefore, since C is a Lyapunov curve,

$$\cos \varphi + \cos \alpha = \cos \varphi - \cos \beta \ll |\beta - \varphi| = \omega \leqslant A r_0^{\lambda}$$

where r_0 is the distance between the points z and z'_0.

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*It follows from (75) and (75′) that

$$\frac{\partial U^+(P'_0)}{\partial n_0} - \frac{\partial U^-(P'_0)}{\partial n_0} = -\sigma(P'_0)$$

and

$$\frac{\partial U^+(z'_0)}{\partial n_0} - \frac{\partial U^-(z'_0)}{\partial n_0} = -\sigma(z'_0).$$

Therefore, Theorem 7 is often called the *theorem on the discontinuities of the normal derivative of the potential of a simple layer.*

In view of the fact that the arc e is sufficiently small, we may assume $\varphi_0 > 1/\sqrt{2}$ on that arc. Now taking into account the fact that $r \geqslant r_0 \sin \varphi_0$ (see Fig. 15), we obtain

$$\int_e \sigma(z) \frac{\cos\varphi + \cos\alpha}{2\pi r} ds \ll \frac{MA}{2\pi} \int_e \frac{r_0^\alpha}{\frac{1}{2} r_0} ds \quad (M = \text{const}).$$

From this it is clear that the integral $\chi(z_0)$ converges uniformly at the point z'_0 under the condition that the point z_0 lies on the normal to the curve C at the point z'_0. Therefore, the function $\chi(z_0)$ is continuous along the normal $\vec{n}_0$ and, since $V(z_0)$ has limiting values from within and without at the points z'_0, the derivative $\partial U(z_0)/\partial n_0$ also possesses this property. By regrouping the terms of the inequalities

$$\frac{\partial U^+(z'_0)}{\partial n_0} + \frac{\sigma(z'_0)}{2} + V(z'_0) = \frac{\partial U^-(z'_0)}{\partial n_0} - \frac{\sigma(z'_0)}{2} + V(z'_0) =$$

$$= \frac{\partial U(z'_0)}{\partial n_0} + V(z'_0)$$

we obtain Eqs. (75′). Since $V(z_0)$ converges uniformly to its limiting values, we may assert that $\partial U(z_0)/\partial n_0$ also converges uniformly to its limiting values from within and without. These limiting values are continuous functions of the point z'_0 on the curve C. Proof of the assertions of the theorem regarding $\partial U(P_0)/\partial n_0$ is the same as that for the case of $\partial U(z_0)/\partial n_0$.

Remark: To be able to assert that not only the normal derivative of the potentials $U(P_0)$ and $U(z_0)$ but their first derivatives in general are continuous on the boundary of the region, we need to require, in addition to the conditions of Theorem 7, that the functions $\sigma(P)$ and $\sigma(z_0)$ be Holder-continuous on S and C respectively.*

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*The following example illustrates this point: Suppose that the circle $x^2 + y^2 < R^2 < 1$ is a portion of a closed Lyapunov surface S. In this circle, let us set

$$\sigma(P) = x \frac{1}{\sqrt{x^2+y^2}\,|\ln\sqrt{x^2+y^2}\,|}.$$

The behavior of

$$\left.\frac{\partial U(x_0, y_0, z_0)}{\partial x_0}\right|_{\substack{x_0=0\\ y_0=0}}$$

at the coordinate origin is determined by the integral

$$\left.\frac{\partial U^*(x_0, y_0, z_0)}{\partial x_0}\right|_{\substack{x_0=0\\ y_0=0}} = \frac{1}{4\pi} \iint\limits_{\sqrt{x^2+y^2}<R} \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}\,|\ln\sqrt{x^2+y^2}\,|\,(x^2+y^2+z_0^2)^{3/2}} =$$

8. Solution of the fundamental boundary-value problems in potential theory by the method of integral equations. Let S (resp. C) denote a closed Lyapunov surface (resp. curve). The theorems proved in the preceding subsections enable us to reduce the basic boundary-value problems of potential theory, both interior and exterior, to Fredholm integral equations of the second kind in a rather simple manner. This approach to the solution of boundary-value problems leads to a number of important results.

Let us examine this question in detail.

(a) *The interior Dirichlet problem and the exterior Neumann problem.* In the three-dimensional case, we seek a solution of the interior Dirichlet problem with the boundary conditions

$$u|_S = \Phi(P), \tag{1'}$$

and a solution of the exterior Neumann problem with the boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_S = \Phi(P) \tag{1''}$$

in the form of a double-layer potential

$$u(P_0) = \iint_S \vartheta(P) \frac{\cos\varphi}{4\pi r^2}\, dS \tag{76}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} \cos^2\varphi\, d\varphi \int_0^R \frac{\rho^2\, d\rho}{|\ln\rho|\,(\rho^2+z_0^2)^{3/2}} =$$

$$= \frac{1}{4}\int_0^R \frac{\rho^2\, d\rho}{|\ln\rho|\,(\rho^2+z_0^2)^{3/2}} > \frac{1}{4}\int_\delta^R \frac{\rho^2\, d\rho}{|\ln\rho|\,(\rho^2+z_0^2)^{3/2}} \to \frac{1}{4}\int_\delta^R \frac{d\rho}{\rho\,|\ln\rho|} =$$

$$= \frac{1}{4}\left|\frac{\ln\delta}{\ln R}\right|.$$

Consequently, for sufficiently small z_0, we have

$$\left.\frac{\partial U(x_0, y_0, z_0)}{\partial x_0}\right|_{\substack{x_0=0\\ y_0=0}} > \frac{1}{2}\ln\left|\frac{\ln\sigma}{\ln R}\right|.$$

This means that

$$\left.\frac{\partial U(x_0, y_0, z_0)}{\partial x_0}\right|_{\substack{x_0=0\\ y_0=0}}$$

increases without bound as we approach the coordinate origin. A detailed account of these questions can be found in the monograph by N. M. Gunther, *La théorie du potential et ses applications aux proolémes fondamentaux de la physique*, Paris, Gauthier-Villars, 1934.

in the first case and in the form of a single-layer potential

$$u(P_0) = \iint_S \sigma(P) \frac{1}{4\pi r} dS \tag{77}$$

in the second case. Here, $\vartheta(P)$ and $\sigma(P)$ are as yet unknown functions, which, like the function $\Phi(P)$, we assume to be continuous.

From Theorem 6 on the limiting values of the potential of a double layer and from Theorem 7 on the limiting values of the normal derivative of a potential of a single layer, we immediately obtain relations which the functions $\vartheta(P)$ and $\sigma(P)$ must satisfy:*

$$\vartheta(P_0') = 2\Phi(P_0') - 2\iint_S \vartheta(P) \frac{\cos\varphi}{4\pi r_0^2} dS, \tag{78}$$

$$\sigma(P_0') = -2\Phi(P_0') - 2\iint_S \sigma(P) \frac{\cos\varphi_0}{4\pi r_0^2} dS . \tag{79}$$

Equation (78) is called the *integral equation for the interior three-dimensional Dirichlet problem;* Eq. (79) is called the *integral equation for the exterior three-dimensional Neumann problem.*

Each of these integral equations is a *Fredholm integral equation of the second kind* with a *kernel* which has a *weak singularity.* The three well-known theorems of Fredholm are applicable to such integral equations.**

Equations (78) and (79) are adjoint to each other. Therefore, according to Fredholm's first theorem, for each of these equations to have a unique continuous solution, it is necessary and sufficient that equation

$$\sigma(P_0') = -2\iint_S \sigma(P) \frac{\cos\varphi_0}{4\pi r_0^2} dS \tag{80}$$

should possess only the trivial solution.

Let us assume the contrary. Suppose $\sigma^*(P)$ is a nonzero solution of Eq. (80). Let us apply Ostrogradskiy's formula

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*The minus sign in front of the first term in the right-hand member of Eq. (79) is due to the fact that, in formulas (75) of Theorem 7, the symbol $\vec{n_0}$ denotes the normal directed into the interior of that portion of space bounded by the surface S. In the boundary conditions (1″), the symbol $\vec{n}$ denotes the normal to S directed into the domain in question, i.e., in the given case, directed into the *exterior* domain, lying *outside* the surface S.

**See, for example, S. G. Mikhlin, *Linear Integral Equations,* Hindustan Publishing Corporation, Delhi, 1960.

$$-\iint\limits_{S_1+S_2} U\frac{\partial U}{\partial n}\,dS=\iiint\limits_{\Omega}(U_x^2+U_y^2+U_z^2)\,d\tau, \tag{81}$$

(where S_2 is a sphere of sufficiently large radius R with center at the coordinate origin, S_1 is a surface enclosing the surface S and satisfying the condition under which Ostrogradskiy's formula is applicable, and Ω is the domain bounded by the surfaces S_1 and S_2) to the potential of a simple layer

$$U(P_0)=\iint\limits_S \sigma^*(P)\frac{1}{4\pi r}\,dS\,.$$

The integral over S_2 approaches zero as $R\to\infty$ since

$$U=O\left(\frac{1}{R}\right),\quad \frac{\partial U}{\partial n}=O\left(\frac{1}{R^2}\right).$$

Let us choose the surface S_1 in such a way that the normals to the surface S meet the surface S_1 at a distance not exceeding some sufficiently small positive number δ.* On the basis of Theorem 7, the normal derivative $\partial U/\partial n$, as we approach S, converges uniformly to its limiting values, and, on the basis of (80) [see also (75)], these limiting values are equal to 0. Therefore, $|\partial U/\partial n|_{S_1}<\varepsilon$, where ε is an arbitrarily small number. From this it follows that $U(P_0)\equiv$ const outside S. Since $U(P_0)|_\infty=0$, it follows that $U(P_0)\equiv 0$ outside S. Furthermore, on the basis of the continuity of $U(P_0)$ on S and the maximum principle, we have $U(P_0)\equiv 0$ inside S. Consequently, $U(P_0)\equiv 0$ throughout all space. Now, from Eqs. (75), we have

$$\sigma^*(P_0')=\frac{\partial U^-(P_0')}{\partial n_0}-\frac{\partial U^+(P_0')}{\partial n_0}\equiv 0.$$

Thus, Eq. (80) has no nontrivial solutions, and Eqs. (78) and (79) each have a unique continuous solution.

After a solution $\vartheta(P_0')$ of the integral equation (78) has been found, we can substitute it into (76) to obtain the solution of the interior Dirichlet problem for the domain G bounded by the given Lyapunov surface S. On the basis of the maximum principle, this solution is unique.

After a solution $\sigma(P_0')$ of the integral equation (79) has been found, we can substitute it into (77) to obtain the solution of the

*This may be done, for example, by taking as the surface S_1 a polyhedron with vertices at the end points of segments of sufficiently small length laid off along the outer normals to the surface S at points of this surface situated at a sufficiently small distance from each other.

exterior Neumann problem for the domain lying outside the given Lyapunov surface S. It follows from formula (81) that, by virtue of the regularity condition $(U|_\infty=0)$, this solution is unique.*

Analogously, in the plane case, we shall seek a solution to the interior Dirichlet problem with boundary conditions

$$u|_C = \Phi(z) \tag{2'}$$

and the solution of the exterior Neumann problem with boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_C = \Phi(z) \tag{2''}$$

in the form of a logarithmic potential of a double layer

$$u(z_0) = \int_C \vartheta(z)\,\frac{\cos\varphi}{2\pi r}\,ds \tag{76'}$$

and a logarithmic potential of a simple layer

$$u(z_0) = \int_C \sigma(z)\,\frac{1}{2\pi}\ln\frac{1}{r}\,ds, \tag{77'}$$

respectively. From Theorems 6 and 7, we obtain integral equations for the interior plane Dirichlet problem and for the exterior plane Neumann problem, respectively, in the form

$$\vartheta(z_0') = 2\Phi(z_0') - 2\int_C \vartheta(z)\,\frac{\cos\varphi}{2\pi r_0}\,ds, \tag{78'}$$

$$\sigma(z_0') = -2\Phi(z_0') - 2\int_C \sigma(z)\,\frac{\cos\varphi_0}{2\pi r_0}\,ds. \tag{79'}$$

Integral equations (78') and (79') are adjoint to each other, and for a unique continuous solution to exist for each of them, it is necessary and sufficient that equation

$$\sigma(z_0') = -2\int_C \sigma(z)\,\frac{\cos\varphi_0}{2\pi r_0}\,ds \tag{80'}$$

should not possess any but the trivial solution.

*To show this, assume the contrary. For the difference of two solutions, use formula (81) and reason just as we have just done regarding the function defined by Eq. (80).

Let us look at this equation in greater detail. Suppose $\sigma^*(z)$ is a solution of Eq. (80′). By integrating along the contour C, we obtain

$$\int_C \sigma^*(z_0')\,ds_0' = -2\int_C \left[\int_C \sigma^*(z)\,\frac{\cos\varphi_0}{2\pi r_0}\,ds\right] ds_0' =$$

$$= -\int_C \sigma^*(z)\left[2\int_C \frac{\cos\varphi_0}{2\pi r_0}\,ds_0'\right] ds = -\int_C \sigma^*(z)\,ds,$$

that is,

$$\int_C \sigma^*(z)\,ds = 0. \tag{80″}$$

Let us now apply Green's formula

$$-\int_{C_1+C_2} U\frac{\partial U}{\partial n}\,ds = \iint_{D_1}(U_x^2 + U_y^2)\,d\sigma, \tag{81′}$$

to the potential of a simple layer

$$U(z_0) = \int_C \sigma^*(z)\,\frac{\cos\varphi}{2\pi r}\,ds.$$

In this case, C_2 denotes a circle of sufficiently large radius R with center at the coordinate origin, C_1 denotes a contour which (1) encompasses the contour C, (2) satisfies the condition under which Green's formula is applicable, and (3) lies at a sufficiently small distance from the contour C as measured along the normals to C; D_1 denotes the domain bounded by the contours C_1 and C_2. Let us recall that, on the basis of (80″), $U \to 0$ and $\partial U/\partial n = O(1/R^2)$ as $R \to \infty$. Then, just as in the three-dimensional case, we conclude with the aid of formula (81′) that $U(z_0) \equiv 0$ throughout the entire plane. From Eqs. (75′) of Theorem 7, we obtain

$$\sigma^*(z_0') = \frac{\partial U^-(z_0')}{\partial n_0} - \frac{\partial U^+(z_0')}{\partial n_0} \equiv 0.$$

This means that Eq. (80′) does not have nontrivial solutions and, consequently, that Eqs. (78′) and (79′) each have a unique solution.

If we substitute the solution of Eq. (78′) into (76′), we obtain the solution of the interior Dirichlet problem. Obviously, this solution is unique. When we substitute the solution of Eq. (79′) into (77′), we obtain the solution of the exterior Neumann problem. However, it is not immediately obvious from formula (77′) whether this

solution is bounded at infinity, i.e., whether it is a solution of the Neumann problem in the sense in which it was posed earlier. To answer this question, let us integrate both members of the integral equation (79′) along the contour C. We obtain

$$\int_C \sigma(z_0')\, ds_0' = 2\int_C \Phi(z_0')\, ds_0' - \int_C \sigma(z)\left[2\int_C \frac{\cos\varphi_0}{2\pi r_0}\, ds_0'\right] ds,$$

that is,

$$\int_C \sigma(z)\, ds = \int_C \Phi(z)\, ds.$$

Therefore, for the solution that we have constructed for the Neumann problem to be bounded at infinity (see subsection 1), it is necessary and sufficient that

$$\int_C \Phi(z)\, ds = 0.$$

Obviously, such a solution of the exterior Neumann problem will be unique up to an arbitrary additive constant which may be added to the right-hand member of Eq. (77′).

(b) *The exterior Dirichlet problem and the interior Neumann problem.* In the three-dimensional case, let us seek the solution of the exterior Dirichlet problem with boundary conditions (1′) in the form of a potential of a double layer (76) and a solution of the interior Neumann problem with boundary conditions (1″) in the form of a potential of a double layer (77).

From the formulas of Theorems 6 and 7, we obtain integral equations for the exterior Dirichlet problem and the interior Neumann problem, respectively, in the form

$$\vartheta(P_0') = -2\Phi(P_0') + 2\iint_S \vartheta(P)\frac{\cos\varphi}{4\pi r_0^2}\, dS. \tag{82}$$

$$\sigma(P_0') = -2\Phi(P_0') + 2\iint_S \sigma(P)\frac{\cos\varphi_0}{4\pi r_0^2}\, dS. \tag{83}$$

Equations (82) and (83) are adjoint to each other. According to the second theorem of Fredholm, homogeneous adjoint integral equations have the same number of eigenfunctions, i.e., of linearly independent nontrivial solutions. This applies, in particular, to the equations

$$\vartheta(P_0') = 2\iint_S \vartheta(P)\frac{\cos\varphi}{4\pi r_0^2}\,dS, \tag{84}$$

$$\sigma(P_0') = 2\iint_S \sigma(P)\frac{\cos\varphi_0}{4\pi r_0^2}\,dS. \tag{84'}$$

It is obvious from the geometrical meaning of a subtended solid angle that Eq. (84) possesses the function $\vartheta^*(P) \equiv 1$. Consequently, Eq. (84′) also possesses an eigenfunction $\sigma^*(P)$. The potential of a single layer

$$U(P_0) = \iint_S \sigma^*(P)\frac{1}{4\pi r}\,dS$$

inside the surface S is identically equal to some constant C since it is continuous and, on the basis of (84′), its normal derivative converges uniformly to 0 as we approach S from within. The significant point for us is that $C \neq 0$. If C were 0, the continuity and the fact that $U(P_0)$ vanishes at infinity would imply that $U(P_0) \equiv 0$ throughout all space. This in turn would, on the basis of Eqs. (75), imply that

$$\sigma^*(P_0') = \frac{\partial U^-(P_0')}{\partial n_0} - \frac{\partial U^+(P_0')}{\partial n_0} \equiv 0,$$

which is impossible since $\sigma^*(P_0)$ is an eigenfunction of Eq. (84′). Let us suppose that $\sigma_1^*(P)$ is a second eigenfunction of Eq. (84′). Then, there exists a constant A such that the potential of a single layer

$$U_2(P_0) = U_1(P_0) - AU(P_0) = \iint_S [\sigma_1^*(P) - A\sigma^*(P)]\frac{1}{4\pi r}\,dS$$

is identically equal to zero throughout all space. In view of Eq. (75), we obtain from this that $\sigma_1^*(P_0') - A\sigma(P_0') \equiv 0$. This means that $\sigma^*(P)$ is the only eigenfunction of Eq. (84′) and $\vartheta^*(P) \equiv 1$ is the only eigenfunction of Eq. (84).

The physical meaning of the function $\sigma^*(P)$ consists of the fact that it is proportional to the density of electric charges that can distribute themselves freely on the boundary S of a conductor occupying a volume G. This is explained by the fact that inside the conductor the potential is always constant, i.e., $\partial U/\partial n\,|_{S=0}$. The function $\sigma^*(P)$ is known as Robin's function.

In what follows, we shall assume that the function $\sigma^*(P)$ satisfies the normalization condition

$$\iint_S \sigma^*(P)\frac{1}{4\pi r}\,dS = 1.$$

On the basis of Fredholm's third theorem, for Eq. (83) to have a solution, it is necessary and sufficient that

$$\iint_S \Phi(P)\,dS = 0. \tag{84''}$$

When this condition is satisfied, the solution is determined uniquely up to an additive term of the form $C\sigma^*(P)$, where C is an arbitrary constant. Consequently, in this case, the interior Neuman problem has a solution and, to obtain this solution in explicit form, we must replace $\sigma(P)$ in (77) with the solution of the integral equation (83). Here, an arbitrary choice of the constant C corresponds to the fact that the solution of the interior Neumann problem is determined up to an arbitrary additive constant.

According to the third Fredholm theorem, the integral equation (82) will have a solution only when

$$\iint_S \Phi(P)\,\sigma^*(P)\,dS = 0. \tag{84'''}$$

When this condition is satisfied, the solution will be determined up to an arbitrary additive constant C. In this case, the exterior Dirichlet problem will have a solution in the form of a potential of a double layer (76). This solution is independent of the choice of the constant C since, outside the surface S,

$$\iint_S \frac{\cos\varphi}{4\pi r^2}\,dS = 0.$$

If the orthogonality condition (84''') is not satisfied, let us seek a solution of the exterior Dirichlet problem in the form

$$U(P_0) = \frac{\alpha}{4\pi r_1} + u(P_0),\quad u(P_0) = \iint_S \vartheta(P)\frac{\cos\varphi}{4\pi r^2}\,dS,$$

where α is a constant and r_1 is the distance from the point P_0 to any fixed point lying inside the surface S. Then,

$$u(P_0)|_S = \Phi(P_0) - \frac{\alpha}{4\pi r_1}\Big|_S,$$

and the integral equation for the exterior Dirichlet problem (82) takes the form

$$\vartheta(P_0') = -2\left[\Phi(P_0') - \frac{\alpha}{4\pi r_1}\right]_S + 2\iint_S \vartheta(P)\frac{\cos\varphi}{4\pi r_0^2}\,dS,$$

and condition (84‴) may be written in the form

$$\iint_S \Phi(P)\,\sigma^*(P)\,dS - \alpha\iint_S \sigma^*(P)\frac{1}{4\pi r_1}\,dS = 0.$$

On the basis of the chosen normalization of the eigenfunction $\sigma^*(P)$, this condition will be satisfied if we set

$$\alpha = \iint_S \Phi(P)\,\sigma^*(P)\,dS.$$

Thus, if the interior Neumann problem has a solution if and only if condition (84″) is satisfied, the exterior Dirichlet problem has a solution with arbitrary continuous boundary conditions on an arbitrary Lyapunov surface S.

In the plane case, we shall seek a solution of the exterior Dirichlet problem with boundary conditions (2′) in the form of a logarithmic potential of a double layer (76′), and we shall seek a solution of the interior Neumann problem with boundary conditions (2″) in the form of a logarithmic potential of a single layer (77′).

With the aid of Theorems 6 and 7, we obtain the integral equation for the exterior plane Dirichlet problem and the integral equation for the interior plane Neumann problem in the forms

$$\vartheta(z_0') = -2\Phi(z_0') + 2\int_C \vartheta(z)\frac{\cos\varphi}{2\pi r_0}\,ds, \tag{82′}$$

$$\sigma(z_0') = -2\Phi(z_0') + 2\int_C \sigma(z)\frac{\cos\varphi_0}{2\pi r_0}\,ds, \tag{83′}$$

respectively.

Integral equations (82′) and (83′) are adjoint to each other. Therefore, the homogeneous equations

$$\vartheta(z_0') = 2\int_C \vartheta(z)\frac{\cos\varphi}{2\pi r_0}\,ds,$$

$$\sigma(z_0) = 2\int_C \sigma(z)\frac{\cos\varphi_0}{2\pi r_0}\,ds$$

have the same number of eigenfunctions. Just as in the preceding case, we see that the only eigenfunction of the first equation is $\vartheta^*(z) \equiv 1$ and the only eigenfunction of the second equation is some function $\sigma^*(z)$.

Therefore, Eq. (83′) has a solution only when

$$\int_C \Phi(z)\,ds = 0, \tag{51′}$$

and this solution is defined up to an additive term of the form $c\sigma^*(z)$, where c is an arbitrary constant. When we substitute this solution into the logarithmic potential of a single layer (77′), we obtain the desired solution of the interior Neumann problem. It is uniquely determined up to an arbitrary additive constant since, inside the contour C, we have*

$$\omega(z_0) = \int_C \sigma^*(z)\frac{1}{2\pi}\ln\frac{1}{r}\,ds = \text{const}\ *.$$

The integral equation for the Dirichlet problem (82′) has a solution only when

$$\int_C \Phi(z)\,\sigma^*(z)\,ds = 0, \tag{85}$$

and this solution is uniquely determined up to an arbitrary additive constant c. When we substitute this solution into (76′), we obtain the solution of the exterior Dirichlet problem. It is uniquely determined since, outside the contour C,

$$\int_C \frac{\cos\varphi}{4\pi r^2}\,ds = 0.$$

If the orthogonality condition (85) is not satisfied, we first find the solution of the Dirichlet problem

*This follows from the fact that, on the basis of the integral equation which $\sigma^*(z)$ satisfies,

$$\left.\frac{\partial\omega^+(z_0)}{\partial n_0}\right|_C = 0.$$

$$U_1(z_0) = \int_C \vartheta(z) \frac{\cos\varphi}{2\pi r} ds, \qquad (85')$$

with boundary conditions

$$U_1(z)|_C = \Phi(z) - A,$$

where A is a constant defined by

$$\int_C [\Phi(z) - A]\sigma^*(z)\, ds = 0.$$

Therefore, the integral equation (82′), with $\Phi(z_0')$ replaced by $\Phi(z_0') - A$, has a solution $\vartheta(z)$ determined up to an additive constant c. When we substitute this solution into (85′), we define the function $U_1(z_0)$ uniquely since, outside the contour C,

$$\int_C \frac{\cos\varphi}{2\pi r} ds = 0.$$

If we set

$$U(z_0) = U_1(z_0) + A,$$

we obtain the solution of the exterior Dirichlet problem with given boundary conditions (2′) which does not satisfy the orthogonality condition (85).

Thus, with the aid of integral equations, we can solve the first and second boundary-value problems of potential theory for an arbitrary closed Lyapunov surface or an arbitrary closed Lyapunov curve with arbitrary continuous boundary conditions. The same method of integral equations can be applied without difficulty to the investigation of solutions of the third boundary-value problem.*

From these results, it follows in particular that there exists a Green's function for the first boundary-value problem of potential theory, and a generalized Green's function for the second boundary-value problem for an arbitrary finite three-dimensional domain bounded by a Lyapunov surface and for an arbitrary finite plane domain bounded by a Lyapunov curve. In the case of the first boundary-value problem this is quite obvious, since in this case finding a Green's function amounts to solving the same boundary-value problem for Laplace's equation with boundary conditions

*The ideas of potential theory and the theory of integral equations have also proved to be quite fruitful in the study of more general boundary-value problems for elliptic systems of equations. For more details on this question, see the monograph by C. Miranda, *Equazioni alle derivate parziali di tipo ellittico*, Berlin, Springer, 1955, although it does not contain a complete bibliography on this question.

$$g(P_0, P)\Big|_S = -\frac{1}{4\pi r}\Big|_S, \qquad r = |P - P_0|$$

in the three-dimensional case and

$$g(z_0, z)\Big|_C = -\frac{1}{2\pi}\ln\frac{1}{r}\Big|_C, \qquad r = |z - z_0|$$

in the plane case.

In the case of the second boundary-value problem for a three-dimensional domain, the generalized Green's function $G(P_0, P) = 1/4\pi r + g(P_0, P)$ must, by definition, be such that the function $g(P_0, P)$ is a solution of the boundary-value problem

$$\Delta g = \frac{1}{\omega}, \quad \frac{\partial g(P_0, P)}{\partial n}\Big|_S = -\frac{\partial}{\partial n}\frac{1}{4\pi r}\Big|_S,$$

where ω is the volume of the domain G for which the generalized Green's function is constructed. If we set

$$g(P_0, P) = g^*(P_0, P) + \frac{1}{6\omega}R^2, \qquad R^2 = x^2 + y^2 + z^2,$$

to find $g^*(P_0, P)$ as a function of P, we arrive at the boundary-value problem

$$\Delta g^* = 0, \quad \frac{\partial g^*(P_0, P)}{\partial n}\Big|_S = -\frac{1}{6\omega}\frac{\partial R^2}{\partial n}\Big|_S - \frac{\partial}{\partial n}\frac{1}{4\pi r}\Big|_S. \qquad (85'')$$

Applying Ostrogradskiy's formula, we obtain

$$\iint_S \frac{\partial g^*}{\partial n}\,dS = \frac{1}{6\omega}\iiint_G \Delta R^2\,d\tau - \iint_S \frac{\partial}{\partial n}\frac{1}{4\pi r}\,dS =$$

$$= \frac{1}{\omega}\iiint_G d\tau - 1 = 0,$$

i.e., condition (84″) for the existence of a solution to the problem (85″) is satisfied. Consequently, Green's function of the second boundary-value problem for the region indicated above exists. Similar reasoning leads to the same conclusion in the case of a plane domain enclosed by a Lyapunov curve.

9. **Applications of potential theory to derive solutions for certain canonical regions.** For the open disk $|z| < R$, we used harmonic conjugate functions to obtain the solution of the interior Neumann problem in the form of Dini's formula (see Sect. 3, subsection 13)

$$u(z_0) = -\frac{1}{\pi}\int_C \Phi(z)\ln\frac{1}{r}\,ds\,ds \tag{62}$$

without completely justifying this procedure. Let us now give a justification of this formula. The right-hand member of Eq. (62) is the logarithmic potential of a simple layer with density $-2\Phi(z)$. Therefore, in accordance with Theorem 7 on the limiting values of the normal derivative, we have

$$\frac{\partial u^+(z_0')}{\partial n} = -\frac{-2\Phi(z_0')}{2} + \int_C -2\Phi(z)\frac{\cos\varphi_0}{2\pi r_0}\,ds = \Phi(z_0') -$$

$$-\frac{1}{2\pi R}\int_C \Phi(z)\,ds, \qquad (2R\cos\varphi_0 = r_0),$$

i.e., formula (62) is valid for an arbitrary continuous function $\Phi(z)$ satisfying the condition

$$\int_C \Phi(z)\,ds = 0. \tag{51'}$$

On the basis of Theorem 7 again, we conclude that, when condition (51') is satisfied, the formula

$$u(z_0) = -\frac{1}{\pi}\int_C \Phi(z)\ln\frac{1}{r}\,ds \tag{86}$$

yields a solution of the exterior Neumann problem for the circle since*

$$\frac{\partial u^-(z_0')}{\partial n_0} = \frac{-2\Phi(z_0')}{2} + \int_C -2\Phi(z)\frac{\cos\varphi_0}{2\pi r_0}\,ds =$$

$$= -\Phi(z_0') - \frac{1}{2\pi R}\int_C \Phi(z)\,ds = -\Phi(z_0') \quad .$$

*Here, we need to recall that in the present case $\vec{n_0}$ is the normal directed into the interior of C but that, in the boundary conditions of the exterior Neumann problem $\partial u/\partial n\,|_C = \Phi(z)$, the normal $\vec{n}$ is directed into the region in question, which, in that case, is the region outside the contour C.

We note that formulas (62) and (86) might have been obtained in a comparatively simple manner by the method of integral equations.*

In Sect. 2, subsection 4, we used the Green's functions and a direction investigation of the limiting values to obtain a solution of the interior Dirichlet problem for the interior of a circle $|z|<R$ and of a sphere $|O-P|<R$ in the forms

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(z)\frac{R^2-r_0^2}{R^2+r_0^2-2Rr_0\cos(\theta-\theta_0)}\,d\theta \tag{40}$$

and

$$u(P_0)=\frac{1}{4\pi R}\iint_S\Phi(P)\frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}}\,dS, \tag{37}$$

respectively. Let us now use Theorem 6 to prove these formulas.

The Green's function for the disk $|z|<R$ takes the form

$$G(z_0, z)=\frac{1}{2\pi}\ln\left(\frac{r^*}{r}\frac{r_0}{R}\right)$$

(see Sect. 2, subsection 3). Keeping formula (19) in mind, let us seek a solution of the interior Dirichlet problem in the form

$$u(z_0)=\int_C\Phi(z)\frac{\partial G(z_0, z)}{\partial n}\,ds=\int_C\Phi(z)\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)ds-$$

$$-\int_C\Phi(z)\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r^*}\right)ds. \tag{87}$$

The right-hand member of this equation is a combination of two logarithmic potentials of a double layer and, consequently, is a

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*Indeed, the integral equations of the interior and exterior Neumann problems (83′) and (79′) in this case take the forms

$$\sigma(z_0')=-2\Phi(z_0')+\frac{1}{2\pi R}\int_C\sigma(z)\,ds,$$

$$\sigma(z_0')=-2\Phi(z_0')-\frac{1}{2\pi R}\int_C\sigma(z)\,ds,$$

respectively. If we integrate both members of these equations along the contour C, we obtain in the first case condition (51′) and the equation $\sigma(z)=-2\Phi(z)$; in the second, assuming condition (51′) is satisfied, i.e., assuming regularity at infinity, we obtain $\sigma(z)=-2\Phi(z)$. When we substitute these values of the density $\sigma(z)$ into the logarithmic potential of a simple layer, we obtain formulas (62) and (86), respectively.

continuous function in the closed disk $|z| \leqslant R$. On the basis of Theorem 6, we have, on the circle $|z'_0| = R$,

$$u^+(z'_0) = \frac{\Phi(z'_0)}{2} + \int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r}\right)_{z'_0} ds -$$

$$-\left[-\frac{\Phi(z''_0)}{2} + \int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r^*}\right)_{z''_0} ds = \Phi(z'_0)\right.$$

$$(z''_0 = z'_0).$$

This means that formula (87) always yields a solution of the interior Dirichlet problem for the open disk $|z| < R$ for an arbitrary continuous function $\Phi(z)$. On the basis of the equation

$$\frac{\partial G(z_0, z)}{\partial n} = \frac{R^2 - r_0^2}{2\pi R\left[R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)\right]}$$

(see Sect. 2, subsection 4) this also applies to formula (40). Since

$$u^-(z'_0) = -\left[-\frac{\Phi(z'_0)}{2} + \int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r}\right)_{z'_0} ds\right] +$$

$$+ \frac{\Phi(z'_0)}{2} + \int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r^*}\right)_{z'_0} ds = \Phi(z'_0)$$

the function

$$u(z_0) = -\int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r}\right) ds + \int_C \Phi(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} \ln \frac{1}{r^*}\right) ds \quad (87')$$

will yield a solution to the exterior Dirichlet problem for $|z| > R$ and, consequently, the function

$$u(z_0) = -\frac{1}{2\pi} \int_0^{2\pi} \Phi(z) \frac{R^2 - r_0^2}{R^2 + r_0^2 - 2Rr_0 \cos(\theta - \theta_0)} d\theta \quad (88)$$

will also be a solution.

We note that formulas (40) and (88) can also be obtained by the method of integral equations. Since $2R \cos \varphi = r_0$, the integral equations of the interior and exterior Dirichlet problem (78′) and (82′) have the form

$$\vartheta(z'_0) = 2\Phi(z'_0) - \frac{1}{2\pi R} \int_C \vartheta(z)\, ds, \quad \vartheta(z'_0) = -2\Phi(z'_0) + \frac{1}{2\pi R} \int_C \vartheta(z)\, ds. \quad (88')$$

If we integrate the first of these equations along the curve C, we obtain

$$\vartheta(z_0') = 2\Phi(z_0') - \frac{1}{2\pi R}\int_C \Phi(z)\,ds.$$

If we substitute this into the logarithmic potential of a double layer, we obtain

$$u(z_0) = \int_C \left[2\Phi(z) - \frac{1}{2\pi R}\int_C \Phi(z_1)\,ds_1\right]\frac{\cos\varphi}{2\pi r}\,ds = \frac{1}{\pi}\int_C \Phi(z)\frac{\cos\varphi}{r}\,ds -$$

$$-\frac{1}{2\pi R^2}\int_C \Phi(z_1)\,ds_1 = \frac{1}{\pi}\int_C \Phi(z)\left(\frac{\cos\varphi}{r} - \frac{1}{2R}\right)ds.$$

From this and the obvious equations

$$\frac{2\cos\varphi}{r} - \frac{1}{R} = \frac{2Rr\cos\varphi - r^2}{r^2R} = \frac{R^2 - r_0^2}{r^2R^2} =$$

$$= \frac{1}{R}\,\frac{R^2 - r^2}{R^2 + r_0^2 - 2Rr_0\cos(\theta - \theta_0)},$$

we immediately obtain (40). If we integrate the second of Eqs. (88′) instead of (85) along the contour C, we obtain the condition

$$\int_C \Phi(z)\,ds = 0 \qquad (88'')$$

and $\vartheta(z) = -2\Phi(z)$. When condition (88″) is satisfied, the solution of the exterior Dirichlet problem can be written in the form

$$u(z_0) = -\frac{1}{\pi}\int_C \Phi(z)\frac{\cos\varphi}{r}\,ds.$$

On the other hand, if this condition is not satisfied, we set

$$u(z_0) = u_1(z_0) + \frac{1}{2\pi R}\int_C \Phi(z)\,ds,\quad u_1\Big|_C = \Phi(z) - \frac{1}{2\pi R}\int_C \Phi(z)\,ds,$$

and obtain

$$u(z_0) = -\frac{1}{\pi}\int_C \left[\Phi(z) - \frac{1}{2\pi R}\int_C \Phi(z_1)\,ds_1\right]\frac{\cos\varphi}{r}\,ds + \frac{1}{2\pi R}\int_C \Phi(z)\,ds.$$

Consequently,

$$u(z_0) = -\frac{1}{\pi}\int_C \Phi(z)\left(\frac{\cos\varphi}{r} - \frac{1}{2R}\right)ds,$$

i.e., formula (88) is valid.

The Green's function for the Dirichlet problem for the sphere $|O - P| < R$ is of the form

$$G(P_0, P) = \frac{1}{4\pi r} - \frac{R}{r_0}\,\frac{1}{4\pi r^*}$$

(see Sect. 2, subsection 3). Therefore, keeping formula (16) in mind, let us seek a solution of the interior Dirichlet problem in the form

$$u(P_0) = \iint_S \Phi(P) \frac{\partial G(P_0, P)}{\partial n} dS = \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r} dS - \\ - \frac{R}{r_0} \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r^*} dS. \tag{89}$$

On the basis of Theorem 6, the right-hand member of this equation is a continuous function in the closed domain $|O-P| \leqslant R$. On the basis of Eqs. (72), we have

$$u^+(P_0') = \frac{\Phi(P_0')}{2} + \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r}\Big|_{P_0'} dS - \\ - \left[-\frac{\Phi(P_0')}{2} + \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r^*}\Big|_{P_0^{*'}} dS = \Phi(P_0') \qquad (P_0^{*'} = P_0').$$

Therefore, formula (89) yields a solution of the interior Dirichlet problem for a sphere; consequently, formula (37) is also a solution.

Analogously, on the basis of the equations

$$u^-(P_0') = -\left[-\frac{\Phi(P_0')}{2} + \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r}\Big|_{P_0'} dS \right] + \frac{\Phi(P_0')}{2} + \\ + \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r^*}\Big|_{P_0'} dS = \Phi(P_0')$$

the function

$$u(P_0) = -\iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r} dS + \frac{R}{r_0} \iint_S \Phi(P) \frac{\partial}{\partial n} \frac{1}{4\pi r^*} dS \tag{90}$$

yields a solution of the exterior Dirichlet problem for $|O-P| > R$, and the function

$$u(P_0) = -\frac{1}{4\pi R} \iint_S \Phi(P) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}} dS, \tag{90'}$$

will also be a solution since

$$\frac{\partial G(P_0, P)}{\partial n} = \frac{R^2 - r_0^2}{4\pi R^2 (R^2 + r_0^2 - 2Rr_0 \cos\gamma)^{3/2}}$$

(cf. Sect. 2, subsection 4).

In the case of the first boundary-value problem for the half-plane $y>0$ and the half-space $z>0$, the Green's functions take the forms (see Sect. 2, subsection 3)

$$G(z_0, z) = \frac{1}{2\pi}\ln\frac{1}{r} - \frac{1}{2\pi}\ln\frac{1}{r^*}, \quad G(P_0, P) = \frac{1}{4\pi r} - \frac{1}{4\pi r^*},$$

respectively. Let us assume that the boundary conditions $u(z)|_{y=0} = \Phi(z)$ and $u(P)|_{z=0} = \Phi(x, y)$ are continuous in every bounded portion of the x-axis and of the xy-plane, respectively, that the function $\Phi(x)$ is bounded as x tends to infinity, and that the function $\Phi(x, y)$ satisfies the condition $\Phi(x, y) = O(1/\sqrt{x^2+y^2})$ as x, y tend to infinity. Keeping formulas (19) and (16) in mind, let us seek solutions of the first boundary-value problem for the half-plane $y>0$ and the half-space $z>0$ in the forms

$$u(z_0) = \int_{-\infty}^{\infty} \Phi(x)\left[\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right) - \frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r^*}\right)\right]dx =$$

$$= \frac{y_0}{\pi}\int_{-\infty}^{\infty} \Phi(x)\frac{dx}{\sqrt{(x-x_0)^2 + y_0^2}}, \tag{91}$$

$$u(P_0) = \int_{-\infty}^{\infty}\int \Phi(x, y)\left[\frac{\partial}{\partial n}\frac{1}{4\pi r} - \frac{\partial}{\partial n}\frac{1}{4\pi r^*}\right]dx\,dy =$$

$$= \frac{z_0}{2\pi}\int_{-\infty}^{\infty}\int \frac{\Phi(x, y)\,dx\,dy}{[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{3/2}}, \tag{92}$$

respectively.

Under the assumptions made regarding the functions $\Phi(x)$ and $\Phi(x, y)$, the right-hand members of Eqs. (91) and (92) can be differentiated under the integral sign. Therefore, they are harmonic functions in the half-plane $y_0>0$ and the half-space $z_0>0$. From Eqs. (72′) and (72), we have

$$u^+(z_0') = \frac{\Phi(x_0')}{2} + \int_{-\infty}^{\infty} \Phi(x)\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)\Big|_{z_0'} dx - \left[-\frac{\Phi(x_0')}{2} + \right.$$

$$\left. + \int_{-\infty}^{\infty} \Phi(x)\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)\Big|_{z_0''} dx\right] = \Phi(x_0'), \qquad (z_0' = z_0'');$$

$$u^+(P_0') = \frac{\Phi(x_0', y_0')}{2} + \int_{-\infty}^{\infty}\int \Phi(x, y) \frac{\partial}{\partial n} \frac{1}{4\pi r}\Big|_{P_0'} dS -$$

$$-\left[-\frac{\Phi(x_0', y_0')}{2} + \int_{-\infty}^{\infty}\int \Phi(x, y) \frac{\partial}{\partial n} \frac{1}{4\pi r^*}\Big|_{P_0^{*\prime}} dS\right] = \Phi(x_0', y_0'),$$

$$(P_0' = P_0^{*\prime}).$$

This means that formulas (91) and (92) do in fact yield the solutions of the Dirichlet problem for the half-plane $y_0 > 0$ and the half-space $z_0 > 0$, respectively.

In just the same way, by using Theorem 7 on the limiting values of the normal derivative, we can show that the integrals

$$u(z_0) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \Phi(x) \ln \frac{1}{\sqrt{(x - x_0)^2 + y_0^2}}\, dx, \tag{91'}$$

$$u(P_0') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int \Phi(x, y) \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z_0^2}}\, dx\, dy \tag{92'}$$

yield the solution of the second boundary-value problem for the half-plane $y_0 > 0$ and the half-space $z_0 > 0$, respectively, whenever the boundary conditions

$$\frac{\partial u}{\partial n}\Big|_{y=0} = \frac{\partial u}{\partial y}\Big|_{y=0} = \Phi(x) \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{z=0} = \frac{\partial u}{\partial z}\Big|_{z=0} = \Phi(x, y)$$

are continuous and are such that the integrals (91') and (92') with y_0 and z_0 positive in the respective cases can be differentiated twice under the integral sign. By the same method, i.e., by using Theorems 6 and 7, we can obtain rigorously justified explicit formulas for the solutions of the basic boundary-value problems of potential theory whenever the corresponding Green's functions are known.

10. Solution of the Neumann problem for the interior and exterior of a sphere. Let us give explicit formulas for the solutions of the second boundary-value problem for a sphere:

$$\Delta U = 0, \quad \frac{\partial U}{\partial n}\Big|_{r=R} = \Phi(P), \quad (r = \sqrt{x^2 + y^2 + z^2}). \tag{93}$$

To simplify the problem, let us note one general property of harmonic functions of three variables.

If a function

$$u(P) = u(r, \theta, \varphi), \qquad u(0) = 0,$$

where r, θ and φ are spherical coordinates of the point $P(x, y, z)$, is harmonic in a region containing the coordinate origin, the function

$$U(P) = U(r, \theta, \varphi) = \int_0^r u(r_0, \theta, \varphi) \frac{dr_0}{r_0} \tag{93'}$$

is also a harmonic function.

Proof: Recalling the expression for the Laplacian in spherical coordinates (see Sect. 3, subsection 1)

$$\Delta_{x,y,z} U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) +$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \varphi^2} = \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} (ru) + \right.$$

$$\left. + \int_0^r \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} \right] \frac{dr_0}{r_0} \right.$$

and making use of $\frac{\partial}{\partial r}(ru)/_{r=0} = 0$, we obtain

$$\frac{\partial}{\partial r}(ru) = \int_0^r \frac{\partial^2}{\partial r_0^2} (r_0 u)\, dr_0 = \int_0^r \left(r_0 \frac{\partial^2 u}{\partial r_0^2} + 2 \frac{\partial u}{\partial r_0} \right) dr_0 =$$

$$= \int_0^r \frac{\partial}{\partial r_0} \left(r_0^2 \frac{\partial u}{\partial r_0} \right) \frac{dr_0}{r_0}.$$

Thus, we obtain

$$\Delta_{x,y,z} U = \frac{1}{r^2} \int_0^r r_0 \left[\frac{1}{r_0^2} \frac{\partial}{\partial r_0} (r_0^2 u) + \frac{1}{r_0^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \right.$$

$$\left. + \frac{1}{r_0^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} \right] dr_0 = 0,$$

i.e., the function $U(P)$ defined by Eq. (93') is indeed harmonic.

Now, it is almost immediately obvious that the solution of the Neumann problem for the sphere (93) can be written in the form

$$U(P_1) = -R \int_0^{r_1} u(r_0, \theta, \varphi) \frac{\partial r_0}{r_0} \left(r_1 = |O - P_1| = \sqrt{x_1^2 + y_1^2 + z_1^2} \right), \tag{93''}$$

where $u(r_0, \theta, \varphi)$ is Poisson's integral for the sphere

$$u(r_0, \theta, \varphi) = \frac{1}{4\pi R} \iint_S \Phi(P) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} dS.$$

This is true since our necessary condition for the solvability of the second boundary-value problem

$$\iint_S \Phi(P)\, dS = 0$$

implies the vanishing of Poisson's integral at the origin, and as a consequence, the function (93″) must be harmonic in $|r_1| < R$. Furthermore, it is immediately obvious that this function is continuous in the closed $|r_1| \leqslant R$, and that on the boundary of this sphere it satisfies the boundary conditions

$$\left.\frac{\partial U}{\partial n}\right|_{r_1 = R} = -\left.\frac{\partial U}{\partial r_1}\right|_{r_1 = R} = u(r_0, \theta, \varphi)|_{r_0 = R} = \Phi(P).$$

This means that the function (93″) is a solution of the interior Neumann problem for a sphere.

Analogously, recalling the behavior of harmonic functions at infinity (see Sect. 3, subsection 6), we obtain the solution of the Neumann problem for the exterior of the sphere $|O—P| > R$ with the boundary function $\Phi(P)$ in the form

$$U(P_1) = R \int_{\infty}^{r_1} u(r_0, \theta, \varphi) \frac{dr_0}{r_0}, \quad (r_1 = |O — P_1| = \sqrt{x_1^2 + y_1^2 + z_1^2}), \tag{94}$$

where $u(r_0, \theta, \varphi)$ is the solution of the first boundary-value problem for the exterior of a sphere with boundary function $\Phi(P)$:

$$u(r_0, \theta, \varphi) = -\frac{1}{4\pi R} \iint_S \varphi(P) \frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0 \cos \gamma)^{3/2}} dS.$$

Formulas (93″) and (94) can be converted to a simpler form. In the case of the interior boundary-value problem, we have

$$\iint_S \Phi(P)\, dS = 0.$$

Therefore, formula (93″) can be written in the form

$$U(P_1)=\frac{-1}{4\pi}\int_0^{r_1}\left\{\iint_S \Phi(P)\left[\frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0\cos\gamma)^{3/2}}-\frac{1}{R}\right]dS\right\}\frac{dr_0}{r_0}=$$

$$=-\frac{1}{4\pi}\iint_S \Phi(P)\left\{\int_0^{r_1}\left[\frac{R^2-r_0^2}{(R^2+r_0^2-2Rr_0^2\cos\gamma)^{3/2}}-\frac{1}{R}\right]\frac{dr_0}{r_0}\right\}dS. \tag{94'}$$

Let us denote the inner integral in the right-hand member of Eq. (94') by A, and let us try to find a simpler expression for it. We introduce the notation

$$R^*=\sqrt{R^2+r_0^2-2Rr_0\cos\gamma}.$$

Then, we have

$$2r_0\frac{\partial R^*}{\partial r_0}=2r_0\frac{r_0-R\cos\gamma}{R^*}=R^*-\frac{R^2-r_0^2}{R^*};$$

i.e.,

$$\frac{R^2-r_0^2}{R^{*3}}=\frac{1}{R^*}+2r_0\frac{\partial}{\partial r_0}\frac{1}{R^*}$$

Therefore,

$$A=\int_0^{r_1}\left[\frac{1}{R^*}+2r_0\frac{\partial}{\partial r_0}\frac{1}{R^*}-\frac{1}{R}\right]\frac{dr_0}{r_0}=\frac{2}{R^*}\Big|_{r_0=0}^{r_1}+$$

$$+\int_0^{r_1}\left(\frac{1}{R^*}-\frac{1}{R}\right)\frac{dr_0}{r_0}=2\left(\frac{1}{\sqrt{R^2+r_1^2-2Rr_1\cos\gamma}}-\frac{1}{R}\right)+$$

$$+\int_0^{r_1}\left(\frac{1}{\sqrt{R^2+r_0^2-2Rr_0\cos\gamma}}-\frac{1}{R}\right)\frac{dr_0}{r_0}.$$

A simple calculation shows* that

$$\int\left(\frac{1}{\sqrt{R^2+r_0^2-2Rr_0\cos\gamma}}-\frac{1}{R}\right)\frac{dr_0}{r_0}=$$

$$=-\frac{1}{R}\ln\left[R-r_0\cos\gamma+\sqrt{R^2+r_0^2-2Rr_0\cos\gamma}\right]+\text{const}^*.$$

*This follows from the formula

$$\int\frac{dx}{\sqrt{1+x^2-2x\cos\gamma}}=\ln\left[x-\cos\gamma+\sqrt{1+x^2-2x\cos\gamma}\right]+\text{const}.$$

Thus, we obtain

$$A = 2\left[\frac{2}{\sqrt{R^2 + r_1^2 - 2Rr_1\cos\gamma}} - \frac{1}{R}\right] -$$

$$- \frac{1}{R}\ln\left[R - r_1\cos\gamma + \sqrt{R^2 + r_1^2 - 2Rr_1\cos\gamma}\right] + \frac{\ln 2R}{R} + \text{const.}$$

Substituting this into Eq. (94′) and recalling that

$$\iint_S \Phi(P)\,dS = 0,$$

we obtain the solution of the Neumann problem for the interior of a sphere (93) in the following form:

$$U(P_1) = -\frac{1}{4\pi}\iint_S \Phi(P)\left[\frac{2}{r} - \frac{1}{R}\ln(R + r - r_1\cos\gamma)\right]dS. \quad (93''')$$

Here, $|O - P_1| = r_1$, $r = |P - P_1| = \sqrt{R^2 + r_1^2 - 2Rr_1\cos\gamma}$, where γ is the angle between the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP}$.

Analogously, by reversing the order of integration in the right-hand member of Eq. (94), we obtain a solution of the second boundary-value problem for the exterior of the sphere in the following form:

$$U(P_1) = -\frac{1}{4\pi}\iint_S \Phi(P)\left[\frac{2}{r} - \frac{1}{R}\ln\frac{R + r - r_1\cos\gamma}{r_1(1 - \cos\gamma)}\right]dS, \quad (94'')$$

where r_1, r, and γ are the same as in formula (93‴).

In conclusion, we point out that, since we know an explicit expression for the solutions of the second interior and exterior boundary-value problems for a sphere, we can also obtain in explicit form the Green's functions corresponding to these boundary-value problems.*

5. The Equation $\Delta u - k^2 u = 0$

1. The maximum principle and its consequences. Consider the equation

$$L(u) = \Delta u - k^2 u = 0, \quad (95)$$

*See S. L. Sobolev, *Equations of Mathematical Physics*, Reading, Addison-Wesley, 1965, pp. 297-300, and also N. S. Koshlyakov, E. B. Gliner, and M. M. Smirnov, *Differential Equations of Mathematical Physics*, North Holland, 1964.

where Δ is the n-dimensional Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

and k is a real constant. The boundary-value problems for this equation are posed as for Laplace's equation and the properties of solutions are in many ways quite analogous to properties of harmonic functions.

Theorem (the maximum principle). *A solution $u(P)$ of Eq. (95) cannot have a positive maximum value (or a negative minimum value) in the interior of a region in which it is twice continuously differentiable.*

Proof: Let us assume the converse, i.e., that a solution $u(P)$ assumes a positive maximum value at an interior point P_0 of a region in which this solution has continuous second derivatives. We obtain

$$\Delta u|_{P_0} \leqslant 0, \quad (\Delta u - k^2 u)_{P_0} < 0.$$

This contradiction proves the theorem.

Corollary 1. The first interior boundary-value problem for the equation

$$L(u) = \Delta u - k^2 u = -F \tag{95'}$$

cannot have more than one solution.

Corollary 2. The first exterior boundary-value problem for Eq. (95) cannot have more than one solution under the condition that the solution vanishes at infinity.

Proof: In these two cases, let us assume that two solutions exist satisfying the same boundary conditions. We conclude, on the one hand, that the difference between these solutions, which we denote by u, is a solution of Eq. (95) and, on the other hand, that this difference must assume a positive maximum value at an interior point of a region in which this difference is twice continuously differentiable. Since this is impossible, $u \equiv 0$.

Theorem. The solutions of the second and third interior boundary-value problems for Eq. (95') in any domain G are unique in the class of functions which have continuously second derivatives in the closed region $G \cup S$.

Proof: From Ostrogradskiy's formula, we have

$$-\overbrace{\int \ldots \int}^{n-1}_{S} u \frac{\partial u}{\partial n}\, dS = \overbrace{\int \ldots \int}^{n}_{G} \left[u\Delta u + \sum_{i=1}^{n} u_{x_i}^2 \right] d\tau.$$

Letting u denote the difference between two solutions of Eq. (95') satisfying identical boundary conditions, we obtain

$$\overbrace{\int\cdots\int}^{n}_{G}\left[k^2u^2+\sum_{i=1}^{n}u_{x_i}^2\right]d\tau=-\overbrace{\int\cdots\int}^{n-1}_{S}u\frac{\partial u}{\partial n}\,dS. \qquad (95'')$$

In the case of the second boundary-value problem (just as in the case of the first), the right-hand member of this last equation is equal to 0; consequently, $u\equiv 0$. In the case of the third boundary-value problem, since $(\partial u/\partial n-\alpha u)_S=0$ for $\alpha>0$, the right-hand member of Eq. (95″) is nonpositive. Therefore, $u\equiv 0$. This completes the proof of the theorem.

2. Integral representations of solutions and potential theory for the equation $\Delta u-k^2u=0$ with three independent variables. Let us consider solutions of the equation

$$L(u)=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}-k^2u=0, \qquad (96)$$

that depend only on

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},$$

where x, y, and z are the coordinates of a variable point P and x_0, y_0, and z_0 are the coordinates of some fixed point P_0. In spherical coordinates (see Sect. 3, subsection 1), the Laplacian for solutions $v(r)$ of Eq. (96) which depend only on r takes the form

$$\Delta v(r)=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)=\frac{1}{r}\frac{d^2(rv)}{dr^2}.$$

If we set $w=rv$, we obtain from Eq. (96)

$$\frac{d^2w}{dr^2}-k^2w=0.$$

From this, we obtain $w=e^{\pm kr}$ and $v(r)=e^{\pm kr}/r$.

The function

$$\delta_k(P,P_0)=\frac{e^{-kr}}{4\pi r} \qquad (97)$$

plays the same role for Eq. (96) as does the unit source function $\delta(P,P_0)=1/4\pi r$ for the three-dimensional Laplace equation. Therefore, we may call the function $\delta_k(P,P_0)$ the unit source function at the point P_0 for Eq. (96). In analogy with the function $\mu(P,P_0)$ of Eq. (6), we may define the function $\mu_k(P_0,P)$ of a unit dipole at the point P_0 with axis $\vec{n}_0$ for Eq. (96). Here, in the right-hand member of Eq. (6), we need to replace $\delta(P,P_0)$ and $\delta(P,P_0)$ respectively with $\delta_k(P,P_0)$ and $\delta_k(P,P_0)$. This gives us a definition of the function $\mu_k(P,P_0)$.

Ostrogradskiy's formula for the Laplace operator

$$-\iint_S\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)dS=\iiint_G(v\Delta u-u\Delta v)\,d\tau, \qquad (12)$$

can be written in the form

$$-\iint_S\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)dS=\iiint_G[vL(u)-uL(v)]\,d\tau. \qquad (98)$$

Here, in analogy with Eq. (12), we set

$$v = v(P_0, P) = \delta_k(P_0, P) + g(P_0, P), \tag{98'}$$

where $g(P_0, P)$, considered as a function of the point $P(x, y, z)$, is a twice continuously differentiable solution of Eq. (96). Proceeding as in the derivation of formula (12″), we see that

$$v(P_0) = \iint_S \left(u \frac{\partial v(P_0, P)}{\partial n} - v(P_0, P) \frac{\partial u}{\partial n} \right) dS - \iiint_G v(P_0, P) L(u) \, d\tau \tag{99}$$

for every twice continuously differentiable function $u(P)$.

If the function $v(P_0, P)$ defined by Eq. (98′) (considered as a function of the point P for every fixed point $P_0 \in G$) satisfies homogeneous boundary conditions of any of the basic types for the domain G, then $v(P_0, P)$ is called a *Green's function corresponding to the boundary-value problem for* G. It is denoted by $C_k(P_0, P)$. We see from formula (99) that, if the Green's function is known in G and on the surface S and if the solution of the corresponding boundary-value problem for the equation

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - k^2 u = -F(P), \tag{96'}$$

satisfies the conditions under which Ostrogradskiy's formula is applicable, we can obtain an explicit representation for this solution. Specifically, in the three boundary-value problems, we have the following equations:

$$u(P_0) = \iint_S \Phi(P) \frac{\partial}{\partial n} G_k(P_0, P) \, dS + \iiint_G G_k(P_0, P) F(P) \, d\tau, \tag{100}$$

$$u(P_0) = \iint_S - G_k(P_0, P) \Phi(P) \, dS + \iiint_G G_k(P_0, P) F(P) \, d\tau, \tag{100'}$$

$$u(P_0) = \iint_S - G_k(P_0, P) \Phi(P) \, dS + \iiint_G G_k(P_0, P) F(P) \, d\tau, \tag{100''}$$

respectively. Here, $G_k(P_0, P)$ should be understood as the Green's function of the first boundary-value problem in Eq. (100), of the second boundary-value problem in Eq. (100′), and of the third boundary-value problem in Eq. (100″). For $k = 0$, these three formulas coincide exactly with formulas (16), (17), and (18), respectively.

By repeating step by step the procedure of Sect. 2, subsection 3, we conclude that the Green's functions of the first, second, and third boundary-value problems for Eq. (96) are symmetric with respect to the points P and P_0, i.e., that

$$G_k(P_0, P) = G_k(P, P_0).$$

In certain special cases, an explicit expression for the Green's function can be found by a suitable arrangement of unit sources and sinks. Thus, for example, the Green's function of the first boundary-value problem for the half-space $z > 0$ takes the form

$$G_k(P_0, P) = \frac{e^{-kr}}{4\pi r} - \frac{e^{-kr^*}}{4\pi r^*},$$

where

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}, \quad r^* = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2}.$$

However, the class of regions for which it is possible to obtain an explicit representation for the Green's function by such a procedure is very restricted. For example, in the case of a sphere, this method does not work.

If we set $g(P_0, P) = 0$ in Eq. (98′), then Eq. (99) takes the form

$$u(P_0) = \iint_S \left[u \frac{\partial}{\partial n}\left(\frac{e^{-kr}}{4\pi r}\right) - \frac{e^{-kr}}{4\pi r}\frac{\partial u}{\partial n}\right] dS - \iiint_G \frac{e^{-kr}}{4\pi r} L(u)\, d\tau. \qquad (99')$$

The integrals

$$U_k(P_0) = \iint_S \sigma(P)\frac{e^{-kr}}{4\pi r}\, dS = \iint_S \sigma(P)\,\delta_k(P_0,P)\, dS, \qquad (101)$$

$$V_k(P_0) = \iint_S \vartheta(P)\frac{\partial}{\partial n}\left(\frac{e^{-kr}}{4\pi r}\right) dS = \iint_S \vartheta(P)\,\mu_k(P_0,P)\, d\tau, \qquad (101')$$

$$I_k(P_0) = \iiint_G \rho(P)\frac{e^{-kr}}{4\pi r}\, d\tau = \iiint_G \rho(P)\,\delta_k(P_0,P)\, d\tau, \qquad (101'')$$

where $\sigma(P)$, $\vartheta(P)$, and $\rho(P)$ are given continuous functions of P, are called potentials corresponding to Eq. (96). The first of these integrals is called the *potential of a single layer*, the second the potential *of a double layer*, and the third the *potential of a volume distribution*. Equation (99′) means that every function $u(P)$ with continuous second derivatives in the closed region $G \cup S$ can be represented as the sum of potentials of a single layer, a double layer, and of a volume, corresponding to Eq. (96). For $k = 0$, formula (99′) coincides with formula (14), which gives a representation of a twice continuously differentiable function as the sum of ordinary potentials $U(P_0)$, $V(P_0)$, and $I(P_0)$, i.e., corresponding to the three-dimensional Laplace equation.

In many ways, the properties of the potentials $U_k(P_0)$, $V_k(P_0)$, and $I_k(P_0)$ are analogous to the properties of the potentials $U(P_0)$, $V(P_0)$, and $I(P_0)$, with which they coincide when $k = 0$.

Each of the potentials $U_k(P_0)$, $V_k(P_0)$, and $I_k(P_0)$ outside the corresponding region of integration has continuous derivatives of all orders and satisfies Eq. (96). This assertion follows from the fact that the integrals of the right-hand members of Eqs. (101), (101′), and (101″) can be differentiated under the integral sign with respect to the variables x_0, y_0, and z_0 if the point $P_0(x_0, y_0, z_0)$ lies outside the corresponding region of integration.

Each of the potentials $U_k(P_0)$, $V_k(P_0)$, and $I_k(P_0)$ can be represented in the following form:

$$U_k(P_0) = U(P_0) + \iint_S \sigma(P)\frac{e^{-kr}-1}{4\pi r}\, dS, \qquad (102)$$

$$V_k(P_0) = V(P_0) + \iint_S \vartheta(P)\frac{\partial}{\partial n}\left(\frac{e^{-kr}-1}{4\pi r}\right) dS, \qquad (102')$$

$$I_k(P_0) = I(P_0) + \iiint_G \rho(P)\frac{e^{-kr}-1}{4\pi r}\, d\tau. \qquad (102'')$$

Each of these integrals converges uniformly at every point. The same is true of the integrals obtained by differentiating them with respect to x_0, y_0, and z_0 under the integral sign. These statements become quite obvious if the function $(e^{-kr}-1)/4\pi r$ is represented in the form of a series of positive powers of r. Therefore, the limit properties of $U_k(P_0)$, $V_k(P_0)$, and $\partial U_k(P_0)/\partial n$ at points of the surface S are determined by the limit properties of $U(P_0)$, $V(P_0)$, and $\partial U(P_0)/\partial n$, respectively. Hence, we may assert that $U_k(P_0)$ is continuous at points of the surface S, and that the limiting values of $V_k(P_0)$ and $\partial U_k(P_0)/\partial n$ as P_0 approaches points of a Lyapunov surface from the inside and from the outside exist, and satisfy

$$V_k^+(P_0') = \frac{\vartheta(P_0')}{2} + V_k(P_0'),\quad V_k^-(P_0') = -\frac{\vartheta(P_0')}{2} + V_k(P_0'), \qquad (103)$$

$$\frac{\partial U_k^+(P_0')}{\partial n_0} = -\frac{\sigma(P_0')}{2} + \frac{\partial U_k(P_0')}{\partial n_0},$$
$$\frac{\partial U_k^-(P_0')}{\partial n_0} = \frac{\sigma(P_0')}{2} + \frac{\partial U_k(P_0')}{\partial n_0}, \qquad (103')$$

where the plus and minus signs indicate, as always, the limiting values as the point P'_0 approaches the surface S from within and without, respectively. Equations (103) and (103′) are completely analogous to Eqs. (72) and (75), with which they coincide when $k = 0$. Equations (103) and (103′) make it possible to reduce the solution of the first and second boundary-value problems for Eq. (96) to the solution of integral equations just as was done in the preceding subsection in connection with the solution of the first and second boundary-value problems for Laplace's equation in three independent variables. For example, if we are seeking the solution of the interior first boundary-value problem for Eq. (96) with boundary conditions $u|_S = \Phi(P)$ in the form of a potential of a double layer

$$u(P_0) = \iint_S \vartheta(P) \frac{\partial}{\partial n}\left(\frac{e^{-kr}}{4\pi r}\right) dS,$$

the first of Eqs. (103) yields the following integral equation for determining the density $\vartheta(P)$ of that potential:

$$\Phi(P_0') = \frac{\vartheta(P_0')}{2} + \iint_S \vartheta(P) \frac{\partial}{\partial n}\left(\frac{e^{-kr}}{4\pi r}\right) dS.$$

If the function $\rho(P)$ is Hölder-continuous, the potential of a volume $I_k(P)$ has continuous second derivatives at points of the region of integration and satisfies the equation $L(I_k(P_0)) = -\rho(P_0)$. To prove this assertion, note that the existence of continuous second derivatives of the potential $I_k(P_0)$ follows from Eq. (102″) because, under the hypotheses stated, the potential $I(P_0)$ is a twice continuously differentiable function of P_0 (x_0, y_0, z_0).

Furthermore, we have

$$L(I_k(P_0)) = L(I_k^*(P_0)) + L(I_k^{**}(P_0)),$$

where

$$I_k^*(P_0) = \iiint_{G-e} \rho(P) \frac{e^{-kr}}{4\pi r} d\tau, \quad I_k^{**}(P_0) = \iiint_e \rho(P) \frac{e^{-kr}}{4\pi r} d\tau,$$

and where e is a sufficiently small sphere with center at the point P_0. But since the point P_0 lies outside the domain $G - e$, we have $L(I_k^*(P_0)) = 0$. Furthermore,

$$L(I_k^{**}(P_0)) = \Delta \iiint_e \rho(P) \frac{1}{4\pi r} d\tau - k^2 \iiint_e \rho(P) \frac{1}{4\pi r} d\tau +$$
$$+ \iiint_e \rho(P)(\Delta - k^2)\left(\frac{e^{-kr} - 1}{4\pi r}\right) d\tau.$$

But the first of the integrals in the right-hand member of this equation is equal to $-\rho(P_0)$ and we can make the other integrals arbitrarily small if the radius of the sphere e is sufficiently small. Thus, since $L(I_k(P_0))$ is independent of the radius of the sphere e, we have $L(I_k(P_0)) = -\rho(P_0)$.

3. Integral representations of solutions and potential theory for the equation $\Delta u - k^2 u = 0$ in two independent variables. To construct a potential theory and integral representations for the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u = 0 \qquad (104)$$

in analogy with what was done for the two-dimensional Laplace equation, let us try to find a solution of Eq. (104) which depends only on $r = \sqrt{(x-x_0)^2+(y-y_0)^2}$, where x and y are the coordinates of a variable point $z = x + iy$ and x_0 and y_0 are the coordinates of some fixed points $z_0 = x_0 + iy_0$. Here, we require that the solution $v = v(r)$ have a logarithmic singularity at $r = 0$. In polar coordinates r and φ (see Sect. 3, subsection 1), the Laplacian is of the form

$$\Delta v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2},$$

and Eq. (104) for determining $v(r)$ yields

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - k^2u = 0.$$

If we introduce a new variable $\rho = kr$, this last equation takes the form

$$L^*(v) = \frac{d^2v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} - v = 0. \tag{105}$$

The ordinary differential equation

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \nu^2)\,u = 0, \tag{106}$$

where ν is a real or complex constant, is called *Bessel's equation of order* ν. Equation (106) is our starting point in the construction of the entire set of special functions known as Bessel (or cylindrical) functions, just as the equation $d^2u/dx^2 + \lambda^2 u = 0$ (where $\lambda^2 = \text{const} > 0$) could have been used as the starting point in the construction of the trigonometric functions. Note that if we were to replace x with iy in this last equation, we would obtain the equation $d^2u/dy^2 - \lambda^2 u = 0$, which defines the exponential functions. Similarly, if we replace x with iy in Eq. (106), this equation takes the form

$$y^2\frac{d^2u}{dy^2} + y\frac{du}{dy} + (y^2 + \nu^2)\,u = 0. \tag{106'}$$

This is Bessel's equation of order ν for imaginary argument.

Obviously, Eq. (105) coincides with the special case of Eq. (106') when $\nu = 0$. Therefore, we shall seek a solution of this equation among Bessel functions of imaginary argument and of order zero. For our present purposes, we are interested in that Bessel function of order zero which has a logarithmic singularity at $\rho = 0$. Let us verify that the function $K_0(\rho)$ defined by

$$K_0(\rho) = \int_1^\infty \frac{e^{-\rho\xi}}{\sqrt{\xi^2-1}}\,d\xi \tag{107}$$

is such a function. We have

$$L^*(K_0) = \int_1^\infty \frac{e^{-\rho\xi}}{\sqrt{\xi^2-1}}\left(\xi^2 - \frac{\xi}{\rho} - 1\right)d\xi = \int_1^\infty e^{-\rho\xi}\sqrt{\xi^2-1}\,d\xi - \int_1^\infty \frac{e^{-\rho\xi}\,\xi\,d\xi}{\rho\sqrt{\xi^2-1}}.$$

If we integrate the second integral on the right by parts, we see that the function $K_0(\rho)$ satisfies Eq. (105). If we set $\rho\xi = \eta$, we have

$$K_0(\rho) = \int_\rho^\infty \frac{e^{-\eta}\,d\eta}{\sqrt{\eta^2-\rho^2}} = \int_\rho^A \frac{d\eta}{\sqrt{\eta^2-\rho^2}} + \int_\rho^A \frac{e^{-\eta}-1}{\sqrt{\eta^2-\rho^2}}\,d\eta + \int_A^\infty \frac{e^{-\eta}\,d\eta}{\sqrt{\eta^2-\rho^2}},$$

where A is some constant. The last two integrals on the right-hand side of this equation remain bounded as $\rho \to 0$ and we have the following expression for the first one:

$$\ln\left(\eta + \sqrt{\eta^2-\rho^2}\right)\Big|_{\rho}^{A} = \ln\frac{A+\sqrt{A^2-\rho^2}}{\rho} = \ln\frac{1}{\rho} + \ldots,$$

where the dots denote quantities that remain bounded as $\rho \to 0$. Therefore, $K_0(\rho)$ has a logarithmic singularity at $\rho=0$. Let us see how the function $K_0(\rho)$ behaves as $\rho \to \infty$. If we set $\rho(\xi-1)=\eta^2$, so that $\xi = 1+\eta^2/\rho$ in Eq. (107), we obtain

$$K_0(\rho) = \int_0^{\infty} \frac{e^{-\eta^2-\rho}}{\sqrt{\left(1+\frac{\eta^2}{\rho}\right)^2-1}}\,\frac{2\eta}{\rho}\,d\eta = e^{-\rho}\sqrt{\frac{2}{\rho}}\int_0^{\infty}\frac{e^{-\eta^2}\,d\eta}{\sqrt{1+\frac{\eta^2}{2\rho}}} =$$

$$= \sqrt{\frac{2}{\rho}}\,e^{-\rho}\left[\int_0^{\infty} e^{-\eta^2}\,d\eta + \varepsilon\,\frac{\sqrt{\pi}}{2}\right].$$

Consequently,

$$K_0(\rho) = \sqrt{\frac{\pi}{2\rho}}\,e^{-\rho}(1+\varepsilon), \tag{107'}$$

where ε is a quantity that approaches 0 as $\rho \to 0$.

The function

$$\delta_k(z, z_0) = \frac{1}{2\pi}K_0(kr) \quad (r = |z - z_0|) \tag{108}$$

plays the same role in connection with Eq. (104) as does the unit source function $\delta(z, z_0) = (2\pi)^{-1}\ln(1/r)$ in connection with the two dimensional Laplace's equation. Therefore, we may call the function (108) the unit source function at the point z_0 for Eq. (104).

Green's formula for the Laplace operator

$$-\int_C\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = \iint_D (v\Delta u - u\Delta v)\,d\sigma \tag{13}$$

may be written in the form

$$-\int_C\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)ds = \iint_D (vL(u) - uL(v))\,d\sigma. \tag{109}$$

In analogy with Eq. (13'), we set

$$v = v(z_0, z) = \delta_k(z_0, z) + g(z_0, z), \tag{109'}$$

where $g(z_0, z)$, being a function of the point $z=x+iy$, is a twice continuously differentiable solution of Eq. (104). Proceeding just as in the derivation of formula (13''), we see that for every function $u(z) = u(x, y)$, which is twice continuously differentiable in the closed region $D \cup C$,

$$u(z_0) = \int_C\left(u\frac{\partial v(z_0, z)}{\partial n} - v(z_0, z)\frac{\partial u}{\partial n}\right)ds - \iint_D v(z_0, z)\,L(u)\,d\sigma. \tag{110}$$

If the function $g(z_0, z)$ in Eq. (109') is chosen in such a way that the function $v(z_0, z)$ as a function of the point z satisfies the homogeneous boundary conditions of one of the

basic boundary-value problems at every fixed point z_0 in the domain D, we say that $v(z_0, z)$ is the Green's function corresponding to the boundary-value problem for the domain D. Let us denote this function by $G_k(z_0, z)$. Let us assume that the conditions under which Green's formula can be applied to the Green's function $G_k(z_0, z)$ and to the solution of the equation

$$\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - k^2 u = -F(z) \tag{104'}$$

are satisfied. Then, we obtain immediately from formula (110) explicit formulas for these solutions. In the cases of the first, second, and third boundary-value problems, these formulas take the forms

$$u(z_0) = \int_C \Phi(z) \frac{\partial}{\partial n} G_k(z_0, z)\, ds + \iint_D G_k(z_0, z) F(z)\, d\sigma, \tag{111}$$

$$u(z_0) = \int_C -G_k(z_0, z) \Phi(z)\, ds + \iint_D G_k(z_0, z) F(z)\, d\sigma, \tag{111'}$$

$$u(z_0) = \int_C -G_k(z_0, z) \Phi(z)\, ds + \iint_D G_k(z_0, z) F(z)\, d\sigma, \tag{111''}$$

respectively. Here, $G_k(z_0, z)$ denotes the Green's function of the first boundary-value problem in Eq. (111), the second boundary-value problem in (111'), and the third boundary-value problem in (111''). These formulas should be regarded as generalizations of formulas (19), (20), and (21), which represent the solutions of the boundary-value problems for the two-dimensional Laplace's equation with the aid of corresponding Green's functions. Similarly, in the case of Laplace's equation, it is possible to show that the Green's functions $G_k(z_0, z)$ of the first, second, and third boundary-value problems for Eq. (104) are symmetric with respect to the points z and z_0, i.e., that

$$G_k(z_0, z) = G_k(z, z_0).$$

In certain cases, an explicit representation for the Green's function can be found without much difficulty by choosing an appropriate distribution of sources. Thus, for example, the Green's function for the first boundary-value problem for the half-plane $y > 0$ takes the form

$$G_k(z_0, z) = \frac{1}{2\pi} K_0(kr) - \frac{1}{2\pi} K_0(kr^*),$$

where $r = |z - z_0|$ and $r^* = |z - \bar{z}_0|$. The Green's function for the second boundary-value problem for this same half-space $y > 0$ is determined by the equation

$$G_k(z_0, z) = \frac{1}{2\pi} K_0(kr) + \frac{1}{2\pi} K_0(kr^*)$$

(same notations).

If we set $g(z_0, z) = 0$ in Eq. (110), this equation takes the form

$$u(z_0) = \frac{1}{2\pi} \int_C \left(u \frac{\partial K_0(kr)}{\partial n} - K_0(kr) \frac{\partial u}{\partial n} \right) ds - \frac{1}{2\pi} \iint_D K_0(kr) L(u)\, d\sigma. \tag{110'}$$

In analogy with logarithmic potentials, the integrals

$$U_k(z_0) = \int_C \sigma(z) \frac{1}{2\pi} K_0(kr)\, ds = \int_C \sigma(z) \delta_k(z_0, z)\, ds, \tag{112}$$

$$V_k(z_0) = \int_C \vartheta(z) \frac{\partial}{\partial n}\left(\frac{1}{2\pi} K_0(kr)\right) ds = \int_C \vartheta(z)\, \mu_k(z_0, z)\, ds, \tag{112'}$$

$$I_k(z_0) = \iint_D \rho(z)\, \frac{1}{2\pi} K_0(kr)\, d\sigma = \iint_D \rho(z)\, \delta_k(z_0, z)\, d\sigma \tag{112''}$$

are called potentials corresponding to Eq. (104). The first is called the potential of a single layer, the second the potential of a double layer, and the third the potential of a plane distribution.

Each of the potentials $U_k(z_0)$, $V_k(z_0)$, and $I_k(z_0)$ has continuous derivatives of all orders outside the region of integration in question, which can be obtained by differentiating under the integral sign, and satisfies Eq. (104). The differences $U_k(z_0) - U(z_0)$, $V_k(z_0) - V(z_0)$, and $I_k(z_0) - I(z_0)$ are expressed by the three integrals

$$\frac{1}{2\pi}\int_C \sigma(z)\left[K_0(kr) - \ln\frac{1}{r}\right] ds, \quad \frac{1}{2\pi}\int_C \vartheta(z)\frac{\partial}{\partial n}\left(K_0(kr) - \ln\frac{1}{r}\right) ds,$$

$$\frac{1}{2\pi}\iint_D \rho(z)\left[K_0(kr) - \ln\frac{1}{r}\right] d\sigma,$$

respectively. But these integrals and the integrals obtained by differentiating them under the integral sign with respect to x_0 and y_0 converge uniformly at every point of the xy-plane. Therefore, $U_k(z_0)$ is continuous at points of the contour C, and the limit properties of $V_k(z_0)$ and $\partial U_k(z_0)/\partial n_0$ at points of a Lyapunov curve are the same as the properties of $V(z_0)$ and $\partial U(z_0)/\partial n_0$, respectively. In particular, at points of the Lyapunov curve C, we have for the limiting values of $V_k(z_0)$ and $\partial U_k(z_0)/\partial n_0$, equations completely analogous to Eqs. (72') and (75'):

$$V_k^+(z_0') = \frac{\vartheta(z_0')}{2} + V_k(z_0'), \quad V_k^-(z_0') = -\frac{\vartheta(z_0')}{2} + V_k(z_0'), \tag{113}$$

$$\frac{\partial U_k^+(z_0')}{\partial n_0} = -\frac{\sigma(z_0')}{2} + \frac{\partial U_k(z_0')}{\partial n_0}, \quad \frac{\partial U_k^-(z_0')}{\partial n_0} = \frac{\sigma(z_0')}{2} + \frac{\partial U_k(z_0')}{\partial n_0}. \tag{113'}$$

These equations make it possible to reduce the solution of the first, second, and third boundary-value problems for Eqs. (104) to the solution of integral equations in the same way as this was done earlier in connection with the two-dimensional Laplace equation (see Sect. 4, subsection 8). For example, if we are seeking the solution of the interior second boundary-value problem for Eq. (104) with boundary conditions $\partial u/\partial n\,|_C = \Phi(z)$ in the form of a potential of a simple layer

$$u(z_0) = \int_C \sigma(z)\, \frac{1}{2\pi} K_0(kr)\, ds,$$

then the first of Eqs. (113') yields the integral equation

$$\Phi(z_0') = -\frac{\sigma(z_0')}{2} + \int_C \sigma(z) \frac{\partial}{\partial n_0}\left(\frac{1}{2\pi} K_0(kr)\right) ds$$

for determining the density $\sigma(z)$.

Reasoning just as at the end of the preceding subsection, we conclude that if the function $\rho(z)$ is Hölder-continuous, the potential $I_k(z_0)$ has continuous second derivatives at points of the region of integration and satisfies the equation $L(I_k(z_0)) = -\rho(z_0)$.

Thus, in conclusion, we can say that in general Eq. (95) hardly differs from Laplace's equation when it comes to the formulation of boundary-value problems or to methods of solving them. Basically, the only difference is that the unit source function in the case of Eq. (95) looks somewhat more complicated than the unit source function in the case of Laplace's equation.

However, the equation

$$\Delta u + k^2 u = 0 \qquad (k^2 > 0), \tag{114}$$

which in form differs from Eq. (95) only in the sign in front of the k^2u, differs significantly from that equation and from Laplace's equation both from the point of view of the general properties of its solutions on the one hand, and of the formulation of boundary-value problems and the uniqueness of their solution on the other. We do not have a maximum principle for solutions of Eq. (114), and the first, second, and third boundary-value problems for this equation do not have unique solutions in general. A very simple example will illustrate this. Consider the equation $u_{xx}+u_{yy}+2u=0$. The function $u=\sin x \sin y$ is a nontrivial solution of this equation. However, this solution vanishes identically on the boundary of the square $0<x,\ y<\pi$. From a physical point of view, the nonuniqueness of the solution to the first boundary-value problem for Eq. (114) can arise, for example, in the following way. In the case of steady-state diffusion of a gas when there is a chain reaction (Chapter 1, Sect. 2, subsection 1), it may happen that the concentration of the gas is equal to zero on the boundary of the region but not in the interior. Conversely, uniqueness of the solution of the first boundary-value problem for Eq. (95) means that this situation cannot arise with the steady-state diffusion of a gas with disintegration (see Chapter 1, Sect. 2, subsection 1).

The most frequently encountered problems associated with Eq. (114) do not arise directly from physical considerations, but rather from the wave equation, when one looks for standing oscillations. A typical problem involving Eq. (114) is the problem of finding nontrivial solutions satisfying homogeneous boundary conditions of the first, second, or third boundary-value problem or the problem of finding the values of the parameter k^2 corresponding to these nontrivial solutions. These values of the parameter k^2 are known as *eigenvalues* and the nontrivial solutions referred to are called *eigenfunctions*.

We shall study problems associated with Eq. (114) later when we examine the wave equation, and again when we take up a particular method of solving boundary-value problems of mathematical physics, namely, the method of separation of variables.

CHAPTER 4

Parabolic Equations. The Basic Boundary-Value Problems. General Properties of Solutions of the Heat-Flow Equation

In this chapter, we shall examine parabolic equations. We shall formulate the basic boundary-value problems and then investigate them by methods based on the transformation of n-dimensional integrals into $(n-1)$-dimensional integrals, and by the use of instantaneous unit source and dipole functions. We shall take up the theory of thermal potentials. Solutions will be given for particular boundary-value problems. We shall introduce the concept of a δ-function, which we shall use to solve the Cauchy problem for the n-dimensional heat-flow equation.

1. Formulation of the Basic Boundary-Value Problems and Their Physical Meaning

One of the basic boundary-value problems for the heat-flow equation

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(P, t)}{k}, \quad a = \sqrt{\frac{k}{\gamma\rho}}, \tag{1}$$

where Δ is the n-dimensional Laplace operator, P is a point in the n-dimensional space, and t is time, is the following, so-called Cauchy problem.

Find a solution of Eq. (1) which has continuous derivatives of second order with respect to the space variables and of first order with respect to t at all points of the n-dimensional space for all $t>0$, which is, furthermore, continuous at $t=0$, and satisfies the initial conditions

$$u|_{t=0} = \psi(P), \tag{2}$$

where $\psi(P)$ is a given continuous bounded function of P.

From a physical point of view, when $n=3$, this is the problem of finding the temperature for $t>0$ at all points of a homogeneous medium which occupies the whole space, if the temperature distribution throughout the space is known at the initial instant $t=0$.

Let G denote a bounded domain in n-dimensional space, S—a piecewise-smooth surface bounding the domain G, $\vec{n}$—the inner normal to S (i.e., directed into the interior of G), Ω—a cylindrical domain in $(n+1)$-dimensional phase space* with base G in the plane $t=t_0=\text{const}$, and S_L—the lateral surface of this cylindrical domain with generators parallel to the t-axis. For example, for $n=1$, Ω degenerates into the half-strip shown in Fig. 16.

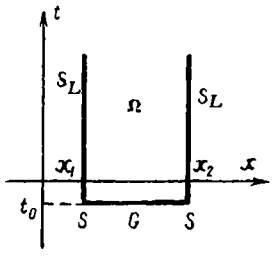

Fig. 16

In addition to the Cauchy problem, one can pose the following basic boundary-value problems for the heat equation (1):

Find a solution $u=u(P, t)$ of Eq. (1), which has continuous second derivatives with respect to the space variables and continuous first derivatives with respect to time in the cylindrical domain Ω, which at $t=t_0$ is continuous in $G \cup S$, satisfies the initial conditions

$$u|_{t=t_0} = \psi(P), \quad P \in G \tag{3}$$

and, in addition, satisfies one of the following boundary conditions:

a) the solution $u=u(P, t)$ is continuous on the lateral surface S_L and

$$u|_{S_L} = \Phi(P, t), \quad t > t_0; \tag{1'}$$

b) the normal derivative of the solution $u=u(P, t)$ exists at every point on the lateral surface S_L and

$$\left.\frac{\partial u}{\partial n}\right|_{S_L} = \Phi(P, t), \quad t > t_0; \tag{1''}$$

c) the solution $u=u(P, t)$ is continuous on the lateral surface S_L, and its normal derivative exists at every point on this surface, and

$$\left(\frac{\partial u}{\partial n} - \alpha u\right)_{S_L} = \Phi(P, t), \quad t > t_0, \ \alpha \geqslant 0, \tag{1'''}$$

where $\psi(P)$ is a given continuous function of P in the closed region $G \cup S$, and where $\Phi(P, t)$ and α are given continuous functions of P and t on the lateral surface S_L for $t \geqslant t_0$.

In the case of boundary conditions (1'), (1''), and (1'''), we speak of the *first, second, and third n-dimensional boundary-value problems, respectively, for the heat equation (1) in the domain G or in the cylindrical domain* Ω. In all these boundary-value problems, the initial conditions are the same [those given by Eq. (3)].

*By $(n+1)$-dimensional phase space, we mean a space in which each point is determined by $n+1$ coordinates, n of these being space coordinates and one being the time t.

If the initial conditions, continued to $P \in S$, and one of the boundary conditions $(1')$, $(1'')$, and $(1''')$, continued to $t=t_0$, are not contradictory, we will say that the initial function $\psi(P)$ and the boundary function $\Phi(P, t)$ satisfy a *compatibility condition*. For example, in the case of the first boundary-value problem, the compatibility condition takes the form $\psi(P)\,|_{P\in S} = \Phi(P, t_0)$. In this case, when the compatibility condition is satisfied in the statement of the boundary-value problems, we may require that the initial conditions be satisfied in $G \cup S$ and that the boundary conditions be satisfied for $t \geqslant t_0$.

Let us look at a physical interpretation of such boundary-value problems. In the heat-flow example, the initial conditions (3) can be interpreted as a specification of the temperature at all points of a body occupying the domain G at the initial instant of time t_0; $\psi(P)$ gives the values of this initial temperature. In the diffusion example, $\psi(P)$ represents the values of the concentration of a solution in the domain G at the initial instant t_0. In the heat-flow example, the boundary conditions $(1')$ represent the values of the temperature of the body on the boundary of the domain G at all instants of time $>t_0$. The boundary conditions $(1'')$ mean that the amount of heat which flows out of the domain G in a unit of time through a unit of the boundary surface S is known and is equal to $k\Phi(P, t)$, where k is the thermal conductivity. If heat is radiated from the surface of the body into space, then, according to Newton's law, the equation $k\partial u/\partial n = h(u-u_0)$ must be satisfied on the surface S; here, u_0 is the given temperature of points of the surrounding space and h is the exterior thermal conductivity. Thus, in this case, we have boundary conditions of type $(1''')$:

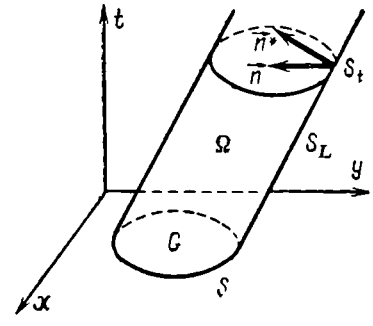

Fig. 17

$$\left(\frac{\partial u}{\partial n} - \frac{h}{k}u\right)_{S_L} = -\frac{h}{k}u_0.$$

Analogous boundary conditions must be used if there is diffusion into a region where the concentration is specified to be u_0. In this case, u should be understood as the concentration of the solution in the domain G and, instead of k and h, we have, respectively, the coefficient of diffusion and some proportionality constant analogous to the exterior thermal conductivity. Furthermore, in the diffusion case, the boundary conditions $(1')$ are the known values of the concentration on the boundary of the domain G at all instants of time $t>t_0$. The boundary conditions $(1'')$ mean that the amount of material leaving the domain G in a unit of time through a unit of area of the surface S is known and equal to $D\Phi(P, t)$, where D is the coefficient

of diffusion. These are simple examples of physical problems which lead to the first, second, and third boundary-value problems for the heat-flow equation.

Sometimes, we need to consider the first, second, and third boundary-value problems when the domain G is unbounded. For example, this will be the case when we study the flow of heat in a semi-infinite rod (coincident with the positive x-axis) if, at the end $x=0$, either the temperature or the amount of heat emitted by the rod into the surrounding space is given, or if there is radiation of heat into space, the temperature of which is given.

These three boundary-value problems and the Cauchy problem are basic and they are the most frequently encountered boundary-value problems for the heat equation (or, in general, for linear second-order parabolic equations).

As a degenerate case of these three problems, we might mention the case in which $t_0=-\infty$. Here, the boundary conditions remain unchanged, but there are no initial conditions. In each of the three such degenerate cases, we speak of the *first, second, or third boundary-value problem without initial conditions.*

For example, let us assume that the earth is half-space and that its temperature depends only on the depth x. If we know the temperature on the surface of the earth, we arrive at the first boundary-value problem for a one-dimensional half-space without initial conditions since, here, we may neglect the initial temperature distribution that existed in the far distant past; that is, we may assume that $t_0=-\infty$ and simply seek the temperature of the earth as the solution of the first boundary-value problem without initial conditions for a one-dimensional half-space $x>0$.

As a generalization of the basic boundary-value problems, we can cite the mixed boundary-value problem for the heat equation for the domain G. This problem is posed in the same way as the first, second, and third boundary-value problems except that on one portion of the boundary S of the domain G the boundary conditions of the first problem are satisfied, on another portion those of the second problem are satisfied, and on a third portion those of the third problem are satisfied.

As a generalization of the first boundary-value problem, we can consider the case in which Ω is a noncylindrical region, i.e., when the intersection of the region Ω by the characteristic plane $t=\text{const}>t_0$ is a region G_t in n-dimensional space with boundary S_t depending on t and when the boundary conditions on the lateral surface S_L (which coincides with the set of points S_t for $t>t_0$) are given by Eq. (1′). In this case, we speak of the *first boundary-value problem with a moving boundary.* We shall also call this problem the *first generalized boundary-value problem.* Let us suppose that boundary conditions (1′) are satisfied on that portion of S_L on which $\cos(\vec{n}^*, t) \neq 0$ (where $\vec{n}^*$ is the inner normal to S_L) and that either the

boundary conditions (1′) or (1″) or (1‴) are satisfied on the remainder of S_L. Here, $\vec{n}$ in Eqs. (1″) and (1‴) denotes the inner normal to the boundary S_t of the region G_t (cf. Fig. 17). In such a case, we may speak of the mixed boundary-value problem for a region with moving boundaries. We shall also call this boundary-value problem the *generalized mixed problem.**

For example, suppose that one end of a rod disintegrates as a result of combustion at a high given temperature and that at the other end, either the temperature is given or there is radiation of heat into space, the temperature of which is given. In the first case, we arrive at the first generalized boundary-value problem and, in the second, we arrive at the mixed generalized boundary-value problem for the one-dimensional heat-flow equation. Here, we assume that there is no heat exchange through the lateral surface of the rod.

2. Uniqueness of Solutions of the First, Second, and Third Boundary-Value Problems

Let u denote the difference between two solutions of the first, second, or third boundary-value problem for the heat equation (1) satisfying the same boundary and initial conditions. Obviously, u will then be a solution of the homogeneous heat equation with homogeneous boundary and initial conditions. Consider the integral

$$I = \frac{1}{2}\iiint_G u^2 d\tau.$$

We have

$$\frac{\partial I}{\partial t} = \iiint_G u \frac{\partial u}{\partial t} d\tau = a^2 \iiint_G u \Delta u \, d\tau.$$

If we set $\vec{v} = u\nabla u$ in Ostrogradskiy's formula

$$-\iint_S v_n \, dS = \iiint_G \operatorname{div} \vec{v} \, d\tau,$$

*One may consider even more general problems in which the boundary conditions of the second or third boundary-value problems are satisfied on that portion of S_L on which $\cos(\vec{n}^*, t) \neq 0$. However, in the following we shall not examine such problems in their most general form.

we obtain

$$-\iint_S u\frac{\partial u}{\partial n}\,dS=\iiint_G u\Delta u\,d\tau+\iiint_G (u_x^2+u_y^2+u_z^2)\,d\tau.$$

Therefore,

$$\frac{\partial I}{\partial t}=-a^2\iint_S u\frac{\partial u}{\partial n}\,dS-a^2\iiint_G (u_x^2+u_y^2+u_z^2)\,d\tau.$$

In view of the homogeneous boundary conditions, the first integral in the right-hand member of this equation is equal to zero in the case of the first and second boundary conditions and will be nonnegative in the case of the third boundary-value problem. Thus, $\partial I/\partial t\leqslant 0$. Since $I\geqslant 0$ and $I|_{t=t_0}=0$, we conclude that $I\equiv 0$. From this it follows that $u\equiv 0$.

Thus, the solution of each of the first, second, and third boundary-value problems is unique in the class of functions satisfying the conditions under which Ostrogradskiy's formula is applicable. We obtain the same conclusion for two-dimensional boundary-value problems in the class of functions satisfying the conditions of applicability of Green's formula. To see this, we need only use Green's formula and consider the integral $I=\frac{1}{2}\iint_G u^2 d\tau$. In the case of one-dimensional boundary-value problems, we consider the integral $I=\frac{1}{2}\int_{x_1}^{x_2} u^2dx$. We have

$$\frac{\partial I}{\partial t}=\int_{x_1}^{x_2} u\frac{\partial u}{\partial t}\,dx=a^2\int_{x_1}^{x_2} uu_x\,dx=a^2\left[u\frac{\partial u}{\partial x}\right]_{x_1}^{x_2}-a^2\int_{x_1}^{x_2} u_x^2\,dx\leqslant 0.$$

Noting that $I|_{t=t_0}=0$, we then obtain $I\equiv 0$ and $u\equiv 0$. This means that the solution of each of the one-dimensional boundary-value problems of the first, second, and third types is unique in the class of functions that are continuously differentiable in the closed region Ω, that is, in the half-strip shown in Fig. 16.

3. The Maximum Principle. Uniqueness and Stability of Solutions of the Cauchy Problem and of the First and the Generalized First Boundary Problems

Theorem. Suppose the right-hand member of the heat equation

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(P, t)}{k} \tag{1}$$

satisfies the condition $f(P, t) \geqslant 0$. *Then, every solution of this equation which is continuous in a closed cylindrical or noncylindrical region* Ω *for* $t_0 \leqslant t \leqslant T < \infty$ *attains its minimum value either on the base of this region* G *corresponding to* $t = t_0$ *or on the lateral surface* S_L *for* $t_0 \leqslant t \leqslant T$.

Proof: Suppose that $m = u(P_1, t_1)$ is the smallest value of the function $u = u(P, t)$ in the closed region Ω for $t_0 \leqslant t \leqslant T$. Suppose that this smallest value is not attained by the function u either in the region $G \cup S$ for $t = t_0$ or on the lateral surface S_L for $t_0 \leqslant t \leqslant T$. Then, for sufficiently small $\varepsilon > 0$, a function of the form

$$v = v(P, t) = u(P, t) + \varepsilon(t - t_1)$$

will, like the function $u(P, t)$, obviously assume its smallest value at some point belonging to $\Omega \cup G_T$. Clearly, at this point $\Delta v \geqslant 0$ and $\partial v/\partial t \geqslant 0$, and the case $\partial v/\partial t > 0$ is possible if the point at which the function v assumes its minimum belongs to the region G_T, i.e., to the upper base of the region Ω (see Fig. 17). At that point, we have

$$0 \leqslant \Delta v - \frac{1}{a^2}\frac{\partial v}{\partial t} = \Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} - \frac{\varepsilon}{a^2} = -\frac{f(P, t)}{k} - \frac{\varepsilon}{a^2} < 0.$$

This contradiction shows that our assumption that the function $u(P, t)$ does not assume its smallest value either in $G \cup S$ for $t = t_0$ or on the lateral surface S_L for $t_0 < t \leqslant T$ cannot hold. This completes the proof of the theorem.

We note that the assertion of this theorem might easily be predicted from physical considerations since a body that is heated as a result of positive heat sources must naturally have its minimum temperature either at an interior point at the initial instant of time or on its boundary at all subsequent instants of time.

Theorem (the maximum principle). *A solution of the homogeneous heat equation*

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0,$$

which is continuous in a closed cylindrical or noncylindrical region Ω *for* $t_0 \leqslant t \leqslant T$ *assumes its maximum and minimum values either on the base of the region* G *for* $t = t_0$ *or on the lateral surface* S_L *of the region* Ω *for* $t_0 \leqslant t \leqslant T$.

The proof of this theorem can be reduced to that of the preceding one simply by changing the sign of the function u.

Corollary 1. A solution of the first boundary-value problem, the first boundary-value problem without initial conditions, or of

the generalized first boundary-value problem is unique and stable in the class of functions that are continuous in the corresponding closed region Ω.

Proof: Let u and u_1 denote two solutions of one of the above-mentioned boundary-value problems. Suppose that both are continuous in the corresponding closed region Ω and that they differ in absolute value from each other by an amount less than $\varepsilon>0$ on the lateral surface S_L of the region Ω and on its lower base $G \cup S$ (if $t_0 \neq -\infty$). Then, according to the maximum principle, we have $-\varepsilon<u-u_1<\varepsilon$ in the region Ω. This means that the solution of each of the above-mentioned boundary-value problems is unique and stable. We note that the last inequality not only answers the question of the stability of the solution from a theoretical standpoint, but also gives a very good quantitative characteristic of this stability.

*Corollary 2. A solution of the Cauchy problem for the heat-flow equation in the class of functions that are bounded throughout all space is unique and stable.**

Proof: Let $u=u(P, t)$ and $u_1=u_1(P, t)$ be two solutions of the Cauchy problem such that their initial conditions differ in absolute value by an amount less than $\varepsilon>0$. For definiteness, let us assume the space to be two-dimensional. Consider the function

$$v_l=\left[\frac{x^2+y^2}{l^2}+\frac{4a^2t}{l^2}\right]M+\varepsilon \quad (M, l=\text{const}, \quad M>2|u|, \ 2|u_1|).$$

This function, like the function $u-u_1$, is a solution of the homogeneous heat equation. Obviously, for $t=0$ and for $x^2+y^2=l^2$, $t\geqslant 0$, the inequality $v_l \geqslant |u-u_1|$ holds. From the maximum principle, this inequality also holds in the closed region Ω constituting the circular cylinder with base $x^2+y^2<l^2$ in the plane $t=0$. If we fix the point (P, t) and let l go to ∞, we obtain

$$|u(P, t)-u_1(P, t)|<\varepsilon.$$

From this it follows that there cannot be two distinct solutions of the Cauchy problem with identical initial conditions. If two solutions have initial conditions differing from each other by an amount less than $\varepsilon>0$ in absolute value, the solutions themselves differ by an amount not exceeding ε. This means that a solution of the Cauchy problem is stable and it gives a quantitative characteristic of this stability. In the case of three-dimensional space or one-dimensional space, the reasoning and the results are the same except that we need to take the auxiliary function v_l in the following forms:

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*A generalization of this theorem to the class of solutions admitting a definite order of growth as $P \to \infty$ is given in the article by A. N. Tikhonov, "Théorèmes d'unicité pour l'équation de la chaleur," *Mat. sb*, 42, 1935, pp. 199-216.

$$v_l = \left[\frac{x^2+y^2+z^2}{l^2} + \frac{6a^2t}{l^2}\right] M + \varepsilon,$$

$$v_l = \left[\frac{x^2}{l^2} + \frac{2a^2t}{l^2}\right] M + \varepsilon,$$

respectively.

4. Instantaneous Unit Source and Dipole Functions for the Heat Equation

1. **The concept of a δ-function.** Let M denote the set of all possible functions in a finite interval $a<x<b$ which are defined, continuous, and square-integrable. Let L denote the set of all functions defined in the same interval and such that $u(x) \in L$ and $f(x) \in M$ together imply that

$$\int_a^b f(x)\, u(x)\, dx$$

exists. We shall say that the sequence of functions

$$\{u_n\} = u_1(x),\ u_2(x),\ \ldots,\ u_n(x),\ \ldots,\ (u_n(x) \in L) \tag{4}$$

converges weakly in the interval (a, b) to the function $u(x) \in L$ if, for an arbitrary function $f(x) \in M$, the following is true:

$$\lim_{n\to\infty} \int_a^b f(x)\, u_n(x)\, dx = \int_a^b f(x)\, u(x)\, dx.$$

In particular, the sequence (4) converges weakly in the interval (a, b) to the function $u(x) \in L$ in the following two cases:

(a) if the sequence consists of continuous functions on the closed interval $[a, b]$ and if it converges uniformly on that interval to the limit function $u(x)$;

(b) if it consists of functions which are square integrable in the interval (a, b) and if it converges in mean to the limit function $u(x)$.

Proof: The limit function $u(x)$ is square-integrable in the interval (a, b) [because of the continuity in case (a), and because of the Riesz-Fischer theorem * in case (b)] and, consequently, belongs to L. If we now apply Bunyakovskiy's inequality to the integral

$$\int_a^b f(x)\ (u_n(x) - u(x))\ dx$$

*See, for example, F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionelle*, 3rd ed., Paris, Gauthier-Villars, 1955.

we see in both cases that

$$\lim_{n\to\infty}\int_a^b f(x)\,u_n(x)\,dx = \int_a^b f(x)\,u(x)\,dx.$$

We shall say that the sequence (4) is a *Cauchy sequence* in the sense of weak convergence in the interval (a, b) if, for every function $f(x) \in M$ and arbitrarily small positive ε, there exists a number N such that

$$\left|\int_a^b f(x)\,[u_n(x) - u_m(x)]\,dx\right| < \varepsilon$$

whenever $n, m > N$.

Obviously, for the sequence (4) to converge weakly in the interval (a, b) to the function $u(x) \in L$, it is necessary that it be a Cauchy sequence in the sense of weak convergence in the interval (a, b).

We shall say that two Cauchy sequences of functions belonging to the set L

$$\{u_n\} = u_1(x),\ u_2(x),\ \ldots,\ u_n(x),\ \ldots,\ \{v_n\} = v_1(x),\ v_2(x),\ \ldots,\ v_n(x),\ \ldots,$$

are equivalent in the sense of weak convergence in the interval (a, b) if, for an arbitrary function $f(x) \in M$, we have

$$\lim_{n\to\infty}\int_a^b f(x)\,[u_n(x) - v_n(x)]\,dx = 0.$$

For example, two Cauchy sequences of functions belonging to the set L

$$\left\{u_n = \frac{1}{n}\right\},\quad \{v_n = \sin nx\}\quad (n = 1, 2, \ldots)$$

are equivalent in the above-defined sense since, as we know from the theory of Fourier series,

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx = 0$$

for an arbitrary function $f(x) \in M$, and, consequently,

$$\lim_{n\to\infty}\int_a^b f(x)\,[u_n(x) - v_n(x)]\,dx = 0.$$

To every Cauchy sequence (4) we assign some element, known as the *limit* of that sequence in the sense of weak convergence in the interval (a, b). By definition, this element must be the same

for all sequences that are equivalent to each other in the sense of weak convergence in the interval (a, b).

For example, for both of the sequences just exhibited $\{u_n = 1/n\}$ and $\{v_n = \sin nx\}$ (for $n = 1, 2, \ldots$), the limit in the sense of weak convergence in the interval (a, b) is the function $u(x) \in L$, which is identically zero in the interval.

Let us show that among the limit elements in the sense of weak convergence in the interval (a, b) there may also be elements that do not belong to the set of functions L. Consider the sequence of functions in L

$$\{\delta_n\} = \delta_1(x, x_0), \delta_2(x, x_0), \ldots, \delta_n(x, x_0), \ldots \qquad (4')$$

satisfying the conditions

$$\delta_n(x, x_0) = \begin{cases} 0 \text{ for } |x - x_0| \geqslant \dfrac{1}{n} \\ \dfrac{n}{2}, \text{ for } |x - x_0| < \dfrac{1}{n} \end{cases} \quad (n = 1, 2, \ldots),$$

where x_0 is a parameter $(a < x_0 < b)$.

It is easy to see that (4′) is a Cauchy sequence in the sense of weak convergence in the interval (a, b) since, for an arbitrary function $f(x) \in M$,

$$\lim_{n \to \infty} \int_a^b f(x)\, \delta_n(x, x_0)\, dx = f(x_0).$$

Let us denote by $\delta(x, x_0)$ the limit element of the sequence (4′) in the sense of weak convergence in the interval (a, b). If this limit element belonged to the set of functions L, we would have, for an arbitrary function $f(x) \in M$,

$$\int_a f(x)\, \delta(x, x_0)\, dx = f(x_0).$$

But it is easy to see that such a function $\delta(x, x_0)$ cannot exist (from the standpoint of the classifical definition of a function as a variable quantity whose values are defined at all points of the interval in question).

Let us now give a definition of weak convergence of a sequence in the interval (a, b) in the general case.

Suppose that the sequence of functions (4) is a Cauchy sequence in the sense of weak convergence in the interval (a, b). Let us denote by $u(x)$ the limit element of this sequence in the sense of weak convergence in the interval (a, b) regardless of its nature (i.e., regardless of whether it is a function in the classical sense or not or whether it does or does not belong to the set of functions L). We shall say that

the sequence of functions (4) converges weakly in the interval (a, b) to $u(x)$ and we shall write $u_n(x) \xrightarrow{\text{сл}} u(x)$. Here when $u(x)$ does not belong to the set L, by the expression

$$\int_a^b f(x)\, u(x)\, dx$$

we mean

$$\lim_{n\to\infty} \int_a^b f(x)\, u_n(x)\, dx.$$

Remark 1: Weak convergence in an interval does not imply convergence in the mean in that interval. For example, the sequence of functions $\{v_n = \sin nx\}$ (for $n = 1, 2, \ldots$) in the interval $(0, \pi)$ converges weakly to zero, but it does not converge in the mean in that interval since

$$\int_0^\pi [\sin nx - \sin mx]^2\, dx = \pi.$$

Remark 2: Weak convergence in an interval does not imply pointwise convergence at any point of that interval. To see this, observe the example cited in Remark 1. Let $x = \alpha\pi$ be any point of the interval $(0, \pi)$. If α is a rational number, it is obvious that the sequence of numbers $\sin n\alpha\pi$ (for $n = 1, 2, \ldots$) does not converge. Suppose that α is an irrational number. Let $[n\alpha]$ denote the fractional part of the number $n\alpha$. In this case, every point of the interval (0, 1) will be a limit point of the sequence of numbers $[n\alpha]$ (for $n = 1, 2, \ldots$).* Therefore, the sequence of numbers $\sin n\alpha\pi$ (for $n = 1, 2, \ldots$) does not converge.

Remark 3: In all our considerations, the significant role was played not by the interval (a, b) itself but by the functions of the set M, which were defined on this interval and which possess precisely defined properties both in themselves and with respect to the terms of the sequence of functions (4) in question.

Remark 4: It is not convenient to speak of convergence of a sequence of functions in an interval when at no point of that interval does the sequence in question (as a sequence of numbers) converge.

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*This can be shown as follows: Let ε be a fixed arbitrarily small positive number. Let us partition the interval (0, 1) into subintervals of equal length not exceeding ε. Let N be the number of these subintervals. Consider the $N+1$ numbers $[n\alpha]$ (for $n = 1, 2, \ldots, N+1$). At least two of these numbers, for example, $[n_1\alpha]$ and $[n_2\alpha]$, fall in the same interval, which is of length not exceeding ε. Therefore, if we set $n_1\alpha = n_1 + [n_1\alpha]$, $n_2\alpha = n_2 + [n_2\alpha]$, where n_1 and n_2 are integers, we have $|(n_1 - n_2)\alpha - (n_1 - n_2)| < \varepsilon$. This means that 0 is a limit point for the sequence of numbers $[n\alpha]$ (for $n = 1, 2, \ldots$). Let $[n^*\alpha] = \delta$, where n^* is an integer and δ is less than the fixed number $\varepsilon > 0$. If we start with the point δ and lay off values $[n\alpha]$ (for $n = n^*+1, n^*+2, \ldots$) from that point (we may think of this as winding a thread of length $n\alpha$ onto a circle of circumference equal to unity), we see that the point δ will be a limit point for our sequence. Continuing this process, we see that all points of the interval (0, 1) will be limit points of this sequence of numbers.

With an eye to what follows, we shall give a generalization of the concept of weak convergence of a sequence of functions in the interval (a, b).

Let T be a domain in an m-dimensional space and suppose M^* is a set of functions defined on T (T need not be bounded). Let

$$\{u_n\} = u_1(P), u_2(P), \ldots, u_n(P), \ldots \tag{5}$$

be a sequence of functions defined on T, where P is a variable point in T. Suppose, finally, that the functions in the set M^* and the functions of the sequence (5) satisfy the following condition: For every function $f(P) \in M^*$, the integrals

$$\int_T f(P)\, u_n(P)\, d\omega, \quad (n = 1, 2, \ldots),$$

where $d\omega$ is an element of volume at the point P, exist.

The above-given concept of weak convergence of the sequence of functions (4) in an interval (a, b) and also other related concepts can be carried over without change to the sequence of functions (5) with respect to the set of functions M^*. Here, instead of integrals from a to b, we consider integrals of the corresponding expressions over the domain T and speak of the weak convergence of the sequence (5) and the other concepts connected with it not in the interval (a, b) but with respect to the set of functions M^*. For example, if the set of functions M^* coincides with the set of functions M, then weak convergence of the sequence of functions (4) with respect to the set of functions M coincides with its weak convergence in the interval (a, b).

If the set M^* consists of a single function $f(x) = \sin x$ defined in the interval $(0, \pi)$, the sequence of functions $u_n(x) = \sum_{k=1}^{n} a_k \sin kx$ where the a_k are arbitrary constants, converges weakly with respect to the set M^* to the function $u(x) = a_1 \sin x$ since

$$\int_0^{\pi} \sin x u_n(x)\, dx = \int_0^{\pi} \sin x\, (a_1 \sin x)\, dx \quad (n = 1, 2, \ldots).$$

This generalization of the concept of weak convergence in an interval enables us to give one of the possible definitions of the so-called δ-function, in our opinion one of the simpler definitions.

Let us suppose that the functions in the sequence (5) depend also on a point P_0 in the region T as a parameter. Setting $u_n = u_n(P) = \delta_n = \delta_n(P, P_0)$, let us write the sequence (5) in the form

$$\{\delta_n\} = \delta_1(P, P_0), \delta_2(P, P_0), \ldots, \delta_n(P, P_0), \ldots . \tag{5'}$$

Suppose that $\delta(P, P_0)$ is the limit of the sequence (5') in the sense of weak convergence with respect to the set of functions M^*. Then,

we shall call this limit the δ-function with respect to the set of functions M^* if, for every function $f(P) \in M^*$, the equation

$$\int_T f(P)\,\delta(P, P_0)\,d\omega = f(P_0) \qquad (5'')$$

holds.*

We shall say that the sequence (5′) is δ-like with respect to the set of functions M^*.

Let us give some examples.

Example 1. Suppose that we have an integral equation

$$\varphi(P_0) = \lambda \int_T K(P, P_0)\,\varphi(P)\,d\omega,$$

for which $\lambda = 1$ is an eigenvalue and $\varphi_1(P)$ is the corresponding eigenfunction. Then, the kernel of this integral equation, $K(P, P_0)$ is a δ-function with respect to the set M^* consisting of the single function $\varphi_1(P)$ (up to a multiplicative constant).

Example 2. Suppose that M^* is the set of all continuous functions of x defined on the interval (a, b). Then, $\delta(x, x_0)$, the limit element of the sequence of functions (4′) in the sense of weak convergence with respect to the set of functions M^*, is the δ-function with respect to the given set M^*.

Example 3. Suppose that M^* is the set of all functions defined in the interval $(-l, l)$ such that each of them can be represented as the sum of a Fourier series. Suppose that we are given the sequence of functions

$$\delta_n(x, x_0) = \frac{1}{2l} + \frac{1}{l}\sum_{k=1}^{n}\left(\cos\frac{k\pi}{l}x_0\cos\frac{k\pi}{l}x + \sin\frac{k\pi}{l}x_0\sin\frac{k\pi}{l}x\right) =$$

$$= \frac{1}{2l} + \frac{1}{l}\sum_{k=1}^{n}\cos\frac{k\pi}{l}(x - x_0) = \frac{1}{2l}\sum_{-n}^{n} e^{\frac{ik\pi}{l}(x - x_0)}$$

$$(n = 1, 2, \ldots, \qquad i = \sqrt{-1}).$$

Then, the limit element $\delta(x, x_0)$ of this sequence in the sense of weak convergence with respect to M^* is the δ-function with respect to M^*. This follows from the fact that, for an arbitrary function $f(x) \in M^*$,

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*Since this definition of the δ-function does not coincide exactly with the definitions with which we are familiar in the literature, we mention that the reader can familiarize himself with the definition of the δ-function as a generalized function or as a functional, and with other definitions of it, by consulting the books by L. Schwartz, *Théorie des distributions*, Vols. 1 and 2, Paris, Hermann, 1950-1951; I. M. Gel'fand and G. E. Shilov, *Generalized functions*, Vol. 1, New York, Academic Press, 1964; D. D. Iwanenko and A. Sokolov, *Klassische Feldtheorie*, Berlin, Akademie-Verlag, 1953. See also the book by N. S. Koshlyakov, *et al., op. cit.*

$$\lim_{n\to\infty}\int_{-l}^{l} f(x)\,\delta_n(x, x_0)\,dx = f(x_0).$$

We may write

$$\delta(x, x_0) = \frac{1}{2l} + \frac{1}{l}\sum_{k=1}^{\infty}\cos\frac{k\pi}{l}(x - x_0),$$

if this equation is understood in the sense explained above.

Example 4. Suppose that M^* is the set of all functions of x that are defined in the infinite interval $(-\infty,\infty)$ and that can be represented by a Fourier integral

$$f(x_0) = \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(x)\cos\lambda(x - x_0)\,dx.$$

Suppose that we are given the sequence of functions

$$\delta_n(x, x_0) = \frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda(x-x_0)}\,d\lambda = \frac{1}{\pi}\int_0^{n}\cos\lambda(x - x_0)\,d\lambda$$

$$(n = 1, 2, \ldots).$$

Then, since

$$f(x_0) = \lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\,\delta_n(x, x_0)\,dx,$$

we may assert that the limit element $\delta(x, x_0)$ of the given sequence in the sense of weak convergence with respect to M^* is the δ-function with respect to M^*. In the sense indicated above, we may write

$$\delta(x, x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-x_0)}\,d\lambda = \frac{1}{\pi}\int_0^{\infty}\cos\lambda(x - x_0)\,d\lambda.$$

Example 5. Suppose that M^* is the set of all continuous bounded functions that are defined in the whole m-dimensional space. Suppose that $\delta(P, P_0, \alpha)$ forms, as the parameter α approaches 0 continuously, a set of functions of a variable point P in m-dimensional space that depend on a point P_0 in this space as a parameter. Let us suppose that this set of functions possesses the property that, for arbitrary values of α,

$$\overbrace{\int\cdots\int}^{m}_{-\infty}^{\infty} \delta\,(P, P_0, \alpha)\,d\omega = 1,$$

$$\overbrace{\int\cdots\int}^{m}_{-\infty}^{\infty} |\,\delta\,(P, P_0, \alpha)\,|\,d\omega \leqslant = M\,\text{const} < \infty$$

$$\lim_{\alpha\to 0} \overbrace{\int\cdots\int}^{m}_{-\infty}^{\prime\infty} |\,\delta\,(P, P_0, \alpha)\,|\,d\omega = 0,$$

where the prime on the integral denotes integration over the entire space except for an arbitrarily small neighborhood of the point P_0.

Then, it is easy to see that, for every function $f(P) \in M^*$,

$$\lim_{\alpha\to 0} \overbrace{\int\cdots\int}^{m}_{-\infty}^{\infty} f(P)\,\delta\,(P, P_0, \alpha)\,d\omega = f(P_0).$$

Therefore, we can say that the limit element of any such sequence of functions generated when $\alpha \to 0$, in the sense of weak convergence with respect to M^*, is a δ-function with respect to M^*. In this case, we can say that $\delta(P, P_0, \alpha)$ is a δ-like function with respect to M^* in the sense that, as α approaches 0, we obtain a sequence of functions that is δ-like with respect to M^*.

Example 6. Let $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_N(x)$ be an orthonormal system of functions in the interval (a, b), i.e., a system of functions satisfying the condition

$$\int_a^b \varphi_k(x)\,\varphi_m(x)\,dx = \begin{cases} 1 & k = m \\ 0 & k \neq m. \end{cases}$$

Let M^* denote the set of functions $f(x)$ that can be represented in the form $f(x) = \sum_{k=1}^{N} c_k \varphi_k(x)$, where the c_k are constants. Then, the δ-function with respect to the set of functions M^* is

$$\delta\,(x, x_0) = \sum_{k=1}^{N} \varphi_k(x)\,\varphi_k(x_0).$$

Example 7. Let $\varphi_1(P)$, $\varphi_2(P)$, ..., $\varphi_n(P)$ denote a system of n linearly independent continuous functions defined in a region T of m-dimensional space and let P be a point in T. Suppose that M^* is the set of functions $F(P)$ which can be represented in the form

$$F(P) = \sum_{k=1}^{n} a_k \varphi_k(P),$$

where the a_k are constants.

We shall say that the system of functions

$$\psi_i(P) = \sum_{k=1}^{n} \alpha_{ik} \varphi_k(P) \qquad (i = 1, 2, \ldots, n),$$

where the α_{ik} are constants, is *biorthonormal* with respect to the system $\varphi_1(P)$, $\varphi_2(P)$, ..., $\varphi_n(P)$ if

$$(\varphi_k, \psi_i) = \int_T \varphi_k(P)\, \psi_i(P)\, d\omega = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

Obviously, such a system of functions $\psi_i(P)$ (for $i=1, 2, \ldots, n$) exists since we only need to determine the constants α_{ik} (for $k=1, 2, \ldots, n$) from the system of equations

$$\sum_{k=1}^{n} \alpha_{ij} (\varphi_i, \varphi_k) = \begin{cases} 1, & k = i \\ 0, & k \neq i. \end{cases}$$

Now, it is easy to see that the δ-function with respect to the given set of functions M^* is

$$\delta(P, P_0) = \sum_{i=1}^{n} \psi_i(P)\, \varphi_i(P_0).$$

When dealing with a δ-function in what follows, we shall always assume that the δ-function is defined with respect to a sufficiently broad class of functions M^*, unless the opposite is stated. Furthermore, in the case of functions of a single variable, we shall also write the δ-function $\delta(x, x_0)$ in the form $\delta(x - x_0)$. Analogously, in the two-dimensional case, we shall sometimes denote the δ-function $\delta(z, z_0)$, where $z=x+iy$ and $z_0=x_0+iy_0$, by $\delta(z - z_0)$.

2. Solution of the Cauchy problem. Instantaneous unit source and dipole functions. It is easy to see that a particular solution of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial u}{\partial t} = 0 \tag{6}$$

is the function

$$e^{-\alpha^2 a^2 t} \cos \alpha x,$$

where α is a real parameter. Consequently, the integral

$$I(x, t) = \int_{-\infty}^{\infty} e^{-\alpha^2 a^2 t} \cos \alpha x \, d\alpha$$

is also a solution of Eq. (6). If we differentiate this integral with respect to the parameter x, we get

$$\frac{\partial I}{\partial x} = -\int_{-\infty}^{\infty} e^{-\alpha^2 a^2 t} \alpha \sin \alpha x \, d\alpha = -\frac{x}{2a^2 t} I,$$

$$I(x, t) = Ce^{-\frac{x^2}{4a^2 t}} \qquad (C = \text{const}).$$

If we set $x=0$ and recall the known value of Poisson's integral, we obtain

$$I|_{x=0} = C = \int_{-\infty}^{\infty} e^{-\alpha^2 a^2 t} \, d\alpha = \frac{1}{a\sqrt{t}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{a\sqrt{t}},$$

i.e.,

$$I(x, t) = \int_{-\infty}^{\infty} e^{-\alpha^2 a^2 t} \cos \alpha x \, d\alpha = \frac{\sqrt{\pi}}{a\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}.$$

Thus, a solution of Eq. (6) is the function

$$\delta(P, P', t-t') = \frac{1}{2\pi} I(x-x', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{r^2}{4a^2(t-t')}} \qquad (6')$$

$$(r = |x-x'|, \quad t > t'),$$

where x' is the coordinate of a fixed point P' and t' is a real parameter. Analogously, solutions of the two- and three-dimensional homogeneous heat equations are

$$e^{-(\alpha^2+\beta^2)a^2 t} \cos \alpha x \cos \beta y,$$

$$e^{-(\alpha^2+\beta^2+\gamma^2)a^2 t} \cos \alpha x \cos \beta y \cos \gamma z,$$

respectively, where α, β, and γ are arbitrary real parameters. The functions

$$\delta(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^2 e^{-\frac{r^2}{4a^2(t-t')}} \qquad (6'')$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2}, \quad t > t'),$$

$$\delta(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^3 e^{-\frac{r^2}{4a^2(t-t')}} \qquad (6''')$$

$$(r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2},\ t > t'),$$

are also solutions of these two equations respectively. Here, P' is a fixed point with coordinates x' and y' in the two-dimensional case and x', y', and z' in the three-dimensional case, and t' is a real parameter. It is easy to see that each of the functions $\delta(P, P', t-t')$ defined by Eqs. (6′), (6″), and (6‴) is a δ-like function with parameter $\alpha = t - t'$. Indeed, since

$$\int_{-\infty}^{\infty} \delta(P, P', t-t')\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4a^2(t-t')}} d\frac{x-x'}{2a\sqrt{t-t'}} = 1,$$

we see that these functions satisfy the conditions of Example 5 in the preceding subsection and hence are δ-like functions with respect to the set M^* of all continuous and bounded functions in the space in question.

We shall also say that the function $\delta(P, P', t-t')$ is a function of an instantaneous unit source at the point P' at the instant t' for the heat equation. Let us give a physical interpretation of it.

For definiteness, let us take the three-dimensional case. Suppose that this space is initially at zero temperature. Then, suppose that at an instant t' a unit amount of heat is produced instantaneously as a result of some external influence at a point P' in that space. As a result of this added heat, the temperature at points in the space will increase. Let us denote this temperature by $u(P, t)$. From the requirement that the amount of heat be conserved, we have

$$\iiint_{-\infty}^{\infty} \gamma\rho u(P, t)\, d\tau = 1,$$

where ρ and γ denote, respectively, the density and specific heat of the medium filling the space. Since the function $u(P, t)$ must satisfy the homogeneous heat equation, we may write

$$u(P, t) = \frac{1}{\gamma\rho}\, \delta(P, P', t-t').$$

Thus, the function

$$\frac{1}{\gamma\rho}\, \delta(P, P', t-t')$$

represents the temperature at the point P at the instant t produced by an instantaneous unit heat source at the point P' at the instant t'

under the hypothesis that the temperature was zero everywhere at the initial instant t'.

It is interesting to note that, if a unit source at a point P' acts continuously beginning at some initial instant t_0 prior to t, we have the following expression for the temperature produced:

$$u(P, t) = \frac{1}{\gamma\rho} \int_{t_0}^{t} \delta(P, P', t - t')\, dt' =$$

$$= \frac{1}{\gamma\rho} \int_{t_0}^{t} \left(\frac{1}{2a\sqrt{\pi(t-t')}} \right)^n e^{-\frac{r^2}{4a^2(t-t')}}\, dt',$$

where r is the distance between the points P and P' and n is the dimension of the space. For example, in three-dimensional space, if we set

$$\frac{1}{\sqrt{t-t'}} = \tau,$$

we obtain

$$u(P, t) = \left(\frac{1}{2a\sqrt{\pi}} \right)^3 \frac{2}{\gamma\rho} \int_{\frac{1}{\sqrt{t-t_0}}}^{\infty} e^{-\frac{r^2}{4a^2}\tau^2}\, d\tau =$$

$$= \frac{1}{2kr\pi^{3/2}} \int_{\frac{r}{2a\sqrt{t-t_0}}}^{\infty} e^{-\xi^2}\, d\xi \qquad \left(a^2 = \frac{k}{\gamma\rho} \right),$$

or

$$u(P, t) = \frac{1}{4\pi kr} \left[1 - \Lambda\left(\frac{r}{2a\sqrt{t-t_0}} \right) \right],$$

where $\Lambda(x)$ is the so-called *error integral*

$$\Lambda(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2}\, dx = erfx.$$

As $t - t_0 \to \infty$, we have

$$u(P, t) \to u(P) = \frac{1}{4\pi kr}.$$

This means that the function due to a unit heat source in three-dimensional space with $k=1$ acting continuously for an infinite length of time degenerates into a unit source function for Laplace's equation or, what amounts to the same thing, into a unit source of steady heat flow.

If we examine the function of a unit instantaneous source for the heat-flow equation, we immediately conclude that heat is propogated with infinite velocity in the space.*

Suppose that an instantaneous heat source of intensity $1/h$ is placed at a point P'' at the instant t' and that a heat sink of the same intensity is simultaneously placed at a point P', h being the distance between P' and P''. Then, the temperature at a point P at an instant t will be given, if we assume that $\gamma\rho=1$ for the space in question, by

$$u(P, t) = \frac{1}{h}[\delta(P, P'', t-t') - \delta(P, P', t-t')].$$

If we let the point P'' approach the point P' along a segment $\vec{n}'$ directed away from the point P' and if we denote the limiting value of the temperature by $\mu(P, P', t-t')$, we obtain

$$\mu(P, P', t-t') = \frac{\partial}{\partial n'}\delta(P, P', t-t'). \tag{7}$$

The function defined by Eq. (7) is called the function of an instantaneous unit dipole for the heat equation at the point P' at the instant t' with axis $\vec{n}'$. This function may be written in the form

$$\mu(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^n \frac{r\cos\varphi'}{2a(t-t')} e^{-\frac{r^2}{4a^2(t-t')}}, \tag{7'}$$

where φ' is the angle between the vectors $\vec{n}'$ and $\vec{P'P}$ (see Fig. 18). We note that in the one-dimensional case, $\cos\varphi' = \pm 1$.

The function of an instantaneous unit source and the function of an instantaneous unit dipole play the same role in the study of the heat equation as the functions of a unit source and of a unit dipole play in potential theory.

The instantaneous unit source function immediately enables us to find the solution of the Cauchy problem for the homogeneous heat equation

*We note that in actuality this is not the case. Experiment shows that heat is propogated with a finite velocity. This discrepancy between theory and experiment is explained by the fact that in setting up the heat equation, we do not take into account the interaction of the molecules of the substance. However, one should not conclude from this that the heat equation derived above does not give an accurate description of the heat-flow process in general.

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0, \quad u\Big|_{t=0} = \psi(P),$$

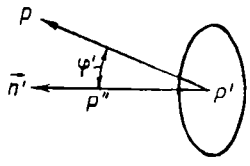

Fig. 18

in closed form, where Δ is the n-dimensional Laplace operator and $\psi(P)$ is a continuous bounded function of the point P in n-dimensional space (for $n=1, 2, 3$). Specifically, the solution of this problem exists and is given in one-, two-, and three-dimensional cases by the equations

$$u(x, t) = \int_{-\infty}^{\infty} \psi(x^2)\frac{1}{2a\sqrt{\pi t}} e^{-\frac{r^2}{4a^2t}}\, dx' \quad (r = |x - x'|), \tag{8}$$

$$u(x, y, t) = \int_{-\infty}^{\infty}\int \psi(x', y')\left(\frac{1}{2a\sqrt{\pi t}}\right)^2 e^{-\frac{r^2}{4a^2t}}\, dx'\, dy', \tag{8'}$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2}),$$

$$u(x, y, z, t) = \int_{-\infty}^{\infty}\int\int \psi(x', y', z')\left(\frac{1}{2a\sqrt{\pi t}}\right)^3 e^{-\frac{r^2}{4a^2t}}\, dx'\, dy'\, dz' \tag{8''}$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}),$$

respectively. The fact that the right-hand members of these equations satisfy the homogeneous heat equation in their respective spaces follows from the fact that the function $\delta(P, P', t)$ possesses this property. The fact that the right-hand members satisfy the initial conditions follows immediately from the property of the δ-function expressed by formula (5'') if we remember that $\delta(P, P', t) = \delta(P', P, t)$. Formulas (8), (8'), and (8'') are called *Poisson's formulas*.

It is natural to seek a solution of the Cauchy problem for the nonhomogeneous heat-flow equation with homogeneous initial conditions

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(P, t)}{k}, \quad u\Big|_{t=0} = 0 \tag{9}$$

in the form

$$u_0(P, t) = a^2 \int_0^t \overbrace{\int_{-\infty}^{\infty} \cdots \int}^{n} \frac{f(P', t')}{k}\, \delta(P, P', t - t')\, d\tau'\, dt', \tag{9'}$$

where the inner integral amounts to integration over the n-dimensional space, n being the number of space coordinates in the heat equation. The chief point is that the desired solution of the Cauchy problem $u_0(P, t)$ can be represented as the temperature at points in the space at the instant t caused by instantaneous heat sources with

intensity $f(P', t')$ acting continuously from the time $t=0$ until the instant $t'=t$, under the hypothesis that the temperature is zero everywhere at the initial instant. We shall now give a mathematical justification of the suggested result. We have

$$\Delta u_0 - \frac{1}{a^2}\frac{\partial u_0}{\partial t} = a^2 \int_0^t \overbrace{\int_{-\infty}^{\infty} \cdots \int}^{n} \frac{f(P', t')}{k}\left(\Delta\delta - \frac{1}{a^2}\frac{\partial \delta}{\partial t}\right) d\tau'\, dt' -$$

$$- \lim_{t' \to t} \overbrace{\int_{-\infty}^{\infty} \cdots \int}^{n} \frac{f(P', t')}{k}\, \delta(P, P', t - t')\, d\tau' = -\frac{f(P, t)}{k}.$$

Consequently, the integral (9′) does give a solution of the Cauchy problem for the nonhomogeneous heat equation with homogeneous initial conditions (9). This solution may be written in the form

$$u_0(x, t) = a^2 \int_0^t \int_{-\infty}^{\infty} \frac{f(x', t')}{k} \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{r^2}{4a^2(t-t')}} dx'\, dt' \qquad (10)$$

$$(r = |x - x'|)$$

$$u_0(x, y, t) = a^2 \int_0^t \int_{-\infty}^{\infty}\int \frac{f(x', y', t')}{k}\left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^2 e^{-\frac{r^2}{4a^2(t-t')}} dx'\, dy'\, dt'$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2}), \qquad (10')$$

$$u_0(x, y, z, t) = a^2 \int_0^t \int_{-\infty}^{\infty}\int \frac{f(x', y', z', t')}{k}\left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^3 \times$$

$$\times e^{-\frac{r^2}{4a^2(t-t')}} dx'\, dy'\, dz'\, dt' \qquad (10'')$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

in the one-, two-, and three-dimensional cases respectively.

Poisson's formulas, together with these last formulas, completely solve the Cauchy problem for the heat-flow equation with one, two, or three space coordinates. These formulas enable us to write down explicitly the solution of the first and second boundary-value problems for the half-space $x>0$ if the right-hand member of the heat equation, the initial function $\psi(P)$, and the boundary function $\Phi(P, t)$ vanish at $x=0$. For this it will be sufficient to extend the initial function $\psi(P)$ and the right-hand member of the heat-flow equation throughout the half-space $x<0$ as an odd function in the case of the first problem and as an even function in the case of the second one, and then apply the formulas given above to the resulting Cauchy problem.

Obviously, these results can be extended without difficulty to the case of many-dimensional spaces.

5. Green's Function. Integral Representations of Solutions of the First, Second, and Third Boundary-Value Problems. Differentiability Properties of Solutions of the Heat-Flow Equation

Suppose that $u=u(P', t')$ and $v=v(P', t')$ are arbitrary functions that are twice continuously differentiable with respect to the space coordinates and once continuously differentiable with respect to the time t' in the closed $(n+1)$-dimensional region Ω (see Sect. 1). In those cases in which the region Ω is not a cylindrical one, that is, when it corresponds to the generalized first or generalized mixed boundary-value problem, we shall assume that every straight line parallel to the t'-axis has a finite number of points or straight line segments in common with the boundary of the region Ω. For definiteness, let us take $n=3$. In accordance with Ostrogradskiy's formula, we have

$$\int_{t_0}^{t}\iiint_{G_{t'}}(v\Delta u - u\Delta v)\,d\tau'\,dt' = -\int_{t_0}^{t}\iint_{S_{t'}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS'\,dt',$$

where $\vec{n}$ is the inner normal to the boundary $S_{t'}$ of the region $G_{t'}$. Integration by parts yields

$$\int_{t_0}^{t}\iiint_{G_{t'}}\frac{\partial uv}{\partial t'}\,d\tau'\,dt' = \iiiint_{\Omega(t_0<t'<t)}\frac{\partial uv}{\partial t'}\,d\tau'\,dt' = -\iiint_{S_L+G_{t_0}+G_t} uv\cos(n^*, t)\,dS',$$

where $\vec{n^*}$ is the inner normal to the boundary of the region Ω for $t_0<t'<t$.* If we multiply this equation by $1/a^2$ and substract the result from the preceding equation, we obtain the formula

*The direction cosines of the inner normal $\vec{n}^*$ to the surface $\varphi(x_1, x_2, \ldots, x_n)=0$ can, by definition, be written in the form

$$\cos(n^*, x_l) = -\frac{\partial\varphi}{\partial x_l}\left(\sum_{k=1}^{n}\varphi_{x_k}^2\right)^{-\frac{1}{2}},$$

and the element of area of this surface is given by the equation

$$dS' = \frac{1}{|\cos(n^*, x_n)|}\,dx_1\,dx_2, \ldots, dx_{n-1}.$$

$$\int_{t_0}^{t}\iiint_{G_{t'}}(vLu-uMv)\,d\tau'\,dt'=-\int_{t_0}^{t}\iint_{S_{t'}}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)dS'\,dt'+$$

$$+\iiint_{G_{t_0}}\frac{uv}{a^2}\,dS'-\iiint_{G_t}\frac{uv}{a^2}\,dS'+\iiint_{S_L}\frac{uv}{a^2}\cos(n^*,t)\,dS', \tag{11}$$

where $Mv=\Delta v+\frac{1}{a^2}\frac{\partial v}{\partial t'}$ is the so-called adjoint operator in the sense of Lagrange to the operator $Lu=\Delta u-\frac{1}{a^2}\frac{\partial u}{\partial t'}$ (where $\Delta=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}$). Similarly, for $n=2$, by applying Green's formula, we obtain

$$\int_{t_0}^{t}\iint_{G_{t'}}(vLu-uMv)\,d\tau'\,dt'=-\int_{t_0}^{t}\int_{S_{t'}}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds'\,dt'+$$

$$+\iint_{G_{t_0}}\frac{uv}{a^2}\,dS'-\iint_{G_t}\frac{uv}{a^2}\,dS'+\iint_{S_L}\frac{uv}{a^2}\cos(n^*,t)\,dS'. \tag{11'}$$

In the case of a single space coordinate, integration by parts yields

$$\int_{G_{t'}}\left(v\frac{\partial^2 u}{\partial x^2}-u\frac{\partial^2 v}{\partial y'^2}\right)dx'=v\frac{\partial u}{\partial x'}\Big|_{x_1}^{x_2}-u\frac{\partial v}{\partial x_1}\Big|_{x_1}^{x_2}=$$

$$=-\left[\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_1}+\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_2}\right],$$

where x_1 and x_2 are, respectively, the left and right end-points of the segment $G_{t'}$. Thus, we obtain

$$\int_{t_0}^{t}\int_{G_{t'}}\left(v\frac{\partial^2 u}{\partial x'^2}-u\frac{\partial^2 v}{\partial x'^2}\right)dx'\,dt'=-\int_{t_0}^{t}\left[\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_1}+\right.$$

$$\left.+\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_2}\right]dt'.$$

Furthermore, we have

$$\frac{1}{a^2}\iint_{\Omega(t_0<t'<t)}\frac{\partial uv}{\partial t'}\,dx'\,dt'=-\frac{1}{a^2}\int_{S_L+G_{t_0}+G_t}uv\cos(n^*,t)\,dS'.$$

If we substract this equation from the preceding one, we obtain the following formula:

$$\int_{t_0}^{t}\int_{G_{t'}}(vLu-uMv)\,dx'\,dt'=-\int_{t_0}^{t}\left[\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_1}+\right.$$

$$\left.+\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)_{x_2}\right]dt'+\int_{G_{t_0}}\frac{uv}{a^2}\,dx'-\int_{G_t}\frac{uv}{a^2}\,dx'+$$

$$+\int_{S_L}\frac{uv}{a^2}\cos(n^*,t)\,ds'. \qquad (11'')$$

We shall refer to formulas (11), (11′), and (11″) as the integral formulas for the heat-equation operator with three, two, and one space coordinate, respectively.

It is easy to see that $\delta(P, P', t-t')$ as a function of the point P' and the parameter t' satisfies the equation

$$Mv=\Delta v+\frac{1}{a^2}\frac{\partial v}{\partial t'}=0,$$

where Δ is the n-dimensional Laplacian operator. Let us take the function $\delta(P, P', t-t')$ as v in formulas (11), (11′), and (11″). Then, from the properties of the δ function, we have in the three cases, respectively, the following equations:

$$\iiint_{G_{t-0}}uv\,dS'=u(P,t);\quad \iint_{G_{t-0}}uv\,dS'=u(P,t);\quad \int_{G_{t-0}}uv\,ds'=u(P,t). \qquad (12)$$

For every function $u=u(P, t)$ with continuous second derivatives with respect to the space variables, and continuous first derivative with respect to time, in the closed region Ω (for $t_0<t'<t$), we obtain from (11), (11′), (11″), by virtue of Eqs. (12), the following integral representation:

$$u(P,t)=-a^2\int_{t_0}^{t}\overbrace{\int\cdots\int}^{n-1}_{S_{t'}}\left(v\frac{\partial u}{\partial n'}-u\frac{\partial v}{\partial n'}\right)dS'\,dt'+\overbrace{\int\cdots\int}^{n}_{G_{t_0}}uv\,d\tau'+$$

$$+\overbrace{\int\cdots\int}^{n}_{S_L}uv\cos(n^*,t)\,dS'-a^2\int_{t_0}^{t}\overbrace{\int\cdots\int}^{n}_{G_{t'}}vLu\,d\tau'\,dt'\quad(v=\delta(P,P',t-t')), \qquad (13)$$

where n is the number of space coordinates, $Lu=\Delta u-\frac{1}{a^2}\frac{\partial u}{\partial t}$, Δ is the n-dimensional Laplace operator with respect to the coordinates of the point P' (for $n=1, 2, 3$), and $\vec{n}'$ is the inner normal to the

boundary $S_{t'}$ of the region $G_{t'}$ at the point P'. In the case of $n = 1$, the integral over $S_{t'}$ should be replaced by the expression

$$\left(v\frac{\partial u}{\partial n'} - u\frac{\partial v}{\partial n'}\right)_{S_{t'}} = \left(v\frac{\partial u}{\partial x'} - u\frac{\partial v}{\partial x'}\right)_{x'=x_1} - \left(\frac{\partial u}{\partial x'} - u\frac{\partial v}{\partial x'}\right)_{x'=x_2},$$

where x_1 and x_2 are the left and right end-points of the interval.

It follows from formula (13) that every solution of the homogeneous heat-flow equation which has continuous second derivatives with respect to the space coordinates and continuous derivative with respect to t is an analytic function of the space coordinates and has derivatives of all orders with respect to t.

Let us show this, confining ourselves for brevity to the one-dimensional case.

Suppose that $x=\xi$, $t=\tau$ is a point in a neighborhood of which $u(x, t)$ is a solution of the homogeneous heat-flow equation in a single space coordinate. Suppose that ξ_1 and ξ_2 are fixed values of the variable x and that ξ, $\xi_1<\xi<\xi_2$ sufficiently close to the point ξ. Taking $t_0<\tau$, we have in accordance with formula (13)

$$u(x, t) = -a^2\int_{t_0}^{t}\left[\left(v\frac{\partial u}{\partial x'} - u\frac{\partial v}{\partial x'}\right)_{\xi_1} - \left(v\frac{\partial u}{\partial x_1'} - u\frac{\partial v}{\partial x'}\right)_{\xi_2}\right]dt' +$$

$$+\int_{\xi_1}^{\xi_2} u(x', t_0)\, v(x, x', t-t_0)\, dx'$$

$$\left(v = \delta(x, x', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}}\, e^{-\frac{(x-x')^2}{4a^2(t-t')}}\right).$$

Each of the integrals resulting from differentiating under the integral sign with respect to x in this expression converges uniformly with respect to the parameter x as x varies in a neighborhood of the point ξ. Making use of the fact that the integrands in all these integrals are analytic functions of x in a neighborhood of the point ξ, we conclude that these integrals and the function $u(x, t)$ are analytic functions of the variable x. Furthermore, it is easy to see that each of the integrals in the right-hand member of the last equation can be differentiated infinitely many times under the integral sign with respect to t. This means that the function $u(x, t)$ has derivatives of all orders with respect to the variable t.

Suppose that $g(P, P', t-t')$ as a function of the point P and the time t is a solution of the homogeneous heat-flow equation. Suppose that it is twice continuously differentiable with respect to P and once with respect to t everywhere in the region Ω, and that it is continuous in the closed region Ω. Then, if the function

$$G(P, P', t-t') = \delta(P, P', t-t') + g(P, P', t-t') \qquad (14)$$

vanishes for $t - t' = +0$ in the region $G_{t'}$, outside the point $P = P'$ and if it satisfies the homogeneous boundary conditions of one or the other of the boundary-value problems for the heat-flow equation for the region Ω on its lateral surface S_L, we shall say that this function is a Green's function of the corresponding boundary-value problem for the region Ω. If the region Ω is cylindrical, we shall also speak of the corresponding Green's function for the region G constituting the lower base of the region Ω.

The Green's function has a simple physical meaning. In each of these boundary-value problems, it represents the temperature at a point P at an instant t caused by an instantaneous unit source of heat at the point P' at the instant t' under the condition that, on the boundary S_t of the region of space at all subsequent instants ($t > t'$), the physical conditions corresponding to homogeneous boundary conditions of the boundary-value problem in question are satisfied, and at the instant t' the temperature of points in the region $G_{t'}$ is zero.

If in addition we assume that the Green's function, as a function of P and t, satisfies the conditions under which the integral formula for the operator heat-equation is applicable, it will be symmetric with respect to the points P and P', i.e.,

$$G(P, P', t) = G(P', P, t). \tag{14'}$$

Proof: In Eqs. (11), (11′), and (11″) [corresponding to the three-, two-, and one-dimensional cases], we set $t = 0$, $u(P', t') = G(P', P_1, t')$ and $v(P', t') = G(P', P_2, t - t')$, where P_1 and P_2 are fixed points. Then, for the operators Lu and Mv with respect to t' and the coordinates of the point P', we have the equations $Lu = 0$ and $Mv = 0$. Because of the boundary conditions, all the integrals in the right-hand members of Eqs. (11), (11′), and (11″) vanish except the integrals over G_{t_0} and G_t. For the first of these, we have $v(P_1, 0) = G(P_1, P_2, t)$ and for the second, we have $u(P_2, t) = G(P_2, P_1, t)$ since $g|_{t-t'=0} = 0$. Consequently, $G(P, P', t) = G(P', P, t)$. From this it follows that the Green's function with respect to the coordinates of the point P' possesses the same properties as it does with respect to the coordinates of the point P. In particular, since the Green's function depends on $t - t'$, it must satisfy the equation $Mv = 0$, which is adjoint to the heat equation, in the primed variables.

If we now assume that the solution $u = u(P, t)$ of the boundary-value problem for the heat-flow equation (1) satisfies the conditions of applicability of the integral formula for the heat-flow equation operator, we can immediately obtain explicit formulas for this equation from formula (13). Specifically, if we set $v = G$ in the case of the first (generalized or not), second, and third boundary-value problems, we obtain, by virtue of the homogeneous boundary conditions for the Green's function, the following three equations respectively:

$$u(P, t) = a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n-1}_{S_{t'}} \Phi(P', t') \frac{\partial G}{\partial n'} dS' dt' + \overbrace{\int \cdots \int}^{n}_{G_{t_0}} \psi(P') G d\tau' + \\ + a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n}_{G_{t'}} \frac{f(P', t')}{k} G d\tau' dt', \tag{15}$$

$$u(P, t) = a_2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n-1}_{S_{t'}} -G\Phi(P', t') dS' dt' + \overbrace{\int \cdots \int}^{n}_{G_{t_0}} \psi(P') G d\tau' + \\ + a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n}_{G_{t'}} \frac{f(P', t')}{k} G d\tau' dt', \tag{16}$$

$$u(P, t) = a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n-1}_{S_{t'}} -G\Phi(P', t) dS' dt' + \overbrace{\int \cdots \int}^{n}_{G_{t_0}} \psi(P') G d\tau' + \\ + a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n}_{G_{t'}} \frac{f(P', t')}{k} G d\tau' dt', \tag{17}$$

where $G = G(P, P', t - t')$ is the Green's function of the first (or generalized first) boundary-value problem for the region Ω in (15) and it is the Green's function of the second and third boundary-value problems for the region Ω in (16) and (17) respectively. An analogous formula is obtained from Eq. (13) for the mixed and generalized mixed boundary-value problems:

$$u(P, t) = a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n-1}_{S_{t'}} \chi\Phi(P', t') dS' dt' + \overbrace{\int \cdots \int}^{n}_{G_{t_0}} \psi(P') G d\tau' + \\ + a^2 \int_{t_0}^{t} \overbrace{\int \cdots \int}^{n}_{G_{t'}} \frac{f(P', t')}{k} G d\tau' dt', \tag{18}$$

where $\chi = \partial G / \partial n'$ on that portion of S_L on which the boundary conditions of the first boundary-value problem are satisfied and where $\chi = -G$ on that portion of S_L on which the boundary conditions of the second or third boundary-value problems are satisfied. Here, $G = G(P, P', t - t')$ is the Green's function of the mixed boundary-value problem or the generalized mixed boundary-value problem respectively.

In those cases in which the Green's function of the boundary-value problem in question is known explicitly, we should consider

the right-hand members of Eqs. (15)-(18) as the assumed result of solving the corresponding boundary-value problem for the heat equation (1) with the initial function $\psi(P)$ and boundary function $\Phi(P, t)$ since we do not know *a priori* that the solution satisfies all the conditions of applicability of the integral formula for the heat-equation operator, nor that the Green's function satisfies these conditions. Therefore, this assumed result must always be investigated by examining the limiting values of the right-hand members of Eqs. (15)-(18). As a rule, these additional investigations lead to an affirmative answer. Thus, the question of the solution of the boundary-value problems is largely reduced to finding the corresponding Green's functions. By virtue of Eq. (14), finding the Green's function of the boundary-value problem in question for the region Ω is reduced in the general case to solving the same boundary-value problem for the same region Ω and is hence associated with considerable difficulties. However, in a number of special cases which are quite important in applications, the Green's function has been found in explicit form.

6. Examples of the Construction of Green's Functions for Particular Regions

1. The Green's function for the first and second boundary-value problems for an n-dimensional half-space. Suppose that $P' = x'$ is a fixed point in the one-dimensional half-space $x>0$. The instantaneous unit source function at the point x'

$$\delta(P, P', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{(x-x')^2}{4a^2(t-t')}}$$

vanishes at $t-t'=+0$, $x \neq x'$ but represents some nonzero temperature at $x=0$, $t>t'$. To compensate for this nonzero temperature, it will obviously be sufficient to put an instantaneous unit sink at

$$-\delta(P, P', t-t') = -\frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{(x+x')^2}{4a^2(t-t')}},$$

the point $P_1' = -x'$, symmetric to the point P' about the coordinate origin. Thus, we obtain the Green's function of the first boundary-value problem for the one-dimensional half-space $x>0$ in the following form:

$$G(P, P', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} - e^{-\frac{(x+x')^2}{4a^2(t-t')}}\right]. \quad (19)$$

If we put an instantaneous unit source at the point $P_1' = -x'$ in the same one-dimensional half-space, we obtain the Green's function of the second boundary-value problem

$$G(P, P', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} + e^{-\frac{(x+x')^2}{4a^2(t-t')}}\right]. \quad (20)$$

Similarly, by placing instantaneous sources of intensity +1 and −1 at the points $P'(x', y')$ and $P_1'(-x', y')$, we obtain the Green's function of the first boundary-value problem for the two-dimensional half-space $x>0$

$$G(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^2\left[e^{-\frac{(x-x')^2+(y-y')^2}{4a^2(t-t')}} - e^{-\frac{(x+x')^2+(y-y')^2}{4a^2(t-t')}}\right]. \quad (19')$$

The Green's function of the first boundary-value problem for the three-dimensional half-space $x>0$ may be written in the form

$$G(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^3\left[e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4a^2(t-t')}} - e^{-\frac{(x+x')^2+(y-y')^2+(z-z')^2}{4a^2(t-t')}}\right]. \quad (19'')$$

If we replace instantaneous unit sinks with instantaneous unit sources in the preceding reasoning, we see that the Green's functions of the second boundary-value problem for the two- and three-dimensional half-spaces $x > 0$ are given by Eqs. (19′) and (19″) except that the second terms should have a plus sign.

Similarly, we can obtain the Green's function of the first boundary-value problem for the quadrant $x > 0$, $y > 0$. To do this, in addition to placing an instantaneous unit source at a point $P'(x', y')$ in this quadrant, we also place a unit instantaneous sink at the point $P_1'(-x', y')$, a unit instantaneous source at the point $P_2'(-x', -y')$, and a unit instantaneous sink at the point $P_3'(x', -y')$ (see Fig. 19). Placing these sources and sinks at the points mentioned will ensure that the temperature on the boundary of the first quadrant will be zero and that it will be zero away from the point $P=P'$ for $t-t'=+0$. Thus, the desired Green's function of the first boundary-value problem for the quadrant $x>0$, $y>0$ will be of the form

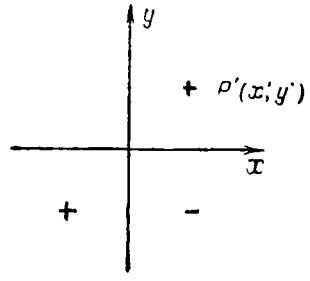

Fig. 19

$$G(P, P', t-t') = \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^2\left[e^{-\frac{(x-x')^2+(y-y')^2}{4a^2(t-t')}} - e^{-\frac{(x+x')^2+(y-y')^2}{4a^2(t-t')}} + e^{-\frac{(x+x')^2+(y+y')^2}{4a^2(t-t')}} - e^{-\frac{(x-x')^2+(y+y')^2}{4a^2(t-t')}}\right]. \quad (21)$$

It is easy to see that this method of instantaneous sources and sinks immediately enables us to find the Green's functions of the first and second boundary-value problems for an arbitrary plane angle of magnitude π/m, where m is any positive integer, and, in the three-dimensional case, for an arbitrary dihedral angle of the same magnitude. In all these cases, the Green's functions represent distributions of a finite number of unit instantaneous sources and sinks.

As examples, we recommend that the reader construct the Green's functions of the first and second three-dimensional boundary-value problems for the region $x>0$, $y>0$, $z>0$.

2. The Green's function of the first and second boundary-value problems for an n-dimensional layer. Let us consider separately the cases in which the layer $0<x<1/2$ is one-, two-, and three-dimensional. In the first case, this will be the interval $0<x<1/2$; in the second case, the strip $0<x<1/2$, $-\infty<y<\infty$; in the third case, the region bounded by the two planes $x=0$ and $x=1/2$, $-\infty<y<\infty$, $-\infty<z<\infty$. We first introduce an auxiliary function. The set of instantaneous unit sources at the instant $t'=0$, located at the points $x'=0$, $x'=\pm 1$, ± 2, ..., yields the following function in the case of one-dimensional space:

$$u(x, t) = \vartheta(x, t) = \sum_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-n)^2}{4a^2 t}}. \tag{22}$$

If we expand this function in a Fourier series, we obtain in the interval $(-1/2, 1/2)$

$$\vartheta(x, t) = A_0 + \sum_{1}^{\infty} A_n \cos 2n\pi x;$$

$$A_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \vartheta(x, t)\, dx, \qquad A_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \vartheta(x, t) \cos 2n\pi x\, dx.$$

Direct calculation yields

$$A_0 = \sum_{-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{(x-n)^2}{4a^2 t}} \frac{1}{2a\sqrt{\pi t}}\, dx = \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}\, dx =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = 1.$$

Furthermore,

$$A_m = \sum_{-\infty}^{\infty} 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{(x-n)^2}{4a^2 t}} \frac{1}{2a\sqrt{\pi t}} \cos(2m\pi x)\, dx =$$

$$= 2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{4a^2 t}} \frac{1}{2a\sqrt{\pi t}} \cos(2m\pi x)\, dx = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} \cos\left(2m\pi 2a\sqrt{t}\,\xi\right) d\xi.$$

But we obtained above

$$I = \int_{-\infty}^{\infty} e^{-a^2\alpha^2 t} \cos \alpha x \, d\alpha = \frac{\sqrt{\pi}}{a\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$$

Therefore, $A_m = 2e^{-4\pi^2 m^2 a^2 t}$. Thus, for the function (22), we have

$$\vartheta(x, t) = 1 + 2\sum_{1}^{\infty} e^{-4\pi^2 n^2 a^2 t} \cos 2n\pi x. \tag{22'}$$

If we set $\tau = 4\pi a^2 it$ (where $i = \sqrt{-1}$), we may write the function (22') in the form

$$\vartheta(x, t) = \vartheta(x/\tau) = 1 + \sum_{1}^{\infty} e^{i\pi\tau n^2} \cos 2n\pi x = \sum_{-\infty}^{\infty} e^{i\pi(n^2\tau + 2nx)}. \tag{22''}$$

The theta function $\vartheta(x/\tau)$ is one of the special elliptic functions and it possesses a number of remarkable properties. For example, if we take the expression

$$\frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} = \sqrt{\frac{i}{\tau}} e^{-\frac{\pi i x}{\tau}},$$

outside the summation sign in (22), we get

$$\vartheta(x, t) = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \left[1 + \sum_{1}^{\infty} e^{-\frac{n^2}{4a^2 t}} \cos 2n \frac{x}{4a^2 it}\right].$$

But, from (22''), the expression in the square brackets on the right-hand side of this last equation is the function $\vartheta[(x/\tau)/(-1/\tau)]$. Thus, we have the following transformation formula for the theta function:

$$\vartheta(x/\tau) = \sqrt{\frac{i}{\tau}} e^{-\frac{i\pi x}{\tau}} \vartheta\left(\frac{x}{\tau} \Big/ -\frac{1}{\tau}\right). \tag{23}$$

This formula is quite important in many problems in mathematics and physics.

By using the theta function $\vartheta(x, t)$, we can easily construct the Green's functions of the first and second boundary-value problems for an n-dimensional layer $0 < x < 1/2$. To obtain the Green's function of the first boundary-value problem in the one-dimensional case, we need only put instantaneous unit sources

$$\vartheta(x - x', t - t') = \sum_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t - t')}} e^{-\frac{(x - x' - n)^2}{4a^2(t - t')}} =$$

$$= 1 + 2\sum_{1}^{\infty} e^{-4\pi^2 n^2 a^2 (t - t')} \cos 2n\pi (x - x'),$$

at the points x', $x' \pm 1$, $x' \pm 2$, ... (for $0 < x' < 1/2$) and to put instantaneous unit sinks

$$-\vartheta(x + x', t - t').$$

at the points x', $-x' \pm 1$, $-x' \pm 2, \ldots$ Thus, the desired Green's function of the first boundary-value problem in the one-dimensional case (for the interval $0 < x < 1/2$) is

$$G(P, P', t - t') = \vartheta(x - x', t - t') - \vartheta(x + x', t - t'). \tag{24}$$

Analogously, in the two- and three-dimensional cases, by putting instantaneous unit sources at the points $P'(x' - n, y')$ and $P'(x' - n, y', z')$, respectively (for $n = 0, \pm 1, \ldots$) and instantaneous unit sinks at the point $P_1'(-x' - n, y')$ and $P_1'(-x' - n,$

y', z') (for $n = 0, \pm 1, \ldots$), we obtain the Green's functions of the first boundary-value problem for two- and three-dimensional layers $0 < x < 1/2$ in the forms

$$G(P, P', t-t') = \\ = [\vartheta(x-x', t-t') - \vartheta(x+x', t-t')] \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{(y-y')^2}{4a^2(t-t')}}, \tag{24'}$$

$$G(P, P', t-t') = \\ = [\vartheta(x-x', t-t') - \vartheta(x+x', t-t')] \left(\frac{1}{2a\sqrt{\pi(t-t')}}\right)^2 e^{-\frac{(y-y')^2+(z-z')^2}{4a^2(t-t')}}, \tag{24''}$$

respectively. Here, $\vartheta(x, t)$ is the function defined by Eq. (22) or, what amounts to the same thing, by Eq. (22').

If we replace sinks with sources in the above discussion, we see that the Green's function of the second boundary-value problem for the layer $0 < x < 1/2$ in the one-, two-, and three-dimensional cases is given respectively by (24), (24'), and (24'') except that in the right-hand member of each of these equations, the second term must be taken with a plus sign.

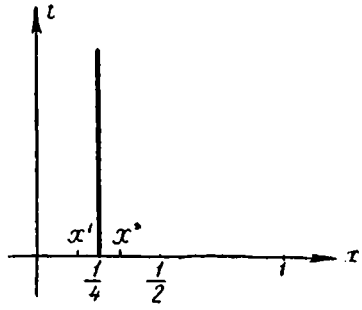

Fig. 20

Similarly, we may obtain the Green's function for an n-dimensional layer $0 < x < 1/4$ in the case of the mixed boundary-value problem if we have the values of the solution at $x=0$ and the values of its normal derivative at $x = 1/4$. Indeed, to ensure the vanishing of the Green's function at $x=0$, it will be sufficient in the one-dimensional case to place, in addition to the instantaneous sources at the points x', $x' \pm 1$, ... (for $0 < x' < 1/4$), instantaneous unit sinks at the points $-x'$, $-x' \pm 1$, ... To ensure the vanishing of the normal derivative of the Green's function at the point $x = 1/4$, we need only put instantaneous unit sources at the points $x^* = 1/2 - x'$, $x^* \pm 1$, $x^* \pm 2$, ... and instantaneous unit sinks at the points $-x^*$, $-x^* \pm 1$, $-x^* \pm 2$ ... (see Fig. 20). Distributing these unit instantaneous sources and sinks as indicated yields the Green's function of this mixed boundary-value problem for the one-dimensional layer (for the interval $0 < x < 1/4$):

$$G(P, P', t-t') = \vartheta(x-x', t-t') - \vartheta(x+x', t-t') + \\ + \vartheta\left(x+x'-\frac{1}{2}, t-t'\right) - \vartheta\left(x-x'+\frac{1}{2}, t-t'\right). \tag{25}$$

In the two- and three-dimensional cases, the Green's functions of this same boundary-value problem for the layers $0 < x < 1/4$ are respectively written in the forms

$$G(P, P', t-t') = \left[\vartheta(x-x', t-t') - \vartheta(x+x', t-t') + \right.$$

$$\left. + \vartheta\left(x+x'-\frac{1}{2}, t-t'\right) - \vartheta\left(x-x'+\frac{1}{2}, t-t'\right)\right] \frac{e^{-\frac{(y-y')^2}{4a^2(t-t')}}}{2a\sqrt{\pi(t-t')}}, \tag{25'}$$

$$G(P, P', t-t') = \left[\vartheta(x-x', t-t') - \vartheta(x+x', t-t') + \right.$$

$$\left. + \vartheta\left(x+x'-\frac{1}{2}, t-t'\right) - \vartheta\left(x-x'+\frac{1}{2}, t-t'\right)\right] \frac{e^{-\frac{(y-y')^2+(z-z')^2}{4a^2(t-t')}}}{(2a\sqrt{\pi(t-t')})^2}. \tag{25''}$$

We note that these formulas make it easy for us to give a representation for the Green's functions of the first and second boundary-value problems for an n-dimensional layer $0 < x < l/2$ and the Green's function of the above-mentioned mixed boundary-value problem for an n-dimensional layer $0 < x < l/4$, where l is an arbitrary positive number.

3. The Green's function of the third boundary-value problem for an n-dimensional half-space. Let us consider first the one-dimensional case, i.e., the case where the half-space coincides with the positive x-axis. We seek the Green's function $G = G(P, P', t-t')$ in the form

$$G = \frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} + Ae^{-\frac{(x+x')^2}{4a^2(t-t')}} + \right. \tag{26}$$

$$\left. + \int_0^\infty a(\xi)\, e^{-\frac{(x+x'+\xi)^2}{4a^2(t-t')}}\, d\xi\right],$$

where A is an undetermined constant and $a(\xi)$ is an unknown function. Noting that $\left.\frac{\partial G}{\partial n}\right|_{x=0} = \left.\frac{\partial G}{\partial x}\right|_{x=0}$, we have

$$\left.\frac{\partial G}{\partial n}\right|_{x=0} = \frac{1}{2a\sqrt{\pi(t-t')}}\left\{(1-A)\frac{x'}{2a^2(t-t')} + \right.$$

$$\left. + \int_0^\infty a(\xi)\frac{\partial}{\partial \xi} e^{-\frac{(x'+\xi)^2}{4a^2(t-t')}}\, d\xi\right\}.$$

If we integrate the last integral on the right by parts and set $x = 0$ in (26), we obtain

$$\left.\frac{\partial G}{\partial n}\right|_{x=0} = \frac{1}{2a\sqrt{\pi(t-t')}}\left\{(1-A)\frac{x'}{2a^2(t-t')} e^{-\frac{x'^2}{4a^2(t-t')}} + \right.$$

$$\left. + a(0)\, e^{-\frac{x'^2}{4a^2(t-t')}} - \int_0^\infty a'(\xi)\, e^{-\frac{(x'+\xi)^2}{4a^2(t-t')}}\, d\xi\right\},$$

$$G|_{x=0} = \frac{1}{2a\sqrt{\pi(t-t')}}\left\{(1+A)\, e^{-\frac{x'^2}{4a^2(t-t')}} + \int_0^\infty a(\xi)\, e^{-\frac{(x'+\xi)^2}{4a^2(t-t')}}\, d\xi\right\}.$$

If we multiply the first of these equations by unity and the second by a constant α and add the results, we obtain

$$\left(\frac{\partial G}{\partial n} - \alpha G\right)_{x=0} = \frac{1}{2a\sqrt{\pi(t-t')}}\left\{(1-A)\frac{x'}{2a^2(t-t')} e^{-\frac{x'^2}{4a^2(t-t')}} + \right.$$

$$\left. + [a(0) + \alpha(1+A)]\, e^{-\frac{x'^2}{4a^2(t-t')}} - \int_0^\infty (a'(\xi) + \alpha a(\xi))\, e^{-\frac{(x'+\xi)^2}{4a^2(t-t')}}\, d\xi\right\}.$$

Therefore, to satisfy the homogeneous boundary conditions of the third boundary-value problem, it will be sufficient to set

$$1 - A = 0,\ a(0) + \alpha(1+A) = 0,\ a'(\xi) + \alpha a(\xi) = 0.$$

Determining A, $a(\xi)$ and $a(0)$, we obtain

$$A = 1,\ a(0) = -2\alpha,\ a(\xi) = -2\alpha e^{-\alpha\xi}.$$

If we substitute these values of A and $a(\xi)$ into Eq. (26), we obtain for $\alpha = \text{const}$ the Green's function of the third boundary-value problem for the one-dimensional half-space $x>0$:

$$G(P, P', t-t') = \frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} + e^{-\frac{(x+x')^2}{4a^2(t-t')}} - \right.$$

$$\left. - 2a\int_0^\infty e^{-a\xi} e^{-\frac{(x+x'+\xi)^2}{4a^2(t-t')}} d\xi\right]. \tag{27}$$

If we replace the one-dimensional instantaneous unit sources in Eqs. (26) with two- and three-dimensional instantaneous unit sources, we obtain in the same way the Green's functions of the third boundary-value problem for the two- and three-dimensional half-spaces $x>0$ for $\alpha = \text{const}$ in the forms

$$G(P, P', t-t') = \frac{e^{-\frac{(y-y')^2}{4a^2(t-t')}}}{(2a\sqrt{\pi(t-t')})^2}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} + e^{-\frac{(x+x')^2}{4a^2(t-t')}} - \right.$$

$$\left. - 2a\int_0^\infty e^{-a\xi} e^{-\frac{(x-x'+\xi)^2}{4a^2(t-t')}} d\xi\right]. \tag{27'}$$

$$G(P, P', t-t') = \frac{e^{-\frac{(y-y')^2+(z-z')^2}{4a^2(t-t')}}}{(2a\sqrt{\pi(t-t')})^3}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} + \right.$$

$$\left. + e^{-\frac{(x+x'+\xi)^2}{4a^2(t-t')}} - 2a\int_0^\infty e^{-a\xi} e^{-\frac{(x+x'+\xi)^2}{4a^2(t-t')}} d\xi\right], \tag{27''}$$

respectively.

This method of instantaneous sources makes it easy, just as in the preceding cases, to find the Green's functions for quite a number of regions. However, this method cannot be used for regions bounded by portions of spheres in three-dimensional space or by arcs of circles in a plane.*

7. Reduction of Boundary-Value Problems for the Heat Equation to Boundary-Value Problems of the Simplest Type. Duhamel's Integral. Examples of Construction of Explicit Formulas for Solutions of Particular Boundary-Value Problems

The solution of every boundary-value problem for the heat-flow equation

*For certain other cases of construction of Green's functions in explicit form for the heat equation, and, in particular, for the Green's function of the third boundary-value problem for a line segment, see H. S. Carslaw, *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, 2nd ed., 1921, London, MacMillan.

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(P,t)}{k},\quad u|_{t=t_0} = \psi(P),\ \chi u|_{S_L} = \Phi(P,t) \qquad (28)$$
$$(P \in G_t,\ t > t_0),$$

where χu is a linear boundary operator (for example, $\chi u = \partial u/\partial n - \alpha u$ in the case of the third boundary-value problem) can be represented as the sum of solutions of the following two boundary-value problems:

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(P,t)}{k},\quad u|_{t=t_0} = \psi(P),\ \chi u|_{S_L} = 0 \qquad (28')$$
$$(P \in G_t,\ t > t_0),$$

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0,\quad u|_{t=t_0} = 0,\ \chi u|_{S_L} = \Phi(P,t) \qquad (28'')$$
$$(P \in G_t,\ t > t_0).$$

A solution of the boundary-value problem (28′) always exists if we know the Green's function, and is given, in accordance with formulas (15)-(18), by the equation

$$u(P,t) = \overbrace{\int\cdots\int}^{n}_{G} \psi(P')\,G(P,P',t-t_0)\,d\tau' +$$
$$+ a^2 \int_{t_0}^{t} \overbrace{\int\cdots\int}^{n}_{G_{t'}} \frac{f(P',t')}{k}\,G(P,P',t-t')\,d\tau'\,dt', \qquad (15')$$

where $G(P,\ P',\ t-t')$ should be understood as the Green's function for the corresponding boundary-value problem.

Indeed, by virtue of the properties of a Green's function, the right-hand member of Eq. (15′) satisfies the homogeneous boundary conditions on the surface S_L and is a solution of the heat-flow equation with the right-hand side equal to $-f(P,t)/k$; the second integral in the right-hand member of this equation vanishes at $t=t_0$ and the first integral is then equal to $\psi(P)$.

In particular, Eq. (15′), which we have proven, enables us to assert the existence of solutions and to write explicit formulas for these solutions in those cases of the boundary-value problem (28′) for which the Green's functions were found in the preceding section. For example, in the case of the first boundary-value problem (28′) for the one-dimensional half-space $x>0$, by taking the Green's function in the form (19), we obtain

$$u(x,t)=\int_0^\infty \psi(x')\,\frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}}-e^{-\frac{(x+x')^2}{4a^2(t-t')}}\right]dx'+$$

$$+a^2\int_{t_0}^{t}\int_0^\infty \frac{f(x',t')}{k}\,\frac{1}{2a\sqrt{\pi(t-t')}}\left[e^{-\frac{(x-x')^2}{4a^2(t-t')}}-e^{-\frac{(x+x')^2}{4a^2(t-t')}}\right]dx'\,dt'. \quad (29)$$

The boundary-value problem (28″) [i.e., any boundary-value problem for the homogeneous heat equation with homogeneous initial conditions] is characterized by the fact that the problem of finding the corresponding Green's function $G=\delta(P, P', t-t')+g$ reduces to this problem:

$$\Delta g-\frac{1}{a^2}\frac{\partial g}{\partial t}=0,\quad g|_{t=t'}=0,\quad \chi g|_{S_L}=-\chi\delta(P,P',t-t')|_{S_L}$$

$$(P\in G_t,\ t>t').$$

The boundary-value problem (28′) and the most general boundary-value problem (28) reduce to the boundary-value problem (28″). For this it is sufficient to set $u=u_1+u_2$, where u_2 is a new unknown function and u_1 is the solution of the Cauchy problem with initial function derived from the function $\psi(P)$ by extending it in some way to the whole space.

Therefore, investigation of the problem (28″) [i.e., investigation of all possible boundary-value problems] for the homogeneous heat-flow equation with homogeneous initial conditions is of fundamental interest.

First of all, let us consider that case of the boundary-value problem (28″) in which we know the corresponding Green's function. In this case, we may formally write the solution of the problem by means of one of the formulas (15), (16), (17), or (18), assuming that the last two integrals on the right-hand side of these formulas vanish. For example, the solution of the first boundary-value problem (28″) for the one-dimensional space $x>0$ can, by use of the Green's function (19), be obtained in the form

$$u(x,t)=a^2\int_{t_0}^{t}\Phi(t')\left.\frac{\partial G}{\partial x'}\right|_{x'=0}dt'=\int_{t_0}^{t}\frac{\Phi(t')}{t-t'}\,\frac{x}{2a\sqrt{\pi(t-t')}}\,e^{-\frac{x^2}{4a^2(t-t')}}dt'. \quad (29')$$

However, in contrast with the case of the problem (28′), with this method of obtaining solutions of the problem (28″), we need to make an additional investigation of the limiting values. As we stated above, such an investigation usually leads to an affirmative answer.

Let us now give certain principles that apply to the problem (28″).

Let us consider all the cases of the problem (28″) in which Ω is a cylindrical region and α is independent of the time t. This, for example, corresponds to the first and second boundary-value problems and also to the third and mixed problems with α independent of t. Let us show that in these cases, the solution of the problem (28″), leads to the solution of the same problem for the case in which the boundary function $\Phi(P, t)$ is independent of the time t, that is, it leads to the solution of a problem of the form

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = 0,\ u|_{t=t_0} = 0,\ \chi u|_S = \Phi(P)\ (P \in G, t > t_0). \tag{30}$$

Theorem. If $U(P, \lambda, t)$ is a solution of the boundary-value problem

$$\Delta U - \frac{1}{a^2}\frac{\partial U}{\partial t} = 0,\ U|_{t=0} = 0,\ \chi U|_S = \Phi(P, \lambda)\ (P \in G, t > t_0), \tag{30'}$$

where λ is a real parameter $t_0 < \lambda < t$, the solution of the boundary-value problem (28″) may be written in the form

$$u(P, t) = \frac{\partial}{\partial t}\int_{t_0}^{t} U(P, \lambda, t-\lambda)\,d\lambda = \int_{t_0}^{t}\frac{\partial}{\partial t}U(P, \lambda, t-\lambda)\,d\lambda =$$

$$= U(P, t_0, t-t_0) + \int_{t_0}^{t}\frac{\partial}{\partial \lambda}U(P, \lambda, t-\lambda')\Big|_{\lambda'=\lambda}\,d\lambda. \tag{31}$$

Proof: Since the function U is continuously differentiable with respect to t, we have

$$\frac{\partial}{\partial t}\int_{t_0}^{t} U(P, \lambda, t-\lambda)\,d\lambda = \int_{t_0}^{t}\frac{\partial}{\partial t}U(P, \lambda, t-\lambda)\,d\lambda +$$

$$+ U(P, \lambda, t-\lambda)|_{\lambda=t} = \int_{t_0}^{t}\frac{\partial}{\partial t}U(P, \lambda, t-\lambda)\,d\lambda\ .$$

Consequently, the first and second expressions for $u(P, t)$, shown in Eq. (31) coincide. From the second expression for $u(P, t)$, we have $u|_{t=t_0} = 0$; from the first expression, we get

$$\chi u|_S = \frac{\partial}{\partial t}\int_{t_0}^{t}\chi U(P, \lambda, t-\lambda)|_S\,d\lambda\ .$$

Since $\chi U|_S = \Phi(P, \lambda)$, we have $\chi u|_S = \Phi(P, t)$. Furthermore, if we assume that $U(P, \lambda, t)$ and its partial derivative with respect to λ are continuous functions of λ, we have

$$\frac{d}{d\lambda} U(P, \lambda, t-\lambda) = \frac{\partial}{\partial \lambda} U(P, \lambda, t-\lambda')\Big|_{\lambda'=\lambda} - \frac{\partial}{\partial t} U(P, \lambda, t-\lambda) \cdot$$

Consequently,

$$u(P, t) = -U(P, \lambda, t-\lambda)\Big|_{t_0}^{t} + \int_{t_0}^{t} \frac{\partial}{\partial \lambda} U(P, \lambda, t-\lambda')\Big|_{\lambda'=\lambda} d\lambda =$$

$$= U(P, t_0, t-t_0) + \int_{t_0}^{t} \frac{\partial}{\partial \lambda} U(P, \lambda, t-\lambda')\Big|_{\lambda'=\lambda} d\lambda,$$

i.e., the third expression for $u(P, t)$ appearing in Eq. (31) is valid. Using this expression, we obtain

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} = \left(\Delta - \frac{1}{a^2}\frac{\partial}{\partial t}\right) U(P, t_0, t-t_0) +$$

$$+ \int_{t_0}^{t} \frac{\partial}{\partial \lambda}\left(\Delta - \frac{1}{a^2}\frac{\partial}{\partial t}\right) U(P, \lambda, t-\lambda')\Big|_{\lambda'=\lambda} d\lambda = 0.$$

This means that Eq. (31) does indeed determine the desired solution of the boundary-value problem (28″). The integrals in the right-hand member of Eq. (31) are known as *Duhamel's integrals*.

We shall give one more general formula for the integral representation of the solution of the problem (28″). Note first that the following equation holds for the solution of the boundary-value problem (30):

$$u(P, t) = V(P) - \int_G V(P') G(P, P', t-t_0)\, d\tau' -$$

$$- a^2 \int_{t_0}^{t} dt' \int_G \rho(P') G(P, P', t-t')\, d\tau', \qquad (32)$$

where $G(P, P', t-t')$ is the corresponding Green's function $\rho(P) = -\Delta V(P)$, and $V(P)$ is an arbitrary twice continuously differentiable function satisfying the boundary condition $\chi V|_S = \Phi(P)$. This is true because, by virtue of the properties of a Green's function, the

right-hand member of Eq. (32) vanishes at $t=t_0$, and the second and third terms in the right-hand member of Eq. (32) taken together satisfy the equation

$$\Delta u - \frac{1}{a^2} u_t = \rho(P).$$

We may now make the following assertion.

*The solution of the boundary-value problem (28″) has the following integral representation:**

$$u(P,t) = V(P,t) - \int_G V(P', t_0) G(P, P', t-t_0)\, d\tau' -$$

$$- a^2 \int_{t_0}^{t} d\lambda \int_G \left[\rho(P', \lambda) + \frac{1}{a^2} \frac{\partial V(P', \lambda)}{\partial \lambda} \right] G(P, P', t-\lambda)\, d\tau', \qquad (33)$$

where $G(P, P', t-t')$ is the corresponding Green's function, where $\rho(P, \lambda) = -\Delta V(P, \lambda)$, and where $V(P, \lambda)$ is an arbitrary twice continuously differentiable function of the point P that has a continuous derivative with respect to the parameter $\lambda > t_0$ and that satisfies the boundary condition $\chi V|_S = \Phi(P, \lambda)$.

Proof: On the basis of (32), the solution of the boundary-value problem (30′) may be written in the form

$$U(P, \lambda, t) = V(P, \lambda) - \int_G V(P', \lambda) G(P, P', t)\, d\tau' -$$

$$- a^2 \int_0^t dt' \int_G \rho(P', \lambda) G(P, P', t-t')\, d\tau',$$

and, in accordance with (31), the solution $u(P, t)$ of the problem (28″) is of the form

$$u(P,t) = \int_{t_0}^{t} \frac{\partial}{\partial t} U(P, \lambda, t-\lambda)\, d\lambda =$$

$$= \int_{t_0}^{t} \int_G - V(P', \lambda) \frac{\partial}{\partial t} G(P, P', t-\lambda)\, d\tau'\, d\lambda -$$

$$- a^2 \int_{t_0}^{t} \frac{\partial}{\partial t} \left[\int_0^{t-\lambda} \int_G \rho(P', \lambda) G(P, P', t-\lambda-t')\, d\tau'\, dt' \right] d\lambda. \qquad (31')$$

Integration of the first term in the right-hand member of this equation by parts yields

*As far as we know, this formula has not appeared before in the literature.

$$\int \left\{ V(P', \lambda) G(P, P', t-\lambda) \Big|_{t_0}^{t} - \int_{t_0}^{t} \frac{\partial V(P', \lambda)}{\partial \lambda} G(P, P', t-\lambda)\, d\lambda \right\} d\tau' =$$

$$= V(P, t) - \int_G V(P', t_0)\, G(P, P', t-t_0)\, d\tau' -$$

$$- \int_{t_0}^{t} \int_G \frac{\partial V(P', \lambda)}{\partial \lambda} G(P, P', t-\lambda)\, d\tau'\, d\lambda.$$

Integrating the second term in the right-hand member of Eq. (31′) by parts, we obtain

$$-a^2 \int_{t_0}^{t} \left[\rho(P, \lambda) + \int_0^{t-\lambda} \int_G \rho(P', \lambda) \frac{\partial}{\partial t} G(P, P', t-\lambda-t')\, d\tau'\, dt' \right] d\lambda =$$

$$= -a^2 \int_{t_0}^{t} \left[\rho(P, \lambda) - \int_0^{t-\lambda} \int_G \rho(P', \lambda) \frac{\partial G}{\partial t'}\, d\tau'\, dt' \right] d\lambda =$$

$$= -a^2 \int_{t_0}^{t} \left[\rho(P, \lambda) - \rho(P, \lambda) + \int_G \rho(P', \lambda)\, G(P, P', t-\lambda)\, d\tau' \right] d\lambda =$$

$$= -a^2 \int_{t_0}^{t} \int_G \rho(P', \lambda)\, G(P, P', t-\lambda)\, d\tau'\, d\lambda.$$

Substituting these equations into (31′), we see that formula (33) is valid.

Let us now prove the converse, specifically, that the right-hand member of Eq. (33), is under the assumptions made concerning the function $V(P, \lambda)$, a solution of the boundary-value problem (28″) corresponding to the given Green's function. By virtue of the properties of a Green's function and, in particular, of a δ-function, we have $u|_{t=t_0} = 0$. It follows from the conditions $\chi V|_S = \Phi(P, \lambda)$ and $\chi G|_S = 0$ that $\chi u|_S = \Phi(P, t)$. Furthermore, since

$$\Delta V = -\rho(P, \lambda), \qquad \left(\Delta - \frac{1}{a^2} \frac{\partial}{\partial t}\right) G = 0,$$

we obtain by differentiation

$$\left(\Delta - \frac{1}{a^2} \frac{\partial}{\partial t}\right) u = -\rho(P, t) - \frac{1}{a^2} \frac{\partial V(P, t)}{\partial t} +$$

$$+ \frac{\partial}{\partial t} \int_{t_0}^{t} \int_G \left[\rho(P', \lambda) + \frac{1}{a^2} \frac{\partial V(P', \lambda)}{\partial \lambda} \right] G(P, P', t-\lambda)\, d\tau'\, d\lambda = 0.$$

Thus, the solution of problem (28″) for a cylindrical region Ω (which corresponds to the first, second, third, and mixed boundary-value problems) in the case in which the Green's function is known can be written down explicitly not only with the aid of formulas (15)-(18) [in which only the first terms are assumed not identically zero] but also with the aid of formula (33). In the present case, formula (33) implies the existence of the desired solution of the problem (28″) without requiring any additional verifications.

It is characteristic that with a different choice of function $V(P, \lambda)$, formula (33) for the solution of a single boundary-value problem (28″) yields explicit expressions that are quite different in form. The identity of these expressions, which is not obvious at first glance, is established because of the uniqueness of the solutions given boundary-value problem (28″).

For example, the solution of the first boundary-value problem (28″) for the one-dimensional half-space $x>0$, which we obtained earlier in the form (29′), can also be obtained from formula (33) by setting, for example, $V(P, \lambda)=\Phi(\lambda)$. By taking the Green's function in the form (19) and remembering that $\rho(x, \lambda)=0$ in the present case, we obtain

$$u(x,t)=\Phi(t)-\int_0^\infty \frac{\Phi(t_0)}{2a\sqrt{\pi(t-t_0)}}\left[e^{-\frac{(x'-x)^2}{4a^2(t-t_0)}}-e^{-\frac{(x'+x)^2}{4a^2(t-t_0)}}\right]dx'-$$

$$-\int_{t_0}^{t}\int_0^\infty \frac{\Phi'(\lambda)}{2a\sqrt{\pi(t-\lambda)}}\left[e^{-\frac{(x'-x)^2}{4a^2(t-\lambda)}}-e^{-\frac{(x'+x)^2}{4a^2(t-\lambda)}}\right]dx'\,d\lambda.$$

Noting that

$$\int_0^\infty \frac{1}{2a\sqrt{\pi(t-\lambda)}}e^{-\frac{(x'-x)^2}{4a^2(t-\lambda)}}dx'=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2a\sqrt{t-\lambda}}}^{\infty}e^{-\xi^2}d\xi=$$

$$=\frac{1}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\xi^2}d\xi-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t-\lambda}}}e^{-\xi^2}d\xi\right]=$$

$$=\frac{1}{2}\left[1-\Lambda\left(\frac{x}{2a\sqrt{t-\lambda}}\right)\right],$$

where $\Lambda(x)$ is the error integral

$$\Lambda(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi,$$

and recalling that $\Lambda(-x) = \Lambda(x)$, we obtain

$$u(x,t) = \Phi(t) - \Phi(t_0)\,\Lambda\left(\frac{x}{2a\sqrt{t-t_0}}\right) - \int_{t_0}^{t} \Phi'(\lambda)\,\Lambda\left(\frac{x}{2a\sqrt{t-\lambda}}\right) d\lambda. \tag{29''}$$

As a second example of the application of formula (33), we present the formula for the solution of the first boundary-value problem (28″) for the one-dimensional layer $0<x<1/2$ (for the interval $0<x$ 1/2). If we set

$$\Phi(t) = \begin{cases} \Phi_1(t) & \text{for } x = 0, \\ \Phi_2(t) & \text{for } x = \frac{1}{2}, \end{cases}$$

we see that, for the function $V(P, \lambda)$, we may take

$$\Phi(x, \lambda) = \Phi_1(\lambda)(1 - 2x) + 2\Phi_2(\lambda)\,x.$$

If we now take the Green's function in the form (24) and remember that $\rho(x, \lambda) = 0$, we obtain the desired solution of the problem in the following form:

$$u(x,t) = \Phi_1(t)(1-2x) + 2\Phi_2(t)\,x -$$

$$- \int_0^{1/2} [\Phi_1(t_0)(1-2x') + 2x'\Phi_2(t_0)]\,[\vartheta(x-x', t-t_0) -$$

$$- \vartheta(x+x', t-t_0)]\,dx' - \int_{t_0}^{t}\int_{t_0}^{1/2} [\Phi'(\lambda)(1-2x') + 2x'\Phi'(\lambda)] \times$$

$$\times [\vartheta(x-x', t-\lambda) - \vartheta(x+x', t-\lambda)]\,dx'\,d\lambda, \tag{34}$$

where $\vartheta(x, t)$ is the theta function defined by Eq. (22) or, what amounts to the same thing, Eq. (22′).

In conclusion, we point out that, in all cases in which the Green's function is known, formula (33) yields an explicit expression for the solution of the boundary-value problem (28″), formula (15′) yields an explicit expression for the solution of the problem (28′), and the sum of these solutions yields an explicit expression for the corresponding problem (28), i.e., for any boundary-value problem for the heat-flow equation for a cylindrical region Ω with α independent of the time.

In particular, all this relates to those boundary-value problems for which explicit expressions for the Green's functions were found in the preceding section:

(1) to the first and second boundary-value problems for the n-dimensional half-space $x>0$ and for a quadrant in two- and an octant in three-dimensional spaces;

(2) to the first and second boundary-value problems for an n-dimensional-space layer;

(3) to the third boundary-value problem for an n-dimensional half-space $x>0$ with $\alpha=\text{const}$.

For these same cases, we may obtain explicit formulas for solving the boundary-value problem (28) without initial conditions. For this, we use formulas (33) and (15′) to write the solutions of the corresponding problems with initial conditions for $t=t_0$ and then let t_0 approach $-\infty$. Here, we recall that, in all the cases mentioned, the Green's function $G(P, P', t-t_0)$ converges uniformly to zero as $t_0 \to -\infty$. When we take the limit, formulas (33) and (15′) take the forms

$$u(P,t) = V(P,t) - a^2 \int_{-\infty}^{t} d\lambda \int_G \left[\rho(P',\lambda) + \right.$$
$$\left. + \frac{1}{a^2} \frac{\partial V(P,\lambda)}{\partial \lambda} \right] G(P, P', t-\lambda)\, d\tau', \tag{33′}$$

$$u(P,t) = a^2 \int_{-\infty}^{t} dt' \int_G \frac{f(P',t')}{k} G(P, P', t-t')\, d\tau', \tag{15″}$$

respectively. The solution of problem (28) without initial conditions is then represented in the form of the sum of the right-hand members of Eqs. (33′) and (15″). This passage to the limit is permissible under rather general hypotheses concerning the right-hand member of the heat equation, $-f(P, t)/k$, and for a rather arbitrary choice of the function $V(P, t)$.

For example, in the case of the first boundary-value problem without initial conditions for the one-dimensional half-space $x>0$, we have

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial u}{\partial t} = -\frac{f(x,t)}{k}, \quad u|_{x=0} = \Phi(t). \tag{35}$$

Formula (15′) becomes Eq. (29), and taking the limit as $t_0 \to -\infty$ with the hypothesis that $f(x, t)=0$ for $t<-N$ (where $N=\text{const}>0$) yields

$$u(x,t) = a^2 \int_{-N}^{t} \int_0^{\infty} \frac{f(x',t')}{k} \frac{1}{2a\sqrt{\pi(t-t')}} \left[e^{-\frac{(x-x')^2}{4a^2(t-t')}} - \right.$$
$$\left. - e^{-\frac{(x+x')^2}{4a^2(t-t')}} \right] dx'\, dt'. \tag{35′}$$

In this case, formula (33) becomes Eq. (29″) and taking the limit as $t_0 \to -\infty$ yields

$$u(x, t) = \Phi(t) - \int_{-\infty}^{t} \Phi'(\lambda)\, \Lambda\left(\frac{x}{2a\sqrt{t-\lambda}}\right) d\lambda. \qquad (35'')$$

Solution of the problem (35) for the one-dimensional half-space $x>0$ without initial conditions may be written in the form of the sum of the right-hand members of Eqs. (35′) and (35″).

8. Thermal Potentials. The Method of Integral Equations

The integrals

$$U(P, t) = \int_{t_0}^{t} \overbrace{\int\cdots\int}^{n-1}_{S} \sigma(P', t')\, a^2 \delta(P, P', t-t')\, dS'\, dt', \qquad (36)$$

$$V(P, t) = \int_{t_0}^{t} \overbrace{\int\cdots\int}^{n-1}_{S} \vartheta(P', t')\, a^2 \mu(P, P', t-t')\, dS'\, dt' =$$

$$= \int_{t_0}^{t} \overbrace{\int\cdots\int}^{n-1}_{S} \vartheta(P', t')\, a^2 \frac{\partial}{\partial n'} \delta(P, P', t-t')\, dS'\, dt' \qquad (37)$$

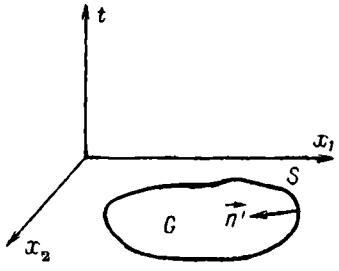

Fig. 21

are called the *thermal potentials* of a single and double layer, respectively. Here, S is the boundary of an n-dimensional domain G, which is the base of a cylindrical domain Ω, $\vec{n}'$ is the inner normal to S, and $\sigma(P', t')$ and $\vartheta(P', t')$ are continuous functions of t' and the point P'. For $n=1, 2, 3$, we speak respectively of the one-, two-, and three-dimensional thermal potentials. In the case of $n=1$, S consists of the left end-point x_1 and the right-end point x_2 of a straight line segment G parallel to the x-axis; in the two-dimensional case, S is a closed curve (see Fig. 21), and in the three-dimensional case, S is a two-dimensional surface. In the one-dimensional case, as always, by the integral over S, we mean the sum of the values of the integrand of the points x_1 and x_2.

Outside the region of integration, i.e., outside the lateral surface of the domain Ω, the thermal potentials are solutions of the homogeneous heat equation, with continuous derivatives of second order with respect to the space coordinates and of first order with respect to t. These thermal potentials vanish at $t=t_0$ and they possess properties analogous to the properties of the potentials of single and double layers.

Let us first consider the case of one-dimensional thermal potentials. Here, as a consequence of (6′) and (7′), the thermal potentials (36) and (37) take the forms

$$U(x, t) = \int_{t_0}^{t} \frac{a^2}{2a\sqrt{\pi(t-t')}} \left[\sigma(x_1, t')\, e^{-\frac{r_1^2}{4a^2(t-t')}} + \sigma(x_2, t)\, e^{-\frac{r_2^2}{4a^2(t-t')}}\right] dt', \qquad (36')$$

$$V(x, t)=\int_{t_0}^{t} \frac{1}{2a\sqrt{\pi(t-t')}}\left[\vartheta(x_1, t')\, r_1 \cos\varphi_1' \, e^{-\frac{r_1^2}{4a^2(t-t')}}+\right.$$

$$\left.+\vartheta(x_2, t')\, r_2 \cos\varphi_2' \, e^{-\frac{r_2^2}{4a^2(t-t')}}\right]\frac{dt'}{2(t-t')}, \tag{37'}$$

where $r_1=|x-x_1|$ and $r_2=|x-x_2|$; φ_1' and φ_2' are the angles between the vectors $\vec{n}'$ and $\overrightarrow{x_1x}$, on the one hand, and $\vec{n}'$ and $\overrightarrow{x_2x}$ on the other. (If $x_1<x<x_2$, then $\cos\varphi_1'=\cos\varphi_2'=1$; if $x<x_1$, then $\cos\varphi_1'=-1$ and $\cos\varphi_2'=1$; if $x>x_2$, then $\cos\varphi_1'=1$ and $\cos\varphi_2'=-1$.)

Theorem 1. *The thermal potential of a single layer (36') is a continuous function of x and t at points on the lateral surface of the domain Ω.*

Proof: Let (x^*, t^*) be a point on the lateral surface S_L, i.e., on the straight line $x=x_1$, $t>t_0$ or the straight line $x=x_2$, $t>t_0$. As an upper bound of the singular integral corresponding to this point, we can take the integral

$$M\int_{t^*-\delta}^{t}\frac{dt'}{\sqrt{t-t'}}, \qquad M=\text{const},$$

where δ is a sufficiently small number. But the last integral can be made arbitrarily small if the point (x, t) lies in a sufficiently small neighborhood of the point (x^*, t^*). This means that the integral (36') converges uniformly at the point (x^*, t^*) and is a continuous function at that point.

Theorem 2. *The thermal potential of a double layer (37') converges to continuous limit functions as (x,t) approaches the points (x^*,t) on the lateral surface of the domain Ω from the interior and the exterior along the normal $\vec{n}'$. These limiting functions are given respectively by*

$$V^+(x^*, t)=\frac{\vartheta(x^*, t)}{2}+V(x^*, t), \quad V^-(x^*, t)=-\frac{\vartheta(x^*, t)}{2}+V(x^*, t), \tag{38}$$

where $V(x^*,t)$ is the result of formally substituting x^* in the integral (37') in place of x.

Proof: For definiteness, let us take $x^*=x_1$. We then have

$$V(x^*, t)=V(x_1, t)=\int_{t_0}^{t}\vartheta(x_2, t')\frac{|x_1-x_2|}{2a\sqrt{\pi(t-t')}}\, e^{-\frac{(x_1-x_2)^2}{4a^2(t-t')}}\frac{dt'}{2(t-t')}.$$

If we now make the change of variables

$$\xi=\frac{|x_1-x_2|}{2a\sqrt{(t-t')}}, \quad t'=t-\frac{(x_1-x_2)^2}{4a^2\xi},$$

we obtain

$$V(x_1, t)=\frac{1}{\sqrt{\pi}}\int_{\frac{|x_1-x_2|}{2a\sqrt{t-t_0}}}^{\infty}\vartheta\left(x_2, t-\frac{(x_1-x_2)^2}{4a^2\xi}\right)e^{-\xi^2}d\xi.$$

It is obvious from this that $V(x_1, t)$ is a continuous function of t. If we set

$$A = \lim_{x \to x_1} [V(x, t) - V(x_1, t)] =$$

$$= \lim_{x \to x_1} \int_{t_0}^{t} \vartheta(x_1, t') r_1 \cos \varphi_1' \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}$$

and make the change of variables

$$\xi = \frac{r_1}{2a\sqrt{t-t'}}, \quad t' = t - \frac{r_1^2}{4a^2\xi^2},$$

we obtain

$$A = \lim_{x \to x_1} \frac{1}{\sqrt{\pi}} \int_{\frac{r_1}{2a\sqrt{t-t_0}}}^{\infty} \vartheta\left(x_1, -\frac{r_1^2}{4a^2\xi}\right) e^{-\xi^2} \cos \varphi_1' \, d\xi =$$

$$= \frac{\vartheta(x_1, t)}{\sqrt{\pi}} \cos \varphi_1' \int_0^{\infty} e^{-\xi^2} d\xi = \frac{\vartheta(x_1, t)}{2} \cos \varphi_1'.$$

But by letting x approach the point x_1 from the interior and the exterior, we obtain $\cos \varphi_1' = \pm 1$. Therefore, in view of the continuity of the function $V(x, t)$, we see that the theorem is valid for $x^* = x_1$. For $x^* = x_2$ the proof is the same.

Theorem 3. *The normal derivative of the thermal potential of a single layer $\partial U(x,t)/\partial n$ approaches continuous limit functions as (x,t) approaches points (x^*,t) of the lateral surface of the domain Ω from the interior and the exterior along the normal $\vec{n}$. These limit functions are given respectively by the equations*

$$\frac{\partial U^+(x^*, t)}{\partial n} = -\frac{\sigma(x^*, t)}{2} + \frac{\partial U(x^*, t)}{\partial n}, \quad \frac{\partial U^-(x^*, t)}{\partial n} = \frac{\sigma(x^*, t)}{2} + \frac{\partial U(x^*, t)}{\partial n}, \tag{39}$$

where $\partial U(x^*,t)/\partial n$ is the result of formally differentiating $U(x,t)$ under the integral sign at the point (x^*,t).

Proof: Taking $x^* = x_1$ for definiteness, we have

$$\frac{\partial U(x_1, t)}{\partial n} = \int_{t_0}^{t} \sigma(x_2, t') \frac{|x_1 - x_2|}{2a\sqrt{\pi(t-t')}} e^{-\frac{|x_1-x_2|^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}.$$

On the basis of the calculations made in the proof of Theorem 2, we then see that the function $\partial U(x_1, t)/\partial n$ is continuous with respect to t. It is then easy to see that

$$\lim_{x \to x_1} \left[\frac{\partial U(x, t)}{\partial n} - \frac{\partial U(x_1, t)}{\partial n} \right] =$$

$$= \lim_{x \to x_1} \int_{t_0}^{t} -\sigma(x_1, t') \frac{r_1 \cos \varphi_1'}{2a\sqrt{\pi(t-t)'}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}.$$

But, just as under the conditions of Theorem 2, the last of these integrals is equal to $-(1/2)\sigma(x_1, t) \cos \varphi_1'$. The validity of the theorem then follows for $x^* = x_1$. In the same way, we can prove the validity of the theorem for $x^* = x_2$.

One can easily see that Theorems 1, 2, and 3 are analogous to the theorems on the continuity of the potential of a single layer, on the limiting values of the potential of a double layer, and on the limiting values of the normal derivative of the potential of a single layer.

These three theorems can be extended to the case of two- and three-dimensional thermal potentials.

In the two-dimensional case, the thermal potential of a single layer (36) and the thermal potential of a double layer (37) take the forms

$$U(P, t) = \int_{t_0}^{t} \int_{S} \sigma(P', t')\, a^2 \left(\frac{1}{2a\sqrt{\pi(t-t')}} \right)^2 e^{-\frac{r^2}{4a^2(t-t')}}\, ds'dt', \qquad (36'')$$

$$V(P, t) = \int_{t_0}^{t} \int_{S} \vartheta(P', t) \left(\frac{1}{2a\sqrt{\pi(t-t')}} \right)^2 e^{-\frac{r^2}{4a^2(t-t')}}\, r\cos\varphi'\, ds' \frac{dt'}{2(t-t')}, \qquad (37'')$$

respectively, where $r = |P - P'| = \sqrt{(x-x')^2 + (y-y')^2}$ and φ' is the angle between the vectors $\overrightarrow{P'P}$ and the normal $\vec{n}'$ to the contour S at the point P' (Fig. 22). Formulas (38) and (39) take the forms

$$V^+(P^*, t) = \frac{\vartheta(P^*, t)}{2} + V(P^*, t) \quad V^-(P^*, t) = -\frac{\vartheta(P^*, t)}{2} + V(P^*, t), \qquad (38')$$

$$\frac{\partial U^+(P^*, t)}{\partial n} = -\frac{\sigma(P^*, t)}{2} + \frac{\partial U(P^*, t)}{\partial n}, \quad \frac{\partial U^-(P^*, t)}{\partial n} = \frac{\sigma(P^*, t)}{2} + \frac{\partial U(P^*, t)}{\partial n}, \qquad (39')$$

respectively, where P^* is a point on a closed Lyapunov curve S; $V(P^*, t)$ is the result of formally substituting the point P^* for P in the integral (37″); $\partial U(P^*, t)/\partial n$ is the result of formal differentiation of the function (36″) under the integral sign in the direction of the inner normal $\vec{n}$ to the contour S at the point P^*.

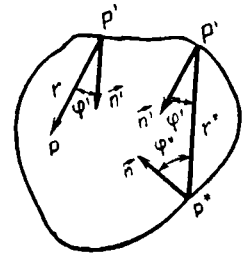

Fig. 22

Indeed, the continuity of the potential (36″) follows from the fact that the change of variables

$$\xi = \frac{r^2}{4a^2(t-t')}, \quad t' = t - \frac{r^2}{4a^2\xi} \qquad (40)$$

transforms this integral into an integral of the form

$$U(P, t) = \int_{S} ds' \int_{\frac{r^2}{4a^2(t-t')}}^{\infty} \sigma\left(P', t - \frac{r^2}{4a^2\xi}\right) e^{-\xi} \frac{1}{4\pi\xi}\, d\xi,$$

and this integral, as one can easily show, is a continuous function of P and t.

The integral (37″) is transformed by the change of variables (40) into the form

$$V(P, t) = \int_{S} \frac{\cos\varphi'}{2\pi r} ds' \int_{\frac{r^2}{4a^2(t-t')}}^{\infty} \vartheta\left(P', t - \frac{r^2}{4a^2\xi}\right) e^{-\xi} d\xi$$

and can be interpreted as the logarithmic potential of a double layer over the contour S with density depending on the point P as a parameter. Assuming this point P coincides with the point $P^* \in S$, let us consider the auxiliary potential

$$V_1(P, t) = \int_{S} \frac{\cos\varphi'}{2\pi r} ds' \int_{\frac{r^{*2}}{4a^2(t-t')}}^{\infty} \vartheta\left(P', t - \frac{r^{*2}}{4a^2\xi}\right) e^{-\xi} d\xi$$

$$(r^* = |P^* - P'|).$$

The density of this potential is a continuous function of the point P'. Therefore, if S is a Lyapunov curve, it follows from the theorem on the limiting values of the logarithmic potential of a double layer that

$$V_1^+(P^*, t) = \frac{\vartheta(P^*, t)}{2} + V_1(P^*, t), \quad V_1^-(P^*, t) = -\frac{\vartheta(P^*, t)}{2} + V_1(P^*, t).$$

Furthermore, when P lies on the normal to the Lyapunov curve S at the point P^*,

$$\int_e \left| \frac{\cos\varphi'}{2\pi r} \right| ds' < M, \quad M = \text{const}, \tag{41}$$

where e is a sufficiently small open subset of S that contains P^* (see Chapter 3, Sect. 4, subsection 6). Therefore, we have

$$\lim_{P\to P^*}(V(P, t) - V_1(P, t)) = \lim_{P\to P^*} \int_S \frac{\cos\varphi'}{2\pi r} ds' \left[\int_{\frac{r^2}{4a^2(t-t_0')}}^{\infty} \vartheta\left(P', t - \frac{r^2}{4a^2\xi}\right) e^{-\xi} d\xi - \right.$$

$$\left. - \int_{\frac{r^{*2}}{4a^2(t-t_0)}}^{\infty} \vartheta\left(P', t - \frac{r^{*2}}{4a^2\xi}\right) e^{-\xi} d\xi \right] = 0.$$

In view of (41) and the fact that $V_1(P^*, t) = V(P^*, t)$, we conclude from this that Eqs. (38′) are valid. Equations (39′) are proven analogously.

In the three-dimensional case, the thermal potential of a single layer (36) and that of a double layer (37) take the forms

$$U(P, t) = \int_{t_0}^{t} \iint_S \sigma(P', t')\, a^2 \left(\frac{1}{2a\sqrt{\pi(t-t')}} \right)^3 e^{-\frac{r^2}{4a^2(t-t')}} dS'\, dt', \tag{36‴}$$

$$V(P, t) = \int_{t_0}^{t} \iint_S \vartheta(P', t') \left(\frac{1}{2a\sqrt{\pi(t-t')}} \right)^3 \times$$

$$\times\, e^{-\frac{r^2}{4a^2(t-t')}} r\cos\varphi'\, dS' \frac{dt'}{2(t-t')}, \tag{37‴}$$

respectively, where $r = |P-P'|$ and φ' is the angle between the vector $\overrightarrow{P'P}$ and the normal $\overrightarrow{n'}$ to the surface S at the point P' (Fig. 22). If we assume that S is a Lyapunov surface, then formulas (38) and (39) take the following forms:

$$V^+(P^*, t) = \frac{\vartheta(P^*, t)}{2} + V(P^*, t), \quad V^-(P^*, t) = -\frac{\vartheta(P, t)}{2} + V(P^*, t) \tag{38″}$$

$$\frac{\partial U^+(P^*, t)}{\partial n} = -\frac{\sigma(P^*, t)}{2} + \frac{\partial U(P^*, t)}{\partial n},$$

$$\frac{\partial U^-(P^*, t)}{\partial n} = \frac{\sigma(P^*, t)}{2} + \frac{\partial U(P^*, t)}{\partial n}, \tag{39″}$$

respectively. Here P^* is a point of the closed Lyapunov surface S, $V(P^*, t)$ is the result of the formal substitution of P^* for P in the expression (37‴), and $\partial U(P^*, t)/\partial n$ is the

result of formal differentiation of the function (36‴) under the integral sign in the direction of the inner normal $\vec{n}$ to the surface S at the point P^*. We shall not give a detailed proof of formulas (38″) and (39″) or a proof of the continuity of the thermal potential of a single layer in the three-dimensional case since these proofs are completely analogous to the corresponding proofs in the two-dimensional case.

These theorems on thermal potentials make it possible to apply integral equations to the solution of the first, second, third, and mixed boundary-value problems for the heat equation. One must look for the solution of the first boundary-value problem in the form of a thermal potential of a double layer, the density of which must be determined from the integral equation obtained from the formulas for the limiting values of a thermal potential of a double layer. The solutions of the second and third boundary-value problems must be sought in the form of thermal potentials of a single layer, the densities of which must be determined from integral equations obtained from the formulas for the limiting values of the normal derivative of the thermal potential of a single layer. The solution of the mixed boundary-value problem must be sought in the form of a sum of the following two quantities: (1) the thermal potential of a double layer over that portion of the surface S on which the boundary conditions of the first boundary-value problem are given; and (2) the thermal potential of a single layer over that portion of the surface S on which the boundary conditions of the second or third boundary-value problems are given.

Let us use the method of integral equations to make a detailed investigation of particular boundary-value problems for the one-dimensional heat-flow equation.

In the case of the first boundary-value problem for the one-dimensional half-space $x>0$,

$$u_{xx} - \frac{1}{a^2} u_t = 0, \quad u|_{t=t_0} = 0, \quad u|_{x=0} = \Phi(t), \tag{42}$$

let us put the desired solution in the form of the potential of a double layer

$$u(x, t) = V(x, t) = \int_{t_0}^{t} \vartheta(t') \frac{x}{2a\sqrt{\pi (t-t')}} e^{-\frac{x^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}.$$

From the first of formulas (38), we obtain the equation for determining the density $\vartheta(t)$:

$$\Phi(t) = \frac{\vartheta(t)}{2} + V(0, t).$$

Noting that $V(0, t)=0$, we obtain from this $\vartheta(t)=2\Phi(t)$. Therefore, the desired solution of the boundary-value problem may be written in the form

$$u(x, t) = \int_{t_0}^{t} \Phi(t') \frac{x}{2a\sqrt{\pi (t-t')}} e^{-\frac{x^2}{4a^2(t-t')}} \frac{dt'}{t-t'}. \tag{42'}$$

This is simply formula (29′), which we obtained earlier from (15) by using the Green's function (19) [see Sect. 7]. However, only now can it be asserted that we have rigorously proved that this formula, and also formula (29″), yields the solution of the first boundary-value problem for the half-line $x>0$ since originally it was obtained without investigating the limiting values on the boundary.

We must seek a solution of the third boundary-value problem for the interval $[0, l]$:

$$u_{xx} - \frac{1}{a^2} u_t = 0, \quad u|_{t=0} = 0, \quad \left(\frac{\partial u}{\partial n} - \alpha_1 u\right)_{x=0} = \Phi_1(t), \left(\frac{\partial u}{\partial n} - \alpha_2 u\right)_{x=l} = \Phi_2(t), \tag{43}$$

where α_1 and α_2 are positive constants, in the form of a thermal potential of a single layer

$$u(x, t) = \int_{0}^{t} \frac{a^2}{2a\sqrt{\pi (t-t')}} \left[\sigma_1(t') e^{-\frac{x^2}{4a^2(t-t')}} + \sigma_2(t') e^{-\frac{(x-l)^2}{4a^2(t-t')}}\right] dt'. \tag{43'}$$

Keeping the first of formulas (39) in mind, we obtain the following system of integral equations for finding the functions $\sigma_1(t)$ and $\sigma_2(t)$:

$$\Phi_1(t) = -\frac{\sigma_1(t)}{2} + \int_0^t \frac{\sigma_2(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{l^2}{4a^2(t-t')}} \times$$

$$\times \left(\frac{l}{2(t-t')} - a^2\alpha_1\right) dt' - \alpha_1 a^2 \int_0^t \frac{\sigma_1(t')}{2a\sqrt{\pi(t-t')}} dt',$$

$$\Phi_2(t) = \frac{-\sigma_2(t)}{2} + \int_0^t \frac{\sigma_1(t)}{2a\sqrt{\pi(t-t')}} e^{-\frac{l^2}{4a^2(t-t')}} \times \qquad (43'')$$

$$\times \left(\frac{l}{2(t-t')} - a^2\alpha_2\right) dt' - \alpha_2 a^2 \int_0^t \frac{\sigma_2(t')}{2a\sqrt{\pi(t-t')}} dt'.$$

This system of Volterra integral equations always has a continuous solution, $\sigma_1(t)$ and $\sigma_2(t)$, as can be shown, for example, by the method of successive approximations. If we substitute this solution into the thermal potential of a single layer (43′), we obtain the desired solution of the third boundary-value problem.

Let us look at another example, that of the mixed boundary-value problem for the interval $[0, l]$:

$$u_{xx} - \frac{1}{a^2} u_t = 0, \quad u|_{t=0} = 0, \quad u|_{x=0} = \Phi_1(t), \quad \left(\frac{\partial u}{\partial n} - \alpha u\right)_{x=l} = \Phi_2(t), \qquad (44)$$

where α is a positive constant. We must seek the solution of this problem in the form of a sum of the thermal potentials of a double and a single layer:

$$u(x, t) = \int_0^t \vartheta(t') \frac{x}{2a\sqrt{\pi(t-t')}} e^{-\frac{x^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')} +$$

$$+ \int_0^t \sigma(t') a^2 \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{(x-l)^2}{4a^2(t-t')}} dt'. \qquad (44')$$

The first of formulas (38) for $x=0$ and the first of formulas (39) for $x=l$ yield the following equations for determining $\vartheta(t)$ and $\sigma(t)$:

$$\Phi_1(t) = \frac{\vartheta(t)}{2} + \int_0^t \sigma(t') a^2 \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{l^2}{4a^2(t-t')}} dt', \qquad (44'')$$

$$\Phi_2(t) = \frac{\sigma(t)}{2} - \alpha a^2 \int_0^t \frac{\sigma(t')\,dt'}{2a\sqrt{\pi(t-t')}} +$$

$$+ \int_0^t \vartheta(t') \frac{1}{2a\sqrt{\pi(t-t')}} e^{-\frac{l^2}{4a^2(t-t')}} \left(\frac{l^2}{2a^2} - \alpha l - 1\right) \frac{dt'}{2(t-t')}.$$

The solution of this system of Volterra integral equations can be obtained by the method of successive approximations. Since the kernels of the given system of integral equations (44″) depends on the difference $t-t'$, one can apply the method of operational calculus

(see Chapter 7, Sect. 2) to the problem of finding a representation for the solution of this system, and to the problem of finding the solution of the system of integral equations (43″).

9. Generalized Thermal Potentials. Examples of the Solution of Boundary-Value Problems With Moving Boundaries

By generalized thermal potentials of a single and double layer, we mean respectively the integrals

$$U(P, t) = \int_{t_0}^{t} \underbrace{\int \dots \int}_{S_{t'}} \sigma(P', t')\, a^2\, \delta(P, P', t-t')\, dS'\, dt', \tag{45}$$

$$V(P, t) = \int_{t_0}^{t} \underbrace{\int \dots \int}_{S_{t'}} \vartheta(P', t')\, a^2\, \mu(P, P', t-t')\, dS'\, dt' =$$

$$= \int_{t_0}^{t} \underbrace{\int \dots \int}_{S_{t'}} \vartheta(P', t')\, a^2\, \frac{\partial}{\partial n'}\, \delta(P, P', t-t')\, dS'\, dt', \tag{46}$$

where $S_{t'}$ is the boundary of the n-dimensional domain $G_{t'}$, which is the intersection of the domain Ω [noncylindrical, in contrast to the cases of (36) and (37)] by the plane $t=t'=\text{const}$, where $\vec{n'}$ is the inner normal to $S_{t'}$ and where $\sigma(P', t')$ and $\vartheta(P', t')$ are continuous functions of P' and t'. Here, in the case of $n=1, 2, 3$, we can speak of a one-, two-, or three dimensional generalized thermal potential, respectively. In the case of $n=1$, the boundary $S_{t'}$ consists of the left end-point $x_1=x_1(t')$ and the right end-point $x_2=x_2(t')$ of a straight line segment $G_{t'}$ parallel to the x-axis. By the integral over $S_{t'}$, we mean the sum of the values of the integrand at the points x_1 and x_2.

Let us make a detailed study of the generalized thermal potentials in the one-dimensional case. Without loss of generality, it will be sufficient for us to consider the case in which $S_{t'}$ is a point $x_1=x_1(t)$, i.e., the case in which Ω coincides with that portion of the xt-plane lying to the right of some smooth curve $x_1=x_1(t)$ for $t>t_0$. In this case, the integrals (45) and (46) may be written in the forms

$$U(x, t) = \int_{t_0}^{t} \sigma(x_1, t')\, a^2\, \frac{1}{2a\sqrt{\pi(t-t')}}\, e^{-\frac{r_1^2}{4a^2(t-t')}}\, dt' \tag{45'}$$

$$V(x, t) = \int_{t_0}^{t} \vartheta(x_1, t')\, \frac{r_1 \cos\varphi_1'}{2a\sqrt{\pi(t-t')}}\, e^{-\frac{r_1^2}{4a^2(t-t')}}\, \frac{dt'}{2(t-t')}, \tag{46'}$$

where $r_1 = |x-x_1(t')|$ and φ_1' is the angle between the vector $\vec{x_1x}$ and the inner normal $\vec{n'}$ to $S_{t'}$, which in the given case coincides with the positive direction of the x-axis, $\cos\varphi_1 = 1$ for $x>x_1$ and $\cos\varphi_1 = -1$ for $x<x_1$.

Theorems 1, 2, and 3 of the previous section can be extended to the generalized thermal potentials (45′) and (46′).

Theorem 1. *The generalized thermal potential of a single layer (45′) is a continuous function of both variables x and t at points (x^*, t) of the lateral surface of the domain Ω, i.e., on the curve $x = x_1(t)$ for $t > t_0$.*

The proof of this theorem does not differ in any way from the proof of Theorem 1 of the preceding section.

Theorem 2. *The generalized thermal potential of a double layer (46′) converges to continuous limit functions as (x,t) approaches points (x^*,t) of the lateral surface of the domain Ω from the interior and the exterior along the normal $\vec{n}'$. These limit functions are given respectively by the equations*

$$V^+(x^*, t) = \frac{\vartheta(x^*, t)}{2} + V(x^*, t), \quad V^-(x^*, t) = -\frac{\vartheta(x^*, t)}{2} + V(x^*, t), \tag{47}$$

where $V(x^*,t)$ is the result of formally substituting $x^* = x_1(t)$ into the integral (46′) in place of x.

Proof: First, suppose that the density $\vartheta(x^*, t)$ is a constant equal to ϑ_0. If we add the integrals

$$V_0(x, t) = \int_{t_0}^{t} \vartheta_0 \frac{r_1 \cos \varphi_1'}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')},$$

$$V_1(x, t) = \int_{t_0}^{t} -\frac{\vartheta_0 x_1'(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} dt'$$

and set

$$\xi = \frac{x - x_1(t')}{2a\sqrt{(t-t')}}, \tag{48}$$

we obtain

$$V_0(x, t) + V_1(x, t) = \frac{\vartheta_0}{4a\sqrt{\pi}} \int_{t_0}^{t} e^{-\frac{r_1^2}{4a^2(t-t')}} \left[\frac{x - x_1(t')}{(t-t')^{3/2}} - \frac{2x_1'(t')}{\sqrt{t-t'}} \right] dt' =$$

$$= \frac{\vartheta_0}{\sqrt{\pi}} \int_{\frac{x-x_1(t_0)}{2a\sqrt{t-t_0}}}^{\xi_0} e^{-\xi^2} d\xi,$$

where

$$\xi_0 = \begin{cases} +\infty & \text{if } x > x^* = x_1(t) \\ -\infty & \text{if } x < x^* = x_1(t) \\ 0 & \text{if } x = x^* = x_1(t). \end{cases}$$

From this we obtain

$$[V_0(x^*, t) + V_1(x^*, t)]^{\pm} - [V_0(x^*, t) + V_1(x^*, t)] = \frac{\vartheta_0}{\sqrt{\pi}} \int_0^{\pm\infty} e^{-\xi^2} d\xi = \pm \frac{\vartheta_0}{2}.$$

Remembering now that $V_1(x, t)$ is continuous, we obtain

$$V_0^{\pm}(x^*, t) = \pm \frac{\vartheta_0}{2} + V_0(x^*, t).$$

If the density $\vartheta(x, t)$ is nonconstant, then

$$V(x, t) = V_0(x, t) + w(x, t),$$

where

$$w(x, t) = \int_{t_0}^{t} (\vartheta(x_1, t') - \vartheta_0) \frac{x - x_1(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')},$$

$$\vartheta_0 = \vartheta(x^*, t), \quad x^* = x_1(t).$$

But the last integral converges uniformly to the point (x^*, t) since the integral

$$w(x, t) - \int_{t_0}^{t} \frac{\vartheta(x_1, t') - \vartheta_0}{2a\sqrt{\pi(t-t')}} x_1'(t') e^{-\frac{r_1^2}{4a^2(t-t')}} dt',$$

does. (This is easily shown by means of the change of variables (48).) Consequently, the function $w(x, t)$ is continuous at the point (x^*, t), so that $w^{\pm}(x^*, t) = w(x^*, t)$. The validity of the relations (47) follows from these equations and the limiting relations for $V_0(x, t)$. Noting that the integral

$$V(x^*, t) = \int_{t_0}^{t} \vartheta(x_1, t') \frac{x_1(t) - x_1(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{[x_1(t)-x_1(t')]^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')},$$

is a continuous function of t, as can be seen by making the change of variables (48), we conclude that the limiting values $V^+(x^*, t)$ and $V^-(x^*, t)$ are also continuous functions of t.

Theorem 3. *The normal derivative of the generalized thermal potential of a single layer (45′) converges to continuous limit functions (x, t) approaches points (x^*, t) of the lateral surface of the domain Ω from the interior and the exterior along the normal $\vec{n}$. These limit functions are given respectively by the equations*

$$\frac{\partial U^+(x^*, t)}{\partial n} = -\frac{\sigma(x^*, t)}{2} + \frac{\partial U(x^*, t)}{\partial n};$$

$$\frac{\partial U^-(x^*, t)}{\partial n} = \frac{\sigma(x^*, t)}{2} + \frac{\partial U(x^*, t)}{\partial n}, \tag{47′}$$

where $\partial U(x^*, t)/\partial n$ is the result of formal differentiation of $\partial U(x, t)$ at the point (x^*, t) (where $x^* = x_1(t)$) under the integral sign.

Proof: We have

$$\frac{\partial U(x, t)}{\partial n} = \frac{\partial U(x, t)}{\partial x} = \int_{t_0}^{t} -\sigma(x_1, t') \frac{x - x_1(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}.$$

This means that $\partial U(x, t)/\partial n$ is the generalized thermal potential of a double layer with density equal to $-\sigma(x_1, t)$. Consequently, on the basis of Theorem 2, the limiting values of $\partial U(x^*, t)/\partial n$ are continuous and are given by Eqs. (47′).

These properties of generalized thermal potentials enable us to apply the method of integral equations to the solution of the generalized first and generalized mixed boundary-value problems in a manner completely analogous to what was done in connection with the first, second, and third boundary-value problems in the preceding section. For example, the solution of the first generalized boundary-value problem in the region $x > x_1(t)$, $t > 0$

$$u_{xx} - \frac{1}{a^2} u_t = 0, \quad u|_{t=0} = 0, \quad u(x_1(t), t) = \Phi(t) \tag{49}$$

must be sought in the form of a generalized thermal potential of a double layer

$$u(x, t) = \int_0^t \vartheta(x_1, t') \frac{x - x_1(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{r_1^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}. \tag{49'}$$

The first of formulas (47) for determining the density $\vartheta(x^*, t) = \vartheta(x_1(t), t)$ yields the Volterra integral equation

$$\Phi(t) = \frac{\vartheta(x_1(t), t)}{2} + \int_0^t \vartheta(x_1(t'), t') \frac{x_1(t) - x_1(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{[x_1(t)-x_1(t')]^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}. \tag{49''}$$

This integral equation always has a continuous solution, substitution of which into the potential of a double layer (49′) yields the solution of the boundary-value problem (49).

Solution of the generalized mixed problem for the region $0<x<x_2(t)$, $t>0$

$$u_{xx} - \frac{1}{a^2} u_t = 0, \quad u|_{t=0} = 0,$$
$$\frac{\partial u}{\partial n}\Big|_{x=0} = \Phi_1(t), \quad u|_{x=x_2(t)} = \Phi_2(t) \tag{50}$$

should be sought in the form of the sum of the thermal potential of a single layer along the line $x=0$ and the generalized thermal potential of a double layer along the curve $x=x_2(t)$

$$u(x, t) = \int_0^t \frac{\sigma(0, t')\, a^2}{2a\sqrt{\pi(t-t')}} e^{-\frac{x^2}{4a^2(t-t')}} dt' +$$
$$+ \int_0^t \vartheta(x_2, t') \frac{x - x_2(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{[x-x_2(t')]^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}. \tag{50'}$$

The first of formulas (47′) for $x=0$ and the first of formulas (47) for $x=x_2(t)$ yield the following system of integral equations for determining the functions $\vartheta(x^*, t) = \vartheta(x_2(t), t)$ and $\sigma(0, t)$:

$$\Phi_1(t) = -\frac{\sigma(0, t)}{2} + \int_0^t \frac{\vartheta(x_2(t'), t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{x_2^2(t')}{4a^2(t-t')}} \times$$
$$\times \left(1 + \frac{x_2^2(t')}{2a^2(t-t')}\right) \frac{dt'}{2(t-t')}, \tag{50''}$$

$$\Phi_2(t) = \frac{\vartheta(x_2(t), t)}{2} + a^2 \int_0^t \frac{\sigma(0, t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{x_2^2(t')}{4a^2(t-t')}} dt' +$$
$$+ \int_0^t \vartheta(x_2(t'), t') \frac{x_2(t) - x_2(t')}{2a\sqrt{\pi(t-t')}} e^{-\frac{[x_2(t)-x_2(t')]^2}{4a^2(t-t')}} \frac{dt'}{2(t-t')}.$$

This system of integral equations always has a solution. Substitution of this solution into (50′) yields the solution of the boundary-value problem (50).*

*Other applications of integral equations and thermal potentials to the heat equation can be found in C. H. Müntz, Zum dynamischen Wärmeleitungsprobleme, *Math. Zeitschrift*, 38, 323–337, 1934.

In conclusion, we point out that the entire theory of the solution of boundary-value problems for the heat equation can be carried over to the so-called generalized heat equation

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t} + cu = -\frac{f(P, t)}{k}, \tag{51}$$

where a and c are constants. As we showed earlier (see Chapter 1, Sect. 2, subsection 1), by introducing the new unknown function $v = ue^{-a^2ct}$, we can transform Eq. (51) into an equation of the following form:

$$\Delta v - \frac{1}{a^2}\frac{\partial v}{\partial t} = -\frac{f(P, t)}{k}e^{-a^2ct}, \tag{51'}$$

i.e., Eq. (51) is transformed into the heat equation.

CHAPTER 5

Hyperbolic Equations. The Basic Boundary-Value Problems. Steady-State Oscillations. Propagation and Distortion of Waves

This chapter will be devoted to the study of hyperbolic equations. Many different kinds of boundary-value problems associated with such equations will be formulated, and their solutions will be investigated by means of methods based on the transformation of n-dimensional integrals into $(n-1)$-dimensional integrals and on the use of δ-like functions (unit impulse functions and Riemann functions). As with elliptic and parabolic equations, we shall again bring in the concept of the Green's function for hyperbolic equations, and we shall give integral representations of the solutions of the boundary-value problems for hyperbolic equations with the aid of Green's functions.

A significant place in this chapter is assigned to questions concerned with steady-state oscillations, radiation conditions, and questions associated with the equation of the characteristic. In analogy with elliptic and parabolic equations, we shall study wave potentials.

1. Formulation of the Simplest Basic Boundary-Value Problems and Their Physical Meaning

Consider the n-dimensional wave equation

$$\Box u = \Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(P, t)}{a^2} \quad (a^2 = \text{const} > 0), \tag{1}$$

where Δ is the n-dimensional Laplace operator with respect to the space coordinates, t denotes time, $f(P, t)$ is a given function of P and t, and where $\Box = \Delta - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}$ is the differential operator known as the *wave operator* or *d'Alembert's operator*.

For the wave equation (1), one can pose the following so-called *Cauchy problem—the initial value problem:*

Find a function $u=u(P, t)$ *which is a solution of Eq. (1) for every* P *and all* $t>0$, *and which satisfies the initial conditions*

$$u|_{t=0} = \psi(P), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(P), \tag{2}$$

where $\psi(P)$ *and* $\psi_1(P)$ *are given functions of the point* P *in* n*-dimensional space.*

In contrast to Laplace's equation and the heat-flow equation, in posing this Cauchy problem we do not impose a priori any continuity or differentiability conditions on the solution or on the initial functions $\psi(P)$ and $\psi_1(P)$. We shall make such restrictions later in studying particular cases of the Cauchy problem and particular questions connected with this problem.

In the one-dimensional case ($n=1$), we may interpret the Cauchy problem as the problem of transverse vibrations of a string: Find the displacement $u=u(x, t)$ of an infinitely long string from its equilibrium position (coinciding with the x-axis) if at the initial instant of time we are given the position of the points of the string

$$u|_{t=0} = \psi(x)$$

and their velocities

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x).$$

Similarly, in the two-dimensional case ($n=2$), we arrive at the Cauchy problem when we seek the transverse vibrations of an infinite membrane (coinciding in its equilibrium position with the xy-plane) if we are given the positions of the points of the membrane and their velocities at the initial instant of time. In the three-dimensional case (for $n=3$), the Cauchy problem can be interpreted as the problem of acoustic vibrations of a gas when the velocity vector

$$\vec{v}|_{t=0} = -\nabla u|_{t=0} = -\nabla\psi(P)$$

and the condensation

$$s\Big|_{t=0} = \frac{1}{a^2}\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{1}{a^2}\psi_1(P),$$

are specified at every point in space at the initial instant. Here, $u=u(P, t)$ is the velocity potential of the acoustic field.

Let G be a bounded domain in n-dimensional space, S—a piecewise-smooth surface bounding it, $\vec{n}$—the inner normal to S, Ω—the

cylindrical domain in the $n+1$-dimensional phase space with base G for $t=t_0=\text{const}$,* and S_L—the lateral surface of this cylindrical region with generators parallel to the t-axis. (For $n=1$, this cylindrical region Ω degenerates into a half-strip.)

Then, in addition to the Cauchy problem, we can pose the following basic boundary-value problems for the wave equation (1): In the cylindrical domain Ω, find a solution $u=u(P, t)$ of the wave equation (1) which satisfies the initial conditions

$$u|_{t=t_0} = \psi(P), \quad \frac{\partial u}{\partial t}\Big|_{t=t_0} = \psi_1(P), \quad P \in G \tag{2'}$$

in G, and in addition satisfies one of the following boundary conditions:

$$u|_{S_L} = \Phi(P, t), \quad t > t_0 \tag{1'}$$

or

$$\frac{\partial u}{\partial n}\Big|_{S_L} = \Phi(P, t), \quad t > t_0, \tag{1''}$$

or

$$\left(\frac{\partial u}{\partial n} - \alpha u\right)_{S_L} = \Phi(P, t), t > t_0, \alpha \geqslant 0, \tag{1'''}$$

where $\psi(P)$ and $\psi_1(P)$ are given functions of P in the closed region $G \cup S$ and where $\Phi(P, t)$ and α are given functions of P and t on the surface S_L for $t \geqslant t_0$. Here, the function α is assumed continuous and nonnegative on the surface S_L for $t \geqslant t_0$.

In the case of the boundary conditions (1'), (1''), and (1'''), we speak of the *first, second, and third n-dimensional boundary-value problems for the wave equation (1) for the domain G or for the cylindrical domain* Ω. The initial conditions for all these problems remain the same and are given by Eqs. (2').

If the initial conditions (2') as applied to $P \in S$ and the boundary condition in question [(1'), (1''), or (1''')] as applied at $t=t_0$ do not contradict each other, we say that the initial functions $\psi(P)$ and $\psi_1(P)$ and the boundary function $\Phi(P, t)$ satisfy a *compatibility condition.* For example, in the case of the first boundary-value problem, the compatibility condition is of the form

$$\psi(P) \quad |_{P \in S} = \Phi(P, t_0),$$

*By a *phase space* we mean a space in which one of the coordinate axes represents the time t.

$$\left.\frac{\partial \Phi(P, t)}{\partial t}\right|_{t=t_0} = \psi_1(P).$$

When the formulation of a particular boundary-value problem is such that the compatibility condition is satisfied, we can require the initial conditions to be satisfied in $G \cup S$, and the boundary conditions to be satisfied for $t \geqslant t_0$.

We note that in posing these three boundary-value problems, just as in the case of the Cauchy problem, we do not impose either on the solutions or on the functions $\psi(P)$, $\psi_1(P)$, and $\Phi(P, t)$ any a priori determined restrictions regarding their differentiability or continuity. Such restrictions will be made later in studying particular cases of these boundary-value problems.

From a physical point of view the initial conditions (2′) can be interpreted in just the same way as the initial conditions in the case of the Cauchy problem, except that here we speak of the corresponding physical processes not throughout all space, but in a finite region G in which the initial conditions apply.

The boundary conditions of the first boundary-value problem describe, in the one-dimensional case, the law of motion of the ends of a string, in the two-dimensional case the law of motion of the edge of a membrane, and in the three-dimensional case the tangential component of the velocity vector, representing the motion of the gas on a surface S enclosing a volume G. The phenomenon that leads most naturally to the boundary conditions of the second boundary-value problem (for $n=1, 2, 3$) is that of the acoustic vibrations of a gas in a container. If we think of S as the walls of the container and denote by $\Phi(P, t)$ the normal component of the velocity of the walls of the container, we obtain the boundary conditions of the second boundary-value problem

$$\left.\frac{\partial u}{\partial n}\right|_{S_L} = \Phi(P, t).$$

For $n=1$, the problem of the longitudinal vibrations of a rod with elastically fastened ends leads to the boundary conditions of the third boundary-value problem. Indeed, the tension in the rod at a point x is given by $T=Es\,\partial u/\partial x$, where E is Young's modulus and s is the cross-sectional area of the rod. At the ends of the rod $x=0$ and $x=l$, this tensile force must be counterbalanced by the elastic forces $hu|_{x=0}$ and $-hu|_{x=l}$, where h is a positive proportionality constant characterizing the stiffness of the fastenings. Thus, at the ends of the rod, we have the boundary conditions of the third boundary-value problem:

$$\left(Es\frac{\partial u}{\partial x} - hu\right)_{x=0} = 0, \quad \left(Es\frac{\partial u}{\partial x} + hu\right)_{x=l} = 0 \qquad (t > t_0).$$

In particular, if the ends of the rod are free, which corresponds to the case $h=0$, these boundary conditions degenerate into the boundary conditions of the second boundary-value problem. On the other hand, if the ends of the rod are fastened or if they move according to some given law, we have the boundary conditions of the first boundary-value problem.

In certain cases it is necessary to consider these boundary-value problems for the wave equation for an unbounded domain G. An example is the case of longitudinal vibrations of a semi-infinite rod (coincident with the positive x-axis) when the displacement of points of the rod from their equilibrium positions and their velocities are given at the initial instant, and when the end $x=0$ is either rapidly clamped, completely free, or elastically constrained.

These three boundary-value problems and the Cauchy problem are among the most basic and typical boundary-value problems for the wave equation and, in general, for linear second-order hyperbolic equations. As a generalization of these problems, we can mention the *mixed boundary-value problem for the wave equation for a cylindrical domain* Ω. This problem is posed like the first, second, and third boundary-value problems except that, for $t>t_0$ the boundary conditions of the first problem are given on one portion of the boundary S of the domain G, those of the second problem are given on another portion, and those of the third problem are given on a third portion. Thus, for example, the question of the longitudinal vibrations of a rod of finite length with one fixed end and the other end either free or elastically constrained leads to a mixed boundary-value problem.

In addition to these boundary-value problems, there are other basic boundary-value problems for the wave equation. The most important of these are the so-called characteristic problem, the problem associated with steady-state oscillations, and the problem associated with the propagation of waves. These problems will be formulated and studied in what follows.

2. Uniqueness and Continuous Dependence of Solutions of the First, Second, and Third Boundary-Value Problems on the Initial Conditions

Let us prove uniqueness for solutions of the first, second, and third boundary-value problems in the class of functions that are twice continuously differentiable in the closed region $G \cup S$ for $t \geqslant t_0$.

First of all, we note an integral identity for solutions of the homogeneous wave equation for $n=3$

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{3}$$

specifically,

$$I(t) = \frac{d}{dt}E(t), \tag{3'}$$

where

$$I(t) = -\iint_S \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\,dS,$$

$$E(t) = \frac{1}{2}\iiint_G \left(u_x^2 + u_y^2 + u_z^2 + \frac{1}{a^2}u_t^2\right)d\tau.$$

This identity follows directly from Ostrogradskiy's formula since*

*The integral $E(t)$ has a simple physical meaning. Confining ourselves for simplicity to the one-dimensional case, let us compute the total energy of a vibrating string with fixed ends at $x = x_1$ and $x = x_2 > x_1$. The kinetic energy of the string is given by the equation

$$K = \frac{1}{2}\int_{x_1}^{x_2} \rho u_t^2\,dx,$$

where ρ is the linear density of the string. Under the influence of the resultant of the tensile forces, $Tu_x|_{x+dx} - Tu_x|_x = Tu_{xx}\,dx$, an element of the string dx moves in a time dt over a path $u_t\,dt$ and the work done by the string in this interval of time is given by

$$dt\int_{x_1}^{x_2} Tu_{xx}u_t\,dx = dt\left(Tu_xu_t\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} Tu_xu_{xt}\,dx\right) = -\,dt\int_{x_1}^{x_2} Tu_xu_{xt}\,dx =$$

$$= -\,dt\frac{d}{dt}\int_{x_1}^{x_2}\frac{Tu_x^2}{2}\,dx.$$

The work done by the string in moving from its equilibrium position to the position given by the function $u = u(x, t)$ is

$$-\frac{1}{2}\int_{x_1}^{x_2} Tu_x^2\,dx.$$

This work, taken with opposite sign, represents the potential energy and, consequently, we have the following expression for the desired total energy:

$$\frac{1}{2}\int_{x_1}^{x_2}(Tu_x^2 + \rho u_t^2)\,dx = T\,\frac{1}{2}\int_{x_1}^{x_2}\left(u_x^2 + \frac{1}{a^2}u_t^2\right)dx = TE(t)\left(a = \sqrt{\frac{T}{\rho}}\right).$$

$$-\iint\limits_S \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\,dS = \iiint\limits_G \operatorname{div}\left(\frac{\partial u}{\partial t}\nabla u\right)d\tau =$$

$$= \iiint\limits_G (u_t\Delta u + u_x u_{xt} + u_y u_{yt} + u_z u_{zt})\,d\tau.$$

If u is the difference between two solutions of the boundary-value problem for Eq. (1) with identical boundary and initial conditions, then u will also be a solution of the same boundary-value problem for the homogeneous equation (3) with homogeneous boundary and initial conditions. In the case of the first and second boundary-value problems, the left-hand member of Eq. (3′) will vanish by virtue of the homogeneous boundary conditions. It follows from this that $E=\text{const}$. Furthermore, $E|_{t=0}=0$. Consequently, $E\equiv 0$ and all the partial derivatives of the function u vanish identically. Since $u|_{t=t_0}=0$, we have $u\equiv 0$. This means that the solutions of the first and second boundary-value problems for $n=3$ are unique. In the case of the third problem for α independent of t, we have, in view of the identity (3′),

$$-\iint\limits_S \frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial n}-\alpha u\right)dS = \frac{d}{dt}E(t) + \alpha\iint\limits_S uu_t\,dS.$$

The left-hand member of this equation is identically equal to zero by virtue of the homogeneous boundary conditions which the function u satisfies. Therefore,

$$E^*(t) = E(t) + \frac{1}{2}\iint\limits_S \alpha u^2\,dS = \text{const}.$$

Since $u|_{t=t_0}=0$ and $\partial u/\partial t\,|_{t=t_0}=0$, we have $E^*(t)\equiv 0$ and, since $\alpha\geqslant 0$, we have $E(t)\equiv 0$. It follows from the last equation and from the fact that $u|_{t=t_0}=0$ that $u\equiv 0$. This means that for $n=3$, the solution of the third boundary-value problem with α independent of t is unique. In the cases of $n=2$ and $n=1$, the reasoning and results remain the same except that in the identity (3′), the integrals $I(t)$ and $E(t)$ take the following forms, respectively, for $n=2$ and $n=1$:

$$I(t) = -\int\limits_S \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\,ds, \quad E(t) = \frac{1}{2}\iint\limits_G \left(u_x^2 + u_y^2 + \frac{1}{a^2}u_t^2\right)d\sigma, \quad (3'')$$

$$I(t) = -\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\Big|_{x=x_1} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial n}\Big|_{x=x_2}\right),$$

$$E(t) = \frac{1}{2}\int\limits_{x_1}^{x_2}\left(u_x^2 + \frac{1}{a^2}u_t^2\right)dx, \quad (3''')$$

also, in the one-dimensional case,

$$\frac{\partial}{\partial n}\Big|_{x=x_1} = \frac{\partial}{\partial x}\Big|_{x=x_1}, \quad \frac{\partial}{\partial n}\Big|_{x=x_2} = -\frac{\partial}{\partial x}\Big|_{x=x_2}.$$

For $n=2$, the identity (3′) is obtained in the same way as for $n=3$ if we use Green's formula instead of Ostrogradskiy's. For $n=1$, this identity follows from the obvious equation

$$I(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}(u_x u_t)\,dx = \int_{x_1}^{x_2} \left(u_x u_{xt} + \frac{1}{a^2} u_t u_{tt}\right) dx.$$

Let us consider the question of the continuous dependence of the first, second, and third (with a independent of t) boundary-value problems on the initial conditions, confining ourselves for simplicity to the one-dimensional case.

Suppose that u_1 and u_2 *are twice continuously differentiable solutions of the first, second, or third boundary-value problem for the one-dimensional wave equation (1) in the closed region* $G \cup S$ *for* $t_0 \leqslant t \leqslant T = \text{const} < \infty$ *under identical boundary conditions. Let* ε *denote an arbitrarily small number. Then,* $|u_1 - u_2| < \varepsilon$ *in the closed region mentioned if the functions*

$$\psi(x) = (u_1 - u_2)_{t=t_0}, \quad \psi_1(x) = \left(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t}\right)_{t=t_0}, \quad \frac{\partial \psi(x)}{\partial x}$$

are sufficiently small in absolute value.

Proof: The function $u = u_1 - u_2 = u(x, t)$ satisfies the homogeneous equation (1). Therefore, the identity (3′) is applicable to it. We have

$$E(t) = \frac{1}{2}\int_{x_1}^{x_2} \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{1}{a^2}\psi_1^2\right] dx + \int_{t_0}^{t} I(t)\,dt. \tag{4}$$

In the case of the first and second boundary-value problems, $I(t) = 0$. Consequently, $E(t)$ is an arbitrarily small quantity.

In the case of the third boundary-value problem,

$$I(t) = -[(au_t u)_{x_1} + (au_t u)_{x_2}] = -\frac{1}{2}\frac{\partial}{\partial t}[(au^2)_{x_1} + (au^2)_{x_2}].$$

Therefore,

$$\int_{t_0}^{t} I(t)\,dt \leqslant \frac{1}{2}\left[(au^2)_{x_1} + (au^2)_{x_2}\right]_{t=t_0} = \frac{1}{2}[a(x_1)\psi^2(x_1) + a(x_2)\psi^2(x_2)].$$

Thus, in this case also, $E(t)$ can be made arbitrarily small.

Furthermore, in the case of the first boundary-value problem, by using Bunyakovskiy's inequality, we obtain

$$|u(x,t)| \leqslant \int_{x_1}^{x} \left|\frac{\partial u}{\partial x}\right| dx \leqslant \sqrt{x_2 - x_1} \sqrt{\int_{x_1}^{x_2} u_x^2\, dx} \leqslant$$

$$\leqslant \sqrt{x_2 - x_1}\, \sqrt{2E(t)} < \varepsilon. \qquad (4')$$

This completes the proof of our assertion for the first problem.

In the case of the second and third, we have, instead of (4′),

$$|u(x,t) - u(x_1,t)| < \varepsilon. \qquad (4'')$$

Let us estimate $u(x_1, t)$. We have

$$u(x_1,t)(x_2 - x_1) = \int_{x_1}^{x_2} u(x,t)\,dx + \int_{x_1}^{x_2} [u(x_1,t) - u(x,t)]\,dx \ll$$

$$\ll \left|\int_{x_1}^{x_2} u(x,t)\,dx\right| + \varepsilon(x_2 - x_1).$$

In view of the inequalities

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{x_2} \frac{\partial u(x,t)}{\partial t}\,dx \leqslant \sqrt{x_2 - x_1} \sqrt{\int_{x_1}^{x_2} u_t^2\,dx} \leqslant$$

$$\leqslant a\sqrt{x_2 - x_1}\,\sqrt{2E(t)},$$

we obtain

$$\left|\int_{x_1}^{x_2} u(x,t)\,dx\right| \leqslant \left|\int_{x_1}^{x_2} u(x,t_0)\,dx\right| + \left|\int_{t_0}^{t} a\sqrt{x_2 - x_1}\,\sqrt{2E(t')}\,dt'\right|.$$

Since the function $\psi(x)$ may be assumed sufficiently small in absolute value, we conclude that the quantity

$$\left|\int_{x_1}^{x_2} u(x,t)\,dx\right|$$

can be made arbitrarily small. Therefore, the quantity $|u(x_1, t)|$ possesses this property also, and it follows from (4″) that so does the quantity $|u(x, t)|$. This proves the assertion for the second and third boundary-value problems.

3. The String Equation. D'Alembert's Solution. The Domain of Determinacy. Physical Conclusions

We shall give the solution of the Cauchy problem for the one-dimensional wave equation

$$\Box u = \frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, t)}{a^2}, \tag{5}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x), \tag{5'}$$

assuming that on every finite interval the initial function $\psi(x)$ is twice continuously differentiable and that the initial function $\psi_1(x)$ has a continuous derivative. To get a better picture of the desired solution, we shall interpret it as the displacement of an infinite string from its equilibrium position (coincident with the x-axis). In this case, the initial conditions represent the given positions of points of the string and their velocities at the initial instant; $f(x, t)$ is the body force, i.e., the force per unit mass applied to the string and acting in a direction perpendicular to it in the xu-plane; $a = \sqrt{T/\rho}$, where T is the tension in the string and ρ is its linear density (see Chapter 1, Sect. 2, subsection 9).

Let us consider first the case of the homogeneous equation, i.e., where $f(x, t) \equiv 0$. This corresponds to free vibrations of the string, and problem (5), (5') is then written in the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0; \tag{6}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x). \tag{6'}$$

To solve this problem we shall use the so-called d'Alembert method of characteristic coordinates, which leads quickly to the desired result. Here, the equation of the characteristics is written in the form

$$dt^2 - \frac{1}{a^2}dx^2 = 0.$$

When we solve this equation, we obtain two families of characteristics in the form of straight lines

$$x - at = \text{const}, \quad x + at = \text{const}.$$

The equation of the string in the characteristic variables

$$\xi = x - at, \quad \eta = x + at$$

is written in the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

The set of all solutions of this equation in the class of continuously differentiable functions is given by the equation

$$u = \varphi(\xi) + \varphi_1(\eta),$$

where $\varphi(\xi)$ and $\varphi_1(\eta)$ are arbitrary twice continuously differentiable functions of their arguments. The general solution of the homogeneous string equation (6) in the class of twice continuously differentiable functions is written in the form

$$u(x, t) = \varphi(x - at) + \varphi_1(x + at). \qquad (6'')$$

This means that every twice continuously differentiable function $u(x, t)$ that satisfies Eq. (6) may be represented in the form (6″), where φ and φ_1 are twice continuously differentiable functions of their arguments. The function $\varphi(x-at)$ is called the *forward wave* and $x-at$ is called the phase of the forward wave; $\varphi_1(x+at)$ is called the *backward wave*, and $x+at$ is called the phase of the backward wave. The physical meaning of the forward wave consists in the fact that it gives the propagation of vibrations with velocity equal to a in the positive direction of the x-axis. Let us suppose that at the initial instant $t=0$ the function $\varphi(x-at)$ is different from zero only in a neighborhood of the point $x=0$, and that its graph takes the form of a hump close to the point $x=0$. Then, at the instant t, the function $\varphi(x-at)$ will be different from zero only in a neighborhood of the point $x=at$. This means that the hump referred to will be displaced without any change in its shape in the positive direction of the x-axis with a velocity equal to a. In the case of the backward wave, the physical meaning remains the same except that it gives the propagation of the oscillations with velocity a not in the positive but in the negative direction of the x-axis.

Since the sum of the forward and backward waves represents the general solution of the homogeneous wave equation in the one-dimensional case, it is natural to seek a solution of the problem (6), (6′) in the form

$$u(x, t) = \varphi(x - at) + \varphi_1(x + at). \qquad (6''')$$

The initial conditions of the problem (6), (6′) yield the equations

$$\varphi(x) + \varphi_1(x) = \psi(x), \quad -a\varphi'(x) + a\varphi_1'(x) = \psi_1(x)$$

for determining the forward and backward waves. From these, we get

$$-a\varphi(x) + a\varphi_1(x) = \int_0^x \psi_1(x)\,dx + aC, \qquad C = \text{const},$$

$$\varphi(x) = \frac{1}{2}\left[\psi(x) - \frac{1}{a}\int_0^x \psi_1(x)\,dx - C\right],$$

$$\varphi_1(x) = \frac{1}{2}\left[\psi(x) + \frac{1}{a}\int_0^x \psi_1(x)\,dx + C\right],$$

$$u(x, t) = \frac{1}{2}\left[\psi(x - at) - \frac{1}{a}\int_0^{x-at} \psi_1(y)\,dy + \psi(x + at) + \right.$$

$$\left. + \frac{1}{a}\int_0^{x+at} \psi_1(y)\,dy\right].$$

Here, if we combine the two integrals, we obtain the so-called *d'Alembert's solution* of the Cauchy problem for the homogeneous string equation (6), (6′):

$$u(x, t) = \frac{\psi(x - at) + \psi(x + at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi_1(y)\,dy. \tag{7}$$

It follows from the very way in which we obtain this solution that it is unique and twice continuously differentiable since the function $\psi_1(x)$ is continuously differentiable and the function $\psi(x)$ is twice continuously differentiable. It follows from (7) that this solution of the Cauchy problem is stable. Specifically, if the initial functions $\psi(x)$ and $\psi_1(x)$ are changed by amounts which are sufficiently small in absolute value in the interval $-\infty < x < \infty$, the changes in the solution that we have found will be arbitrarily small at points in the region $-\infty < x < \infty$, $0 \leqslant t \leqslant T < \infty$.

It is obvious from the integral (7) that the value of the solution at the point Q with coordinates x and t is uniquely determined by the values of the initial functions $\psi(x)$ and $\psi_1(x)$ on the interval $[A, B]$ constituting the base of a triangle bounded by the x-axis and the characteristics passing through the point Q (see Fig. 23). Therefore, the segment $[A, B]$ is called the *domain of dependence of the point* Q. Obviously, the initial conditions on this segment

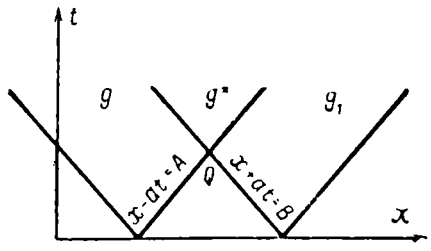

Fig. 23.

determine uniquely the solution at points of the triangle mentioned. Therefore, this triangle is called the *domain of determinacy* of the segment $[A, B]$. It follows from (7) that the initial conditions on the segment $[A, B]$ have an effect on the values of the solution not only in the domain of determinacy corresponding to it but also in the region bounded by the segment $[A, B]$ and the characteristics $x+at=A$ and $x-at=B$ (see Fig. 23). Therefore, this last region is called the *domain of influence* of the segment $[A, B]$. In Fig. 23, the domain of dependence for the point Q is shown in the form of the segment $[AB]$, the domain of determinacy of the interval $[A, B]$ is shown in the form of the triangle QAB, and the domain of influence of the segment $[A, B]$ in the form of the same triangle QAB together with the adjacent regions g, g_1 and g^*. These concepts of domain of dependence, domain of determinacy, and domain of influence bring out the nature of the dependence of the solution of the Cauchy problem on the initial conditions. For example, to know the solution in the triangle QAB of the xt-plane, it will be sufficient to know the initial conditions only for the segment $[A, B]$, with regard to which the triangle referred to is the domain of determinacy.

In contrast to the previously made assumptions, let us suppose that the point B is a point of discontinuity of the first kind for the initial functions $\psi(x)$ and $\psi_1(x)$ and their derivatives. In this case, the solution of the Cauchy problem also exists and is given by the d'Alembert formula (7). However, it will not be a twice continuously differentiable function on the characteristics which pass through the point B. If $\gamma=[\psi]_B$ is the jump in the initial function $\psi(x)$ at the point B, then, as we can see from (7), the solution $u(x, t)$ on the characteristics issuing from the point B in the direction of increasing x will have jump equal to $\gamma/2$. This means that, in the xt-plane, the *discontinuities in the solutions of the Cauchy problem are propagated along the characteristics*. It turns out that the jump in the solution $u=u(x, t)$ that exists at the initial instant at the point $x=B$ splits in two and is propagated along the x-axis in both directions from the point $x=B$ with velocity equal to a.

If we differentiate d'Alembert's formula (7), we obtain

$$\frac{\partial u}{\partial x}=\frac{\psi'(x-at)+\psi'(x+at)}{2}+\frac{\psi_1(x+at)-\psi_1(x-at)}{2a},$$

$$\frac{\partial u}{\partial t}=a\frac{-\psi'(x-at)+\psi'(x+at)}{2}+\frac{\psi_1(x+at)+\psi_1(x-at)}{2}.$$

It is clear from these equations that the derivatives of the solutions of the Cauchy problem have jumps on the characteristics passing

through the point B. These jumps can be expressed quite simply in terms of the jumps in $\partial\psi/\partial x$ and $\psi_1(x)$ at the point B. Specifically, in the characteristic $x+at=B$,

$$\left[\frac{\partial\psi}{\partial x}\right]=\frac{1}{2}[\psi'(x)]+\frac{1}{2a}[\psi_1(x)],\quad \left[\frac{\partial u}{\partial t}\right]=\frac{a}{2}[\psi'(x)]+\frac{1}{2}[\psi_1(x)]$$

and on the characteristic $x-at=B$,

$$\left[\frac{\partial u}{\partial x}\right]=\frac{1}{2}[\psi'(x)]-\frac{1}{2a}[\psi_1(x)],\quad \left[\frac{\partial u}{\partial t}\right]=$$
$$=-\frac{a}{2}[\psi'(x)]+\frac{1}{2}[\psi_1(x)],$$

where the symbol [ ] denotes the jump of the function in the square brackets as the argument of that function increases. When $\partial^2\psi/\partial x^2$ and $\partial\psi_1/\partial x$ are discontinuous, analogous conclusions hold with regard to the propagation of the discontinuities of the second derivatives of the solution of the Cauchy problem along the characteristics.

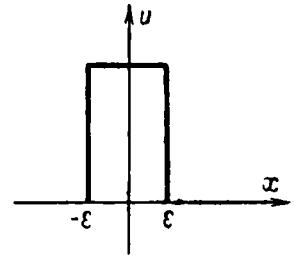

Fig. 24.

Let us consider some particular cases. Suppose that the initial functions $\psi(x)$ and $\psi_1(x)$ are of the form

$$\psi(x)=\begin{cases} h, & |x|\leqslant\varepsilon,\\ 0, & |x|>\varepsilon,\end{cases}\qquad \psi_1(x)\equiv 0,$$

where ε is a sufficiently small positive number and h is a positive constant. In this case, the solution of the Cauchy problem may, in accordance with (7), be written in the form

$$u(x,t)=\frac{\psi(x-at)+\psi(x+at)}{2}=\begin{cases}\frac{1}{2}h, & |x\mp at|\leqslant\varepsilon,\\ 0, & |x\mp at|>\varepsilon.\end{cases}\qquad (7')$$

Figure 24 shows the position of the string at the initial instant; Fig. 25 shows its position at the instant t. Here, at every instant of time, the disturbance has a *leading wavefront*, i.e., a value of x separating that portion of the x-axis where the disturbance has not yet arrived from the portion where it has already arrived. Similarly, there is a *trailing wavefront* consisting of that value of x which separates the portion of the x-axis on which the disturbance is

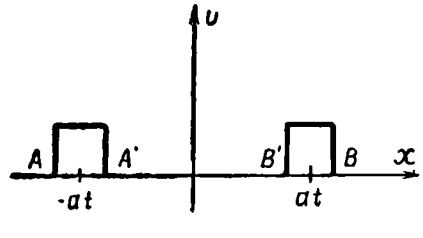

Fig. 25.

still going on from that portion on which the disturbance has stopped. In Fig. 25, the leading wavefront is denoted by the points A and B, the trailing wavefront by the points A' and B'.

Suppose that the initial functions $\psi(x)$ and $\psi_1(x)$ are of the form

$$\psi(x) \equiv 0, \quad \psi_1(x) = \begin{cases} h, & |x| \leqslant \varepsilon, \\ 0, & |x| > \varepsilon, \end{cases}$$

where ε and h are the same as in the preceding case. Then, the solution of the Cauchy problem may, in accordance with d'Alembert's formula (7), be written in the form

$$u(x, t) = \frac{1}{2a} \int_{x-at}^{x+at} \psi_1(y)\, dy = \psi^*(x + at) - \psi^*(x - at), \qquad (7'')$$

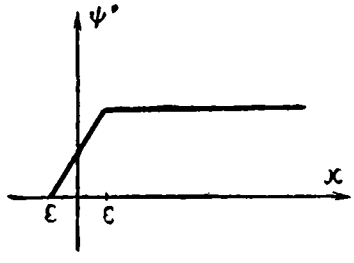

Fig. 26.

where

$$\psi^*(x) = \frac{1}{2a} \int_{-\infty}^{x} \psi_1(y)\, dy.$$

Figure 26 shows the graph of the function $\psi^*(x)$. Figure 27 shows the position of the string at the instant t. It is characteristic that here only the leading front of the wave, indicated in Fig. 27 by the points A and B, is sharply defined; there is no trailing wavefront or, as we say, there is *wave diffusion*, meaning that the trailing edge is being washed out. This means that a disturbance, having arrived at a given point x, never completely vanishes at that point.

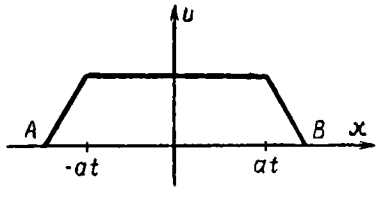

Fig. 27.

Suppose that the portion of the string $|x-x_0| < \varepsilon$, which is at rest at the instant t_0, is given a velocity equal to h. Then, in accordance with (7''), the displacement of points of the string from the equilibrium position is given by the equation

$$u(x, t) = \begin{cases} \dfrac{1}{2a} \displaystyle\int_{x_0-\varepsilon}^{x_0+\varepsilon} h\, dy, & |x - x_0| \leqslant a(t - t_0), \\ 0, & |x - x_0| > a(t - t_0). \end{cases} \qquad (8)$$

We may assume that the velocity h is created by an impulse due to a body force $f(x, y)$ applied to the segment $|x-x_0| < \varepsilon$ in the course of an infinitesimal interval of time from $t=t_0$ to $t=t_0+\Delta t_0$; that is,

$$I = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \int_{t_0}^{t_0+\Delta t_0} \rho f(y, t)\, dt\, dy = 2\varepsilon\rho h. \tag{9}$$

Therefore, at the point x_0, the density of the impulse acting during the infinitesimal interval of time from $t=t_0$ to $t=t_0+\Delta t_0$ is given by the equation

$$\rho^*(x_0, t_0) = \lim_{\varepsilon\to 0} \frac{I}{2\varepsilon} = \rho \int_{t_0}^{t_0+\Delta t_0} f(x_0, t)\, dt = \rho h.$$

In particular,

$$h(x_0, t_0) = f(x_0, t_0)\,\Delta t_0,$$

where $h(x_0, t_0)$ is the velocity at the point x_0 that arises at the instant t_0 as a result of the impulse referred to. From d'Alembert's formula (7), the displacement of the string from its equilibrium position as a result of this impulse may be written in the form

$$\varphi_0(x, t, t_0) = \frac{1}{2a} \int_{x-a(t-t_0)}^{x+a(t-t_0)} h(x_0, t_0)\, dx_0 = \Delta t_0 \frac{1}{2a} \int_{x-a(t-t_0)}^{x+a(t-t_0)} f(x_0, t_0)\, dx_0. \tag{10}$$

This equation makes it possible, by means of purely physical considerations—by the *"method of impulses"*—to find the solution of the Cauchy problem for the nonhomogeneous wave equation with homogeneous initial conditions

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, t)}{a^2},$$
$$u\Big|_{t=0} = 0 \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \tag{11}$$

Specifically, if the body force is represented as the superposition of continuously acting impulses of density $\rho^*(x_0, t_0)$, it is natural to expect the desired solution to be a superposition of solutions of the form (10) corresponding to different values of the parameter t_0, which varies from 0 to t. If we write the sum of these solutions and take the limit as $\Delta t_0 \to 0$, we obtain the following expression for the desired solution:

$$v(x, t) = \int_0^t \varphi(x, t, t_0)\, dt_0, \quad \varphi(x, t, t_0) = \frac{1}{2a} \int_{x-a(t-t_0)}^{x+a(t-t_0)} f(x_0, t_0)\, dx_0. \tag{10'}$$

Now, it only remains to verify that Eq. (10′) actually is a solution of problem (11). It is obvious from the d'Alembert integral that the function $\varphi(x, t, t_0)$ is a solution of the Cauchy problem for the homogeneous wave equation with initial conditions

$$\varphi\Big|_{t=t_0} = 0, \quad \frac{\partial \varphi}{\partial t}\Big|_{t=t_0} = f(x, t_0).$$

Therefore, we have

$$v\Big|_{t=0} = 0, \quad \frac{\partial v}{\partial t} = \varphi(x, t, t) + \int_0^t \varphi_t(x, t, t_0)\, dt_0 = \int_0^t \varphi_t(x, t, t_0)\, dt_0,$$

$$\frac{\partial v}{\partial t}\Big|_{t=0} = 0,$$

i.e., the initial conditions of problem (11) are satisfied. Furthermore,

$$\frac{\partial^2 v}{\partial t^2} = \varphi_t(x, t, t) - \int_0^t \varphi_{tt}(x, t, t_0)\, dt_0 = f(x, t) + \int_0^t \varphi_{tt}(x, t, t_0)\, dt_0,$$

$$\frac{\partial^2 v}{\partial x^2} = \int_0^t \varphi_{xx}(x, t, t_0)\, dt_0$$

and, consequently,

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} = -\frac{f(x, t)}{a^2} + \int_0^t \left[\varphi_{xx}(x, t, t_0) - \right.$$

$$\left. - \frac{1}{a^2}\varphi_{tt}(x, t, t_0)\right] dt_0 = -\frac{f(x, t)}{a^2}.$$

This means that the function (10′) is indeed a solution of the Cauchy problem (11); by making the change of variables $(t-t_0)\, a = r$, the solution (10′) that we have found may be written in the form

$$v(x, t) = \frac{1}{2a^2}\int_0^{at} dr \int_{x-r}^{x+r} f\left(x_0, t - \frac{r}{a}\right) dx_0. \tag{12}$$

The function defined by this equation is called a *retarded potential.*

To find the general solution of the Cauchy problem for the nonhomogeneous wave equation (5), (5′), we need only add d'Alembert's solution (7) to the retarded potential (12).

Let us show that, since the sum of the forward and backward waves (6″) is a solution of the homogeneous string equation, the solution of the first, second, and third boundary-value problems for this equation can be found by suitable choice of the functions φ and φ_1. Here, the initial conditions may, without loss of generality, be assumed to be homogeneous since we can always arrange for this to be the case by subtracting from the desired solution the solution of the Cauchy problem with given nonhomogeneous initial conditions.

Thus, for example, the solution of the first boundary-value problem for the positive x-axis

$$u_{xx} - u_{tt} = 0,$$

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = \Phi(t)$$

is immediately found in the following form:

$$u(x, t) = \begin{cases} \Phi(t-x), & t-x \geqslant 0, \\ 0, & t-x \leqslant 0. \end{cases}$$

The solution of the second boundary-value problem for the positive x-axis

$$u_{xx} - u_{tt} = 0,$$

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \Phi(t)$$

takes the form

$$u(x, t) = \begin{cases} \Phi^*(t-x), & t-x \geqslant 0, \\ 0, & t-x < 0, \end{cases} \qquad \Phi^*(t-x) = -\int_0^{t-x} \Phi(\xi)\, d\xi.$$

The solution of the first boundary-value problem for the interval $[0, l]$

$$u_{xx} - u_{tt} = 0, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,$$

$$u|_{x=0} = 0, \quad u|_{x=l} = \Phi(t) \quad (t > 0)$$

must, by virtue of the homogeneous boundary condition for $x=0$, be represented in the form

$$u(x, t) = \varphi(t-x) - \varphi(t+x) \quad (0 \leqslant x \leqslant l, \ t > 0),$$

where φ is a function to be determined. By virtue of the homogeneous initial conditions, we have

$$\varphi(-x) - \varphi(x) = 0, \quad \varphi'(-x) - \varphi'(x) = 0 \quad (0 \leqslant x \leqslant l).$$

If we differentiate the first of these equations, we see that, for $0 \leqslant x \leqslant l$, we have $\varphi'(x) = 0$ and $\varphi'(-x) = 0$. Consequently, $\varphi(x) = \text{const}$. We may, without loss of generality, assume the last constant equal to 0. Thus, we can take

$$\varphi(x) = 0 \quad (-l \leqslant x \leqslant l).$$

The boundary condition for $x=l$ yields

$$\varphi(t-l) - \varphi(t+l) = \Phi(t), \ t > 0.$$

This equation defines the function $\varphi(x)$ in the interval $-l \leqslant x < \infty$ since we know the values of $\varphi(x)$ for $-l \leqslant x \leqslant l$ and the values of $\Phi(x)$ are given for $x > 0$. Thus, the desired solution of the first boundary-value problem takes the form

$$u(x, t) = \varphi(t - x) - \varphi(t + x),$$

where

$$\varphi(x) = \begin{cases} 0, & -l \leqslant x \leqslant l, \\ \varphi(x - 2l) - \Phi(x - l), & l < x < \infty. \end{cases}$$

4. Green's Function. Integral Representations of the Solutions of the First, Second, and Third Boundary-Value Problems

Let us return to the function $u(x, t)$ defined by Eq. (8) [see Sect. 3]. Since this function depends on x_0 and t_0, we shall denote it by $\omega(x, t, x_0, t_0)$. Let us suppose that the impulse of the body forces (9) is equal to unity and that the length of the segment $|x - x_0|$ on which it acts tends to zero. Then, when we take the limit, Eq. (8) yields a function of x and t

$$\omega(x, t, x_0, t_0) = \begin{cases} h_1, & |x - x_0| < a(t - t_0), \\ 0, & |x - x_0| > a(t - t_0), \end{cases} \qquad (8')$$

where

$$h_1 = \lim_{\varepsilon \to 0} \frac{1}{2a} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} h \, dy = \lim_{\varepsilon \to 0} \frac{1}{2a} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \frac{l}{2\varepsilon\rho} \, dy = \frac{1}{2a\rho}.$$

The solution $\omega(x, t, x_0, t_0)$ that we have obtained for the homogeneous string equation can be interpreted as the disturbance at a point x at the instant t caused by a concentrated unit impulse applied at the point x_0 at the instant t_0. Therefore, this solution may be called a *unit impulse function*. Obviously, this function as a function of x_0 and t_0 for fixed values of x and t will be equal to $1/2a\rho$ when $|x - x_0| \leqslant a(t - t_0)$ (i.e., in the angle bounded by the characteristic passing through the point (x, t) in the downward direction) and will be equal to 0 outside that angle. The function $v(x, t, x', t') = 2a\rho\omega(x, t, x', t')$ may be called the *Riemann function for the one-dimensional wave equation (5)*.*

It is possible to arrive at the unit impulse function $\omega(x, t, x_0, t_0)$ by a different procedure. Let $\omega_\varepsilon(x, t)$ denote a solution of the Cauchy problem,

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*The concepts of a unit impulse function and of a Riemann function will be presented from a more general point of view in Sect. 8.

$$\frac{\partial^2\omega_\varepsilon}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2\omega_\varepsilon}{\partial t^2}=-\frac{f_\varepsilon(x,t)}{a^2},$$

$$\omega_\varepsilon\Big|_{t=t_0}=0,\ \frac{\partial\omega_\varepsilon}{\partial t}\Big|_{t=t_0}=0,$$

where $f_\varepsilon(x, t)$ is everywhere equal to zero except in a neighborhood of a point $(x_0, t_0,)$ defined by the inequalities $-\varepsilon<x-x_0<\varepsilon$, $t_0\leqslant t\leqslant t_0+\Delta t$ and satisfying

$$\int_{t_0}^{t_0+\Delta t}dt\int_{x_0-\varepsilon}^{x_0+\varepsilon}\rho f_\varepsilon(x,t)\,dx\equiv 1. \tag{8''}$$

This last equation means that the impulse of the force applied to the interval $|x-x_0|<\varepsilon$ is equal to unity. In accordance with formula (10′), we have

$$\omega_\varepsilon(x,t)=\frac{1}{2a}\int_{t_0}^{t}d\tau\int_{x-a(t-\tau)}^{x+a(t-\tau)}f_\varepsilon(x',t-(t'-t_0))\,dx'.$$

From this and the fact that $f_\varepsilon(x, t)\neq 0$ is different from zero only in a neighborhood of the point (x_0, t_0), we have

$$\omega_\varepsilon(x,t)=\begin{cases}\dfrac{1}{2a}\displaystyle\int_{t_0}^{t_0+\Delta t}d\tau\int_{x_0-\varepsilon}^{x_0+\varepsilon}f_\varepsilon(x',\tau)\,dx' & \text{for } |x-x_0|<a(t-t_0)\\ 0 & \text{for } |x-x_0|>a(t-t_0).\end{cases}$$

If we now take the limit as $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\omega_\varepsilon(x,t)=\omega(x,t,x_0,t_0).$$

On the basis of this, we may assert that the unit impulse function $\omega(x, t, x_0, t_0)$ with respect to the variables x and t satisfies the differential equation

$$\Box\,\omega=\frac{\partial^2\omega}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2\omega}{\partial t^2}=-\frac{1}{a^2\rho}\delta(x,x_0)\,\delta(t,t_0). \tag{13}$$

Here, $\delta(x, x_0)$ and $\delta(t, t_0)$ are the δ-functions of the variables x and t, respectively. On the basis of (8″), they possess the property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x,t)\,\delta(x,x_0)\,\delta(t,t_0)\,dx\,dt=\Phi(x_0,t_0),$$

where $\Phi(x, t)$ is a continuous bounded function of its arguments (see Chapter 4, Sect. 4).

The real meaning of Eq. (13) consists in the fact that

$$\Box\omega_\varepsilon = -\frac{1}{a^2} f_\varepsilon(x, t),$$

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Box\,\omega_\varepsilon \Phi(x, t)\, dxdt = -\frac{1}{a^2\rho}\Phi(x_0, t_0).$$

The unit impulse function and the Riemann function should be regarded as a close analog of the instantaneous unit source function for parabolic equations.* Carrying the analogy with elliptic and parabolic equations further, we can introduce the concept of a Green's function for hyperbolic equations and construct integral formulas giving the solution of the first, second, and third boundary-value problems for various regions when we know the corresponding Green's function. From a physical point of view, the Green's function $G(P, t, P_0, t_0)$ for the n-dimensional wave equation (1) must represent the disturbance at the point P at the instant t caused by a concentrated impulse of a known magnitude at the point P_0 at the instant $t_0<t$ under the condition that the homogeneous boundary conditions of the corresponding boundary-value problem are satisfied on the boundary of the given region and under the condition that $G(P, t, P_0, t_0)$ and $\partial G(P, t, P_0, t_0)/\partial t$ vanish for $t<t_0$. This means that there are no disturbances in the region in question up to the instant t_0.

For the string equation

$$\Box u = \frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, t)}{a^2}, \tag{5}$$

the Green's function $G(x, t, x_0, t_0)$ may be defined by the equation

$$G(x, t, x_0, t_0) = \rho\omega(x, t, x_0, t_0) + g(x, t, x_0, t_0),$$

where ρ is the linear density, $\omega(x, t, x_0, t_0)$ is the unit impulse function, and $g(x, t, x_0, t_0)$ as a function of x and t satisfies the homogeneous string equation, so that $G(x, t, x_0, t_0)$ satisfies the homogeneous boundary conditions of the corresponding boundary-value problem at the ends of the interval in question $(0, l)$ and the functions $G(x, t, x_0, t_0)$ and $\partial G(x, t, x_0, t_0)/\partial t$ vanish in the interval $(0, l)$ for $t<t_0$.

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*In some books on the equations of mathematical physics, the Riemann function, or what amounts to almost the same thing, the unit impulse function, is called Green's function for hyperbolic equations (cf., for example, A. G. Webster and G. Szegö, *Partial Differential Equations of Mathematical Physics*).

The function $G(x, t, x_0, t_0)$ as a function of the variables x and t satisfies the equation

$$\Box G = \frac{\partial^2 G}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 G}{\partial t^2} = -\frac{1}{a^2}\delta(x, x_0)\,\delta(t, t_0). \tag{13'}$$

It is easy to point out the properties of the Green's function $G(x, t, x_0, t_0)$ as a function of x_0 and t_0. We have the so-called reciprocity relation

$$G(x, t, x_0, t_0) = G(x_0, -t_0, x, -t). \tag{14}$$

To interpret this relation, it is convenient to set $t_0=0$. Then, we obtain $G(x, t, x_0, 0)=G(x_0, 0, x, -t)$. This means that the disturbance at a point x at an instant t caused by an impulse at the point x_0 at the instant $t_0=0$ is equal to the disturbance at the point x_0 at the instant $t_0=0$ caused by an impulse acting at the point x at the instant $-t$.

To prove Eq. (14), let us write the equations that $G(x, t, x_0, t_0)$ and $G(x, -t, x_1, -t_1)$, as functions of the variables x and t, satisfy:

$$\Box G(x, t, x_0, t_0) = -\frac{1}{a^2}\delta(x, x_0)\,\delta(t, t_0),$$

$$\Box G(x, -t, x_1, -t_1) = -\frac{1}{a^2}\delta(x, x_1)\,\delta(-t, -t_1).$$

If we multiply the first equation by $G(x, -t, x_1, -t_1)$ and the second by $G(x, t, x_0, t_0)$, subtract the first of the resulting products from the second, and integrate with respect to x from 0 to l and with respect to the time t from $-\infty$ to $t'>t_1$, we obtain

$$\int_{-\infty}^{t'} dt \int_0^l [G(x, t, x_0, t_0)\,\Box G(x, -t, x_1, -t_1) - G(x, -t, x_1, -t_1) \times$$

$$\times \Box G(x, t, x_0, t_0)]\,dx = \frac{1}{a^2}G(x_0, -t_0, x_1, -t_1) - \frac{1}{a^2}G(x_1, t_1, x_0, t_0). \tag{14'}$$

Integration by parts yields the identity

$$\int_{-\infty}^{t'} dt \int_0^l (v\,\Box u - u\,\Box v)\,dx = \int_{-\infty}^{t'} \left[v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right]_0^l dt -$$

$$-\frac{1}{a^2}\int_0^l \left[v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\right]_{-\infty}^{t'} dx. \tag{14''}$$

Therefore, Eq. (14′) can be written in the form

$$\frac{1}{a^2}G(x_0, -t_0, x_1, -t_1) - \frac{1}{a^2}G(x_1, t_1, x_0, t_0) =$$

$$= \int_{-\infty}^{t'} \left[G(x, t, x_0, t_0) \frac{\partial G(x, -t, x_1, -t_1)}{\partial x} - \right.$$

$$\left. - G(x, -t, x_1, -t_1) \frac{\partial G(x, t, x_0, t_0)}{\partial x} \right]_0^l dt -$$

$$- \frac{1}{a^2} \int_0^l \left[G(x, t, x_0, t_0) \frac{\partial G(x, -t, x_1, -t_1)}{\partial t} - \right.$$

$$\left. - G(x, -t, x_1, -t_1) \frac{\partial G(x, t, x_0, t_0)}{\partial t} \right|_{-\infty}^{t'} dx. \qquad (14''')$$

The first of the integrals on the right is equal to 0 since in both cases the Green's function at $x = 0$ and $x = l$ satisfies the homogeneous boundary conditions of the corresponding boundary-value problem (first, second, or third). The integrand in the second integral on the right-hand side of Eq. (14‴) vanishes at $t = -\infty$ since

$$G(x, -\infty, x_1, -t_1) = 0,$$

and

$$\frac{\partial G(x, -t, x_1, -t_1)}{\partial t}\Big|_{t=-\infty} = 0,$$

and this integrand vanishes at $t = t'$ since

$$G(x_1, -t', x_1, -t_1) = 0,$$

and

$$\frac{\partial G(x_1, -t, x_1, -t_1)}{\partial t}\Big|_{t=t'} = 0$$

(because the instant $-t'$ precedes the instant $-t_1$). Consequently, the right-hand side of Eq. (14‴) is equal to 0 and the reciprocity relation (14) is valid.

As a consequence of the reciprocity relation, we conclude that the Green's function $G(x, t, x_0, t_0)$ as a function of x_0 and t_0 satisfies the equation

$$\square_0 G(x, t, x_0, t_0) = \frac{\partial^2 G(x, t, x_0, t_0)}{\partial x_0^2} - \frac{1}{a^2} \frac{\partial^2 G(x, t, x_0, t_0)}{\partial t_0^2} =$$

$$= -\frac{1}{a^2} \delta(x_0, x) \delta(-t_0, -t). \qquad (13'')$$

We shall now give integral representations of the solutions of the boundary-value problems for the string equation (5) with the aid of Green's function. Here, we shall assume for definiteness that the initial conditions are given for $t=0$:

$$u\Big|_{t=0} = \psi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x).$$

In Eq. (5), we replace x and t with x_0 and t_0 and we multiply both members of the equation by $G(x, t, x_0, t_0)$. From the resulting equation, we subtract the product obtained by multiplying both members of Eq. (13″) by $u(x_0, t_0)$. When we integrate the result with respect to x_0 from 0 to l and with respect to t_0 from 0 to $t+0$, we obtain

$$\int_0^{t+0} dt_0 \int_0^l [G(x, t, x_0, t_0) \square_0 u(x_0, t_0) - u(x_0, t_0) \square_0 G(x, t, x_0, t_0)] dx_0 =$$

$$= -\frac{1}{a^2} \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0) dx_0 + \frac{1}{a^2} u(x, t).$$

From (14″) we then obtain

$$u(x, t) = a^2 \int_0^{t+0} \left[G(x, t, x_0, t_0) \frac{\partial u(x_0, t_0)}{\partial x_0} - \right.$$

$$\left. - u(x_0, t_0) \frac{\partial G(x, t, x_0, t_0)}{\partial x_0} \right]_{x_0=0}^{x_0=l} dt_0 - \int_0^l \left[G(x, t, x_0, t_0) \frac{\partial u(x_0, t_0)}{\partial t_0} - \right.$$

$$\left. - u(x_0, t_0) \frac{\partial G(x, t, x_0, t_0)}{\partial t_0} \right]_{t_0=0}^{t_0=t+0} dx_0 +$$

$$+ \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0) dx_0.$$

The integrand in the second integral on the right-hand side of this equation vanishes at $t_0 = t+0$ since

$$G(x, t, x_0, t_0)\Big|_{t_0=t+0} = G(x_0, -t_0, x, -t)\Big|_{t_0=t+0} = 0,$$

$$\frac{\partial G(x, t, x_0, t_0)}{\partial t_0}\Big|_{t_0=t+0} = \frac{\partial G(x_0, -t_0, x, -t)}{\partial t_0}\Big|_{t_0=t+0} = 0.$$

In view of this, we obtain the desired formula for the integral representations of the solutions of the boundary-value problems for the string equation (5) with the aid of the Green's function:

$$u(x, t) = a^2 \int_0^{t+0} \Big[G(x, t, x_0, t_0) \frac{\partial u(x_0, t_0)}{\partial x_0} -$$

$$- u(x_0, t_0) \frac{\partial G(x, t, x_0, t_0)}{\partial x_0} \Big]_{x_0=0}^{x_0=l} dt_0 - \int_0^l \Big(G(x, t, x_0, 0)\, \psi_1(x_0) -$$

$$- \frac{\partial G(x, t, x_0, t_0)}{\partial t_0}\Big|_{t_0=0} \psi(x_0) \Big) dx_0 +$$

$$+ \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0)\, dx_0. \qquad (15)$$

In this formula, the last two integrals in the right-hand member are known and the first integral is completely determined by the boundary conditions of the problem.

In the case of the first boundary-value problem,

$$u(0, t_0) = \varphi_1(t_0), \quad u(l, t_0) = \varphi_2(t_0),$$
$$G(x, t, x_0, t_0) = G(x_0, -t_0, x, -t) = 0 \text{ for } x_0 = 0 \; x_0 = l$$

and formula (15) takes the form

$$u(x, t) = a^2 \int_0^{t+0} \Big(\frac{\partial G(x, t, x_0, t_0)}{\partial x_0}\Big|_{x_0=0} \varphi_1(t_0) -$$

$$- \frac{\partial G(x, t, x_0, t_0)}{\partial x_0}\Big|_{x_0=l} \varphi_2(t_0) \Big) dt_0 -$$

$$- \int_0^l \Big(G(x, t, x_0, 0)\, \psi_1(x_0) - \frac{\partial G(x, t, x_0, t_0)}{\partial t_0}\Big|_{t_0=0} \psi(x_0) \Big) dx_0 +$$

$$+ \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0)\, dx_0. \qquad (15')$$

In the case of the second boundary-value problem,

$$\left.\frac{\partial u(x_0, t)}{\partial x_0}\right|_{x_0=0} = \varphi_1(t_0), \quad -\left.\frac{\partial u(x_0, t_0)}{\partial x_0}\right|_{x_0=l} = \varphi_2(t_0),$$

$$\frac{\partial G(x, t, x_0, t_0)}{\partial x_0} = \frac{\partial G(x_0, -t_0, x, -t)}{\partial x_0} = 0 \text{ при } x_0 = 0,\ x_0 = l$$

and formula (15) takes the form

$$u(x,t) = -a^2 \int_0^{t+0} (G(x, t, 0, t_0)\varphi_1(t_0) + G(x, t, l, t_0)\varphi_2(t_0))\, dt_0 -$$

$$- \int_0^l \left(G(x, t, x_0, 0)\psi_1(x_0) - \left.\frac{\partial G(x, t, x_0, t_0)}{\partial t_0}\right|_{t_0=0} \psi(x_0) \right) dx_0 +$$

$$+ \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0)\, dx_0. \qquad (15'')$$

In the case of the third boundary-value problem,

$$\left(\frac{\partial u(x_0, t_0)}{\partial x_0} - \alpha u(x_0, t_0)\right)_{x_0=0} = \varphi_1(t_0),$$

$$\left(-\frac{\partial u(x_0, t_0)}{\partial x_0} - \alpha u(x_0, t_0)\right)_{x_0=l} = \varphi_2(t_0),$$

$$\left(\frac{\partial G(x, t, x_0, t_0)}{\partial x_0} - \alpha G(x, t, x_0, t_0)\right)_{x_0=0} =$$

$$= \left(\frac{\partial G(x_0, -t_0, x, -t)}{\partial x_0} - \alpha G(x_0, -t_0, x, -t)\right)_{x_0=0} = 0,$$

$$\left(-\frac{\partial G(x, t, x_0, t_0)}{\partial x_0} - \alpha G(x, t, x_0, t_0)\right)_{x_0=l} =$$

$$= \left(-\frac{G(x_0, -t_0, x, -t)}{\partial x_0} - \alpha G(x_0, -t_0, x, -t)\right)_{x_0=l} = 0$$

and formula (15) takes the form

$$u(x, t) = -a^2 \int_0^{t+0} (G(x, t, 0, t_0)\varphi_1(t_0) + G(x, t, l, t_0)\varphi_2(t_0))\, dt_0 -$$

$$- \int_0^l \left(G(x, t, x_0, 0)\psi_1(x_0) - \left.\frac{\partial G(x, t, x_0, t_0)}{\partial t_0}\right|_{t_0=0} \psi(x_0) \right) dx_0 +$$

$$+ \int_0^{t+0} dt_0 \int_0^l G(x, t, x_0, t_0) f(x_0, t_0)\, dx_0. \qquad (15''')$$

In connection with formulas (15′), (15″), (15‴), it should be noted that the right-hand members of these formulas should be

regarded as the expected result of the solution of the corresponding boundary-value problems. The point is that we could not in advance impose on the Green's function or on the desired solution of the boundary-value problem the restrictions under which we can, in accordance with the rules of mathematical analysis, use the identity (14″) or Ostrogradskiy's formula. Therefore, in each particular case, the solutions obtained with the aid of formulas (15′), (15″), (15‴) must be checked. The check usually yields affirmative results.

The results that we have obtained concerning integral representations of the solutions of the first, second, and third boundary-value problems for the string equation with the aid of the Green's function can without any difficulty be generalized to the case of the n-dimensional wave equation (1). Here, we only need to choose the magnitude of the concentrated impulse at the point P_0 at the instant t_0 in such a way that $G(P, t, P_0, t_0)$ as a function of the coordinates of the point P and the time t will satisfy not Eq. (13′) but Eq. (1) with right-hand member equal to $-a^2\ \delta(P, P_0)\ \delta(t, t_0)$ and, instead of using (14″), we need to use the equation

$$\int_{-\infty}^{t'} dt \int_G (v\Box u - u\Box v)\, dx_1\, dx_2 \ldots dx_n =$$

$$= -\int_{-\infty}^{t'} dt \int_S \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS - \frac{1}{a^2}\int_G \left[v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\right]_{t=-\infty}^{t-t'} dx_1 dx_2 \ldots dx_n,$$

which follows directly from Ostrogradskiy's formula applied to the cylindrical domain Ω (see Sect. 1).

As a result, we conclude that the Green's function $G(P, t, P_0, t_0)$ satisfies the reciprocity relation (14), where we need to replace x and x_0 with the points P and P_0, respectively. For the solutions of the first, second, and third boundary-value problems for Eq. (1) with initial conditions given for $t=0$, we have the formula for the integral representations

$$u(P, t) = -a^2 \int_0^{t+0} dt_0 \int_S \left(G(P, t, P_0, t_0)\frac{\partial u(P_0, t_0)}{\partial n_0} - \right.$$

$$\left. - u(P_0, t_0)\frac{\partial G(P, t, P_0, t_0)}{\partial n_0}\right) dS_0 -$$

$$- \int_G \left(G(P, t, P_0, 0)\psi_1(P_0) - \frac{\partial G(P, t, P_0, t_0)}{\partial t_0}\Big|_{t_0=0}\psi(P_0)\right) d\omega_0 +$$

$$+ \int_0^{t+0} dt_0 \int_G G(P, t, P_0, t_0) f(P_0, t_0)\, d\omega_0, \qquad (16)$$

where $\vec{n_0}$ denotes the inner normal to S at the point P_0, dS_0 is an element of area at that point, $d\omega_0$ is an element of volume of the domain G at the point P_0, and $G(P, t, P_0, t_0)$ is the Green's function of the corresponding boundary-value problem.

An example of the construction of the Green's function for the wave equation and an application of formula (16) will be given in Chapter 6, Sect. 1, subsection 1.

However, explicit construction of Green's functions for hyperbolic equations, even for some restricted class of domains, is associated with considerable difficulties. Here, in contrast with the situation with elliptic and parabolic equations, the Green's function does not lead to a comparatively simple method of obtaining explicit solutions of the first, second, and third boundary-value problems for even a moderately broad class of regions.

5. Solution of the Cauchy Problem for the Three- and Two-Dimensional Wave Equations. Physical Conclusions. Invariance of the Wave Equation Under the Lorentz Transformation

Suppose that $u(x, y, z, t)$ is an arbitrary twice continuously differentiable function of the variables x, y, and z. Then, its mean value

$$Q(P, r, t) = \frac{1}{4\pi r^2} \iint\limits_{S_{r,P}} u(\alpha_1, \alpha_2, \alpha_3, t)\, d\sigma, \tag{17}$$

where r is a positive number and $S_{r,P}$ is the sphere of radius r with center at the point $P(x, y, z)$, satisfies the so-called Darboux equation

$$\Delta Q - Q_{rr} - \frac{2}{r} Q_r = 0 \quad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \tag{18}$$

and the additional conditions

$$Q|_{r=0} = u(x, y, z, t), \quad Q_r|_{r=0} = 0. \tag{18'}$$

Proof: By virtue of the mean-value theorem, the first of conditions (18') is satisfied since the area of the sphere is equal to $4\pi r^2$. Furthermore, if we denote by $S_{1,P}$ a sphere of unit radius, by $d\sigma_1$ an element of its area, and by β_1, β_2, and β_3 the direction cosines of a unit vector, we obtain

$$Q_r = \frac{1}{4\pi}\iint\limits_{S_{1,P}} \sum_{i=1}^{3} u_{\alpha_i}\beta_i\, d\sigma_1 = \frac{1}{4\pi r^2}\iint\limits_{S_{r,P}} \sum_{i=1}^{3} u_{\alpha}\beta_i\, d\sigma = \frac{1}{4\pi r^2}\iiint\limits_{e_r} \Delta u\, d\tau,$$

where e_r is the sphere of radius r with center at the point $P(x, y, z)$. If we now recall that the volume of the sphere is equal to $4\pi r^3/3$, we see that the second of conditions (18′) is satisfied. If we differentiate the last equation, we obtain

$$Q_{rr} = -\frac{2}{r}Q_r + \frac{1}{4\pi r^2}\iint\limits_{S_{r,P}} \Delta u\, d\sigma.$$

If we compare this with the equation

$$\Delta Q = \Delta \frac{1}{4\pi}\iint\limits_{S_{1,P}} u(\alpha_1, \alpha_2, \alpha_3, t)\, d\sigma_1 = \frac{1}{4\pi r^2}\iint\limits_{S_{r,P}} \Delta u\, d\sigma,$$

we see that the function $Q(P, r, t)$ satisfies Eq. (18).

Suppose now that $u(x, y, z, t)$ is a twice continuously differentiable solution of the Cauchy problem for homogeneous three-dimensional wave equation

$$\Delta u - \frac{1}{a^2}\frac{\partial u}{\partial t^2} = 0, \tag{19}$$

$$u|_{t=0} = \psi(P), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(P). \tag{19′}$$

Then, by virtue of (18), the mean value of the solution $u(x, y, z, t)$ determined by Eq. (17) satisfies the equation

$$Q_{rr} - \frac{1}{a^2}Q_{tt} + \frac{2}{r}Q_r = 0 \quad \left(\Delta Q = \frac{1}{a^2}Q_{tt}\right).$$

If we multiply by r, this equation takes the form

$$\frac{\partial^2 rQ}{\partial r^2} - \frac{1}{a^2}\frac{\partial^2 rQ}{\partial t^2} = 0$$

and, consequently,

$$rQ(P, r, t) = F(at + r) + F_1(at - r),$$

where F and F_1 are arbitrary twice continuously differentiable functions of their arguments. If we set $r=0$, we have

$$F(at) + F_1(at) = 0, \text{ т. е. } Q(P, r, t) = \frac{1}{r}[F(at + r) - F(at - r)].$$

If we let r approach 0 and take the first of conditions (18′) into account, we obtain

$$u(x, y, z, t) = 2\frac{dF(at)}{d\,at} = 2F'(at).$$

Now, it only remains to express $2F'(at)$ in terms of the initial conditions that the function $u(x, y, z, t)$ satisfies. We have

$$\frac{\partial rQ}{\partial r} = F'(at + r) + F'(at - r), \frac{\partial rQ}{\partial t} = [F'(at + r) - F'(at - r)]a.$$

Therefore, by setting $t=0$, we obtain

$$2F'(r) = \left[\frac{\partial rQ}{\partial r} + \frac{1}{a}\frac{\partial rQ}{\partial t}\right]_{t=0} =$$

$$= \left[\frac{\partial}{\partial r}\frac{1}{4\pi}\iint\limits_{S_{r,P}}\frac{u}{r}d\sigma + \frac{1}{4\pi a}\iint\limits_{S_{r,P}}\frac{\frac{\partial u}{\partial t}}{r}d\sigma\right]_{t=0}.$$

Consequently, if a twice continuously differentiable solution of the Cauchy problem (19), (19′) exists, it is given by the equation

$$u(x, y, z, t) = \frac{\partial}{\partial t}\frac{1}{4\pi a^2}\iint\limits_{S_{at,P}}\frac{\psi(\alpha_1, \alpha_2, \alpha_3)}{t}d\sigma +$$

$$+ \frac{1}{4\pi a^2}\iint\limits_{S_{at,P}}\frac{\psi_1(\alpha_1, \alpha_2, \alpha_3)}{t}d\sigma. \tag{20}$$

This equation can be written in the form

$$u(x, y, z, t) = \frac{\partial}{\partial t}U_\psi + U_{\psi_1}, \tag{21}$$

where

$$U_\psi = \frac{1}{4\pi a^2}\iint\limits_{S_{at,P}}\frac{\psi(\alpha_1, \alpha_2, \alpha_3)}{t}d\sigma. \tag{21′}$$

To prove the existence of a twice continuously differentiable solution of the Cauchy problem (19), (19′), let us suppose that the initial function $\psi_1(x, y, z)$ is twice continuously differentiable and

that the initial function $\psi(x, y, z)$ is three times continuously differentiable. If we write the function U_{ψ_1} in the form

$$U_{\psi_1} = \frac{t}{4\pi} \iint\limits_{S_{1,P}} \psi_1 (x + at\beta_1,\ y + at\beta_2,\ z + at\beta_3)\, d\sigma_1,$$

we note that it is twice continuously differentiable and satisfies the condition $U_{\psi_1}|_{t=0} = 0$.

Furthermore, we obtain

$$\frac{\partial U_{\psi_1}}{\partial t} = \frac{1}{4\pi} \iint\limits_{S_{1,P}} \psi_1 (x + at\beta_1,\ y + at\beta_2,\ z + at\beta_3)\, d\sigma_1 +$$

$$+ \frac{ta}{4\pi} \iint\limits_{S_{1,P}} \sum_{k=1}^{3} \psi_{1\alpha_k} \beta_k\, d\sigma_1.$$

From this it follows that

$$\left.\frac{\partial U_{\psi_1}}{\partial t}\right|_{t=0} = \psi_1 (x, y, z)$$

and, in addition,

$$\frac{\partial U_{\psi_1}}{\partial t} = \frac{U_{\psi_1}}{t} + \frac{1}{4\pi at} \iint\limits_{S_{at,P}} \sum_{k=1}^{3} \psi_{1\alpha_k} \beta_k\, d\sigma = \frac{U_{\psi_1}}{t} + \frac{I}{4\pi at},$$

$$I = \iiint\limits_{e_{at,P}} \sum_{k=1}^{3} \psi_{1\alpha_k \alpha_k}\, d\tau.$$

From this we get

$$\frac{\partial^2 U_{\psi_1}}{\partial t^2} = -\frac{U_{\psi_1}}{t^2} + \frac{1}{t}\left[\frac{U_{\psi_1}}{t} + \frac{'I}{4\pi at^2}\right] - \frac{I}{4\pi at^2} + \frac{1}{4\pi at}\frac{\partial I}{\partial t} =$$

$$= \frac{1}{4\pi at}\frac{\partial I}{\partial t} = \frac{1}{4\pi t} \iint\limits_{S_{at,P}} \sum_{k=1}^{3} \psi_{1\alpha_k \alpha_k}\, d\sigma.$$

Simple calculation yields

$$\Delta U_{\psi_1} = \frac{\partial^2 U_{\psi_1}}{\partial x^2} + \frac{\partial^2 U_{\psi_1}}{\partial y^2} + \frac{\partial^2 U_{\psi_1}}{\partial z^2} =$$

$$= \Delta \frac{t}{4\pi} \iint\limits_{S_{1,P}} \psi_1 (x + at\beta_1,\ y + at\beta_2,\ z + at\beta_3)\, d\sigma_1 =$$

$$= \frac{1}{4\pi a^2 t} \iint\limits_{S_{at,P}} \sum_{k=1}^{3} \psi_{1\alpha_k \alpha_k}\, d\sigma$$

Consequently, U_{ψ_1} is a twice continuously differentiable solution of the homogeneous wave equation (19) with initial conditions

$$U_{\psi_1}|_{t=0} = 0,$$

$$\frac{\partial U_{\psi_1}}{\partial t}\bigg|_{t=0} = \psi_1 (x, y, z).$$

In just the same way, the function U_ψ can be shown to be a three times continuously differentiable solution of the same equation with the initial conditions

$$U_\psi|_{t=0} = 0,$$

$$\frac{\partial U_\psi}{\partial t}\bigg|_{t=0} = \psi (x, y, z).$$

Therefore, the function $V = \partial U_\psi / \partial t$ is a twice continuously differentiable solution of Eq. (19) with the initial conditions

$$V|_{t=0} = \psi (x, y, z),\ \frac{\partial V}{\partial t}\bigg|_{t=0} = \frac{\partial^2 U_\psi}{\partial t^2}\bigg|_{t=0} = a^2 \Delta U_\psi|_{t=0} = 0.$$

Consequently, the function defined by Eq. (21) is indeed a twice continuously differentiable solution of the Cauchy problem (19), (19′). Thus, we have proven the existence and the uniqueness of the desired solution and have given an explicit expression for it in the form of formula (21). The right-hand side of the last formula is often referred to as *Poisson's solution.*

Let us now solve the Cauchy problem for the three-dimensional nonhomogeneous wave equation with the homogeneous initial conditions

$$\Delta u - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = -\frac{f(P, t)}{a^2}, \tag{22}$$

$$u|_{t=0} = 0,\ \frac{\partial u}{\partial t}\bigg|_{t=0} = 0. \tag{22′}$$

Assuming that the function $f(P, t) = f(x, y, z, t)$ is twice continuously differentiable, in analogy with formula (15), which was obtained by

the method of impulses, let us seek a solution of the problem (22), (22′) in the form

$$v(P, t) = \int_0^t \varphi(P, t, t_0)\, dt_0, \tag{22″}$$

$$\varphi(P, t, t_0) = \frac{1}{4\pi a^2} \iint\limits_{S_{a(t-t_0),P}} \frac{f(\alpha_1, \alpha_2, \alpha_3, t_0)}{t - t_0}\, d\sigma.$$

On the basis of Poisson's integral (21), the function $\varphi(P, t, t_0)$ is a twice continuously differentiable solution of the homogeneous wave equation and it satisfies the initial conditions

$$\varphi(P, t_0, t_0) = 0, \quad \frac{\partial \varphi}{\partial t}\Big|_{t=t_0} = f(P, t_0).$$

From this we obtain

$$v|_{t=0} = 0, \quad \frac{\partial v}{\partial t} = \int_0^t \varphi_t(P, t, t_0)\, dt_0, \quad \frac{\partial v}{\partial t}\Big|_{t=t_0} = 0.$$

Furthermore, we have

$$\frac{\partial^2 v}{\partial t^2} = \varphi_t(P, t, t) + \int_0^t \varphi_{tt}(P, t, t_0)\, dt_0 = f(P, t) + \int_0^t \varphi_{tt}(P, t, t_0)\, dt_0,$$

$$\Delta v = \int_0^t \Delta\varphi(P, t, t_0)\, dt_0.$$

Therefore, by setting $\Box = \Delta - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}$, we obtain

$$\Box v = \int_0^t \Box\, \varphi(P, t, t_0)\, dt_0 - \frac{1}{a^2}\varphi_t(P, t, t) = -\frac{f(P, t)}{a^2}.$$

This means that Eq. (22″) does indeed determine a twice continuously differentiable solution of the Cauchy problem (22), (22′). If we set $t_0 = t - r/a$, this equation may be written in the form

$$v(x, y, z, t) = \frac{1}{4\pi a^2} \iiint\limits_{r<at} \frac{f\left(\alpha_1, \alpha_2, \alpha_3, t - \frac{r}{a}\right)}{r} d\tau \tag{23}$$

$$(r = \sqrt{(x-\alpha_1)^2 + (y-\alpha_2)^2 + (z-\alpha_3)^2}).$$

The last integral is called a *retarded potential.*

The solution of the Cauchy problem for the three-dimensional nonhomogeneous equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, y, z, t)}{a^2}, \tag{23'}$$

$$u|_{t=0} = \psi(x, y, z), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x, y, z)$$

may be written in the form of the sum of Poisson's integral (21) and the retarded potential (23).

To obtain the solution of the Cauchy problem for the two-dimensional equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, y, t)}{a^2}, \tag{23''}$$

$$u|_{t=0} = \psi(x, y), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x, y),$$

we need only suppose that all the functions in the formulas which give the solution of the Cauchy problem for the three-dimensional equation, depend only on the two space variables x and y. In this case, the function U_ψ defined by Eq. (21′) takes the form

$$U_\psi = U_\psi(x, y, t) = \frac{1}{4\pi a^2} \iint\limits_{S_{at,P}} \frac{\psi(\alpha_1, \alpha_2)}{t} d\sigma,$$

where P is a point with coordinates x and y. If we denote by $K_{at,P}$ the circle of radius at with center at the point P and recall that

$$d\sigma = \frac{at}{\sqrt{a^2t^2 - \rho^2}} d\alpha_1 d\alpha_2, \quad \rho = \sqrt{(x-\alpha_1)^2 + (y-\alpha_2)^2},$$

we may replace the integral over the sphere $S_{at,P}$ with two integrals over the circle $K_{at,P}$. Thus, we obtain

$$U_\psi(x, y, t) = \frac{1}{2\pi a} \iint\limits_{K_{at,P}} \frac{\psi(\alpha_1, \alpha_2)\, d\alpha_1 d\alpha_2}{\sqrt{a^2t^2 - \rho^2}}. \tag{24}$$

In view of this, the retarded potential (22″) takes the form

$$v(x, y, t) = \int_0^t \varphi(x, y, t, t_0)\, dt_0,$$

$$\varphi(x, y, t, t_0) = \frac{1}{2\pi a} \iint\limits_{K_{a(t-t_0)}, P} \frac{f(\alpha_1, \alpha_2, t_0)\, d\alpha_1\, d\alpha_2}{\sqrt{a^2(t-t_0)^2 - \rho^2}}$$

or, after we make the change of variables $(t - t_0)a = r$,

$$v(x, y, t) = \frac{1}{2\pi a^2} \int_0^{at} dr \iint\limits_{\rho<r} \frac{f\left(\alpha_1, \alpha_2, t - \frac{r}{a}\right)}{\sqrt{r^2 - \rho^2}}\, d\alpha_1\, d\alpha_2. \tag{25}$$

Thus, the solution of the Cauchy problem for two-dimensional equation (23″) is the sum of the right-hand member of Eq. (21), in which the function U_ψ is defined by Eq. (24), and retarded potential (25).

In the case of the Cauchy problem for the one-dimensional equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f(x, t)}{a^2}, \tag{5}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x) \tag{5′}$$

the function (21′) takes the form

$$U_\psi(x, t) = \frac{1}{4\pi a^2} \iint\limits_{S_{at}, P} \frac{\psi(\alpha_1)}{t}\, d\sigma = \frac{1}{2a} \int_{x-at}^{x+at} \psi(\alpha_1)\, d\alpha_1, \tag{26}$$

since the element of area of the sphere intercepted by the planes $x = \alpha_1$ and $x = \alpha_1 + d\alpha_1$ is equal to $2\pi at\, d\alpha_1$. In this case, formula (21) coincides with d'Alembert's solution (7) since

$$\frac{\partial}{\partial t}\frac{1}{2a}\int_{x-at}^{x+at} \psi(\alpha_1)\, d\alpha_1 = \frac{\psi(x - at) + \psi(x + at)}{2}.$$

By virtue of (26), the retarded potential (22″) may be written in the form

$$v(x, t) = \int_0^t \varphi(x, t, t_0)\, dt_0, \quad \varphi(x, t, t_0) = \frac{1}{2a} \int_{x-a(t-t_0)}^{x+a(t-t_0)} f(\alpha_1, t_0)\, d\alpha_1,$$

which, after the change of variables $(t-t_0)\,a=r$, coincides with formula (12) [see Sect. 3].

Let us now draw some physical conclusions from the solution of the Cauchy problem for the homogeneous wave equation. In the one-dimensional case, Poisson's integral coincides with d'Alembert's formula (7) and here, as we saw in the preceding section, diffusion of waves is possible. This is explained by the fact that at the point (x, t) the value determined by d'Alembert's formula depends on the initial values on a segment of the x-axis, which constitutes the domain of dependence of the point (x, t). Wave diffusion also takes place in the two-dimensional case. To see this, suppose that the initial functions $\psi(x, y)$ and $\psi_1(x, y)$ are different from zero only in a neighborhood of the coordinate origin of the xy-plane. Then, their mean values $Q_\psi(P, at, 0)$ and $Q_{\psi_1}(P, at, 0)$, and hence the value determined by Poisson's formula at the point P with coordinates x and y, will be zero up to the instant t when the circle $K_{at, P}$ of radius at with center at the point P covers the region where the values of the initial functions are not zero. At that instant, we observe at the point P the leading front of the wave, which is being propagated from the neighborhood of the coordinate origin with velocity equal to a. Beginning at this instant, Poisson's integral, although it decreases with the passage of time, will remain different from zero since the region of integration $K_{at, P}$ will always contain the region of the initial disturbances. This means that in this case there will be no trailing wavefront.

The picture is quite different in the three-dimensional case. Now, if the initial functions $\psi(P)$ and $\psi_1(P)$ are different from zero only in a neighborhood of the coordinate origin, then their mean values $Q_\psi(P, at, 0)$ and $Q_{\psi_1}(P, at, 0)$ and, consequently, the value given by Poisson's formula at the point P with coordinates x, y, and z, will be different from zero only for those instants of time t at which the sphere $S_{at, P}$ of radius at with center at the point P intersects the region where the initial functions are not zero. Thus, in this case, we have not only a well-defined leading wavefront but also a well-defined trailing wavefront; i.e., there will be no wave diffusion. An observer at the point P will see disturbances that are propagated from the neighborhood of the coordinate origin with velocity a only during the course of a well-defined interval. This phenomenon is called *Huygens' principle for the wave equation.* It brings out the fact that an initial sharply localized disturbance is propagated through space in such a way that it appears later in another place in the form of a disturbance just as sharply restricted in time. If the initial disturbance is concentrated at a single point, its effect at another point is concentrated at a definite instant of time corresponding to the distance from the original point. This last phenomenon is corroborated by the fact that the beginning and end of a period of audibility of a sound emitted at any point away from the observer are always sharply defined.

It is obvious from Eq. (17) that if the function $u(x, y, z)$ is odd or even with respect to the variable z, its mean value will be given respectively by the equation $Q_u|_{z=0} = 0$, $\partial Q_u/\partial z|_{z=0} = 0$. Therefore, with the aid of Poisson's formula (21) and the retarded potential, we are able to find the solution of the first and second boundary-value problems for the wave equation for a half-space or for a layer, but only under homogeneous initial conditions. For example, to obtain the solution of the first boundary-value problem for the homogeneous equation for the half-space $z>0$ with homogeneous boundary conditions $u|_{z=0} =$, we need only extend the initial functions of the problem, $\psi(P)$ and $\psi_1(P)$, which are given for $z \geqslant 0$, as odd functions into the half-space $z<0$ and then substitute them into Poisson's formula (21). Analogous results are obtained by means of such an extension of the functions in the case of the second problem for the half-space and also in the case of the first and second problems for the layer $0<z<l$.

In conclusion, let us point out a method of solving the Cauchy problem for the three-dimensional wave equation in the case in which the initial conditions u and $\partial u/\partial t$ are given not for $t=0$ but on some hyperplane L passing through the coordinate origin $x=y=z=t=0$ and inclined at an angle greater than arctan a to the t-axis. Without loss of generality, we may assume that the equation of the hyperplane L is of the form* $x/t = a^2/v > a$ since we can always achieve this by making an orthogonal transformation of coordinates under which, as is easy to see, the three-dimensional Laplacian operator is mapped into the three-dimensional Laplacian operator in the new independent variables. If we now introduce new variables in the wave equation (1) by making the so-called Lorentz transformation**

*Here, v is a parameter introduced for convenience; $v<a$.

**In general, the term "Lorentz transformation" of the variables, x, y, z, and t is applied to any linear homogeneous transformation of these variables with real coefficients under which the quadratic form $t^2 - x^2 - y^2 - z^2$ remains invariant, i.e., is mapped into the form $t_1^2 - x_1^2 - y_1^2 - z_1^2$, where t_1, x_1, y_1 and z_1 are the new variables. It can be shown that every Lorentz transformation is a combination of (1) an orthogonal transformation of the variables x, y, and z with t left unchanged (2) a transformation of the form

$$x' = \frac{x - \beta t}{\sqrt{1-\beta^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \beta x}{\sqrt{1-\beta^2}},$$

where β is a number smaller in absolute value than unity, and (3) possibly a change of sign of some of the variables (reflection). Every nonsingular linear transformation of the variables x, y, z, and t such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

is a combination of a Lorentz transformation, a displacement of the coordinate origin in $xyzt$-space, and a similarity transformation in that space. For further remarks, see I. G. Petrovskiy, *Lectures on Partial Differential Equations*, New York, Interscience, 1950.

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}},\quad y' = y,\quad z' = z,\quad t' = \frac{t - \frac{v}{a^2}x}{\sqrt{1-\beta^2}}\quad \left(\beta = \frac{v}{a}\right), \tag{26'}$$

then the equation of the hyperplane L, namely, $x/t = a^2/v$, becomes $t'=0$ and the values of u and $\partial u/\partial t'$ for $t'=0$ are determined in terms of the initial conditions on that hyperplane.

A remarkable property of the Lorentz transformation (26′) is the fact that under this transformation, the three-dimensional wave equation remains invariant; i.e.,

$$\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} + \frac{\partial^2 u}{\partial z'^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t'^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{26''}$$

To see this, note that

$$\frac{\partial}{\partial x} - \frac{1}{u}\frac{\partial}{\partial t} = \frac{1}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'} + \frac{-\frac{v}{a^2}}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'} -$$

$$-\frac{1}{a}\left[\frac{-v}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'} + \frac{1}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'}\right] =$$

$$= \frac{1+\frac{v}{a}}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'} - \frac{1}{a}\frac{1+\frac{v}{a}}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'},$$

$$\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t} = \frac{1-\frac{v}{a}}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'} + \frac{1}{a}\frac{1-\frac{v}{a}}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'}.$$

Therefore, we have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)u = \left(\frac{\partial}{\partial x} - \frac{1}{a}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t}\right)u =$$

$$= \left(\frac{\partial^2}{\partial x'^2} - \frac{1}{a^2}\frac{\partial^2}{\partial t'^2}\right)u.$$

Consequently, since

$$\frac{\partial^2 u}{\partial y'^2} = \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial z'^2} = \frac{\partial^2 u}{\partial z^2},$$

Eq. (26″) is valid. Thus, the solution of the Cauchy problem for the three-dimensional wave equation with initial conditions on the hyperplane $x/t = a^2/v > a$ leads to a solution of the same problem with initial conditions given for $t'=0$.

6. Integral Representations of Solutions of the Wave Equation

1. Ostrogradskiy's formula for the wave operator. Let us consider the question of integral representations of twice continuously differentiable solutions of the wave equation

$$\Box u = \Delta u - \frac{\partial^2 u}{\partial t^2} = -f(P, t) \tag{27}$$

and, in particular, of the homogeneous wave equation

$$\Box u = \Delta u - \frac{\partial^2 u}{\partial t^2} = 0, \tag{27'}$$

where P is a point in n-dimensional space with coordinates $x_1, x_2, \ldots, x_n$, where Δ is the n-dimensional Laplace operator with respect to the coordinates of the point P, and where $f(P, t)$ is a given function of its arguments.

In contrast with Sect. 4, we shall now look at solutions of the wave equation not in cylindrical domain, but in domains of a more general nature.

Suppose S^* is an arbitrary piecewise-smooth n-dimensional surface which bounds an $(n+1)$-dimensional domain Ω^* in the phase space $x_1, x_2, \ldots, x_n, t$. Suppose also that S^* has the property that straight lines parallel to the coordinate axes have only finitely many points or segments in common with S^*. Let us denote by $\vec{n}^*$ the inner normal to the surface S^*. (For $n = 2$, the domain Ω^* degenerates to a volume in xyt-space; for $n=1$, it degenerates into the xt-plane).

It is easy to verify the identity

$$v\Box u - u\Box v = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\right) + \frac{\partial}{\partial t}\left(-v\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial t}\right). \tag{28}$$

This identity means that the expression $v\Box u - u\Box v$ for arbitrary functions u and v represent the divergence of the $(n+1)$-dimensional vector with components

$$P_i = v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}, \quad P_{n+1} = -v\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial t} \quad (i = 1, 2, \ldots, n).$$

If we apply Ostrogradskiy's formula to Eq. (28), we obtain

$$\int\cdots\int_{\Omega^*}(v\Box u - u\Box v)\,dx_1\ldots dx_n\,dt =$$

$$= -\int\cdots\int_{S^*}\left[\sum_{i=1}^{n}\left(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\right)\cos n^*x_i - \left(v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\right)\cos n^*t\right]dS. \tag{29}$$

Formula (29) can also be obtained by integration by parts.

Indeed, if we assume for simplicity that straight lines parallel to the coordinate axes intersect the surface S^* at two points, we have*

$$\int\cdots\int_{\Omega^*}\frac{\partial P_1}{\partial x_1}\,dx_1\,dx_2\ldots dx_n\,dt =$$

$$= -\int\cdots\int_{S^*}P_1\frac{dx_1}{dn^*}\,dS = -\int\cdots\int_{S^*}P_1\cos n^*x_1\,dS.$$

By making analogous calculations for the other terms in the right-hand member of Eq. (29), we verify its validity.

If we introduce the direction $\vec{N}$, symmetric with the inner normal $\vec{n}^*$ to S^* about the plane $t=0$, i.e.,

$$\cos n^*x_i = \cos Nx_i, \quad -\cos n^*t = \cos Nt \quad (i = 1, 2, \ldots, n),$$

formula (29) may be written in the form

$$\int\cdots\int_{\Omega^*}(v\Box u - u\Box v)\,dx_1\ldots dx_n\,dt = -\int\cdots\int_{S^*}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right)dS. \tag{30}$$

The direction $\vec{N}$ is called the *conormal*. Equation (30) can be regarded as *Ostrogradskiy's formula for the wave operator*. Let us consider this formula for $n=1$, $n=2$, and $n=3$.

2. The one-dimensional case. The fundamental integral formula. *Solution of the characteristic problem and the problem of the propagation of sound from a moving source.* We shall say that a domain Ω^* in the $x\,t$-plane *possesses property* α if its boundary S^* consists, in addition to segments of the

*If we use the fact that $\varphi(x_1, x_2, \ldots, x_n) = 0$ on the surface, we have, by definition

$$dS = \frac{1}{|\cos n^*x_n|}\,dx_1\ldots dx_n, \qquad \cos n^*x_n = \frac{dx_n}{dn^*} = -\varphi_{x_n}\left(\sum_{k=1}^{n}\varphi_{x_k}\right)^{-1},$$

characteristics, only of a curve C such that any pair of characteristics passing through the same point in the domain Ω^* intersects the curve C at a single point. Examples of domain Ω^* possessing this property are shown in Fig. 28. In cases a, b, c, and d, the curve C coincides with S^*; in case e, the curve C forms a part of S^*.

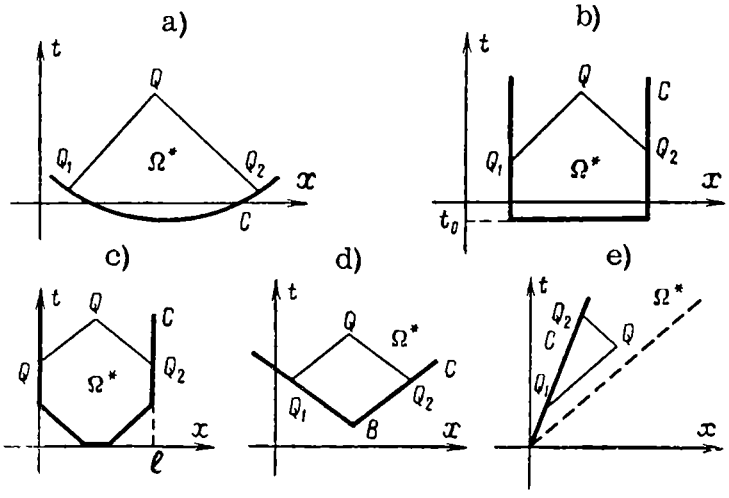

Fig. 28.

Suppose $Q(x, t)$ is a point in the domain Ω^*, Q_1 and Q_2 are points of intersection of characteristics extending from the point Q to the curve C, C_Q is that portion of C lying between the points Q_1 and Q_2, and Ω_Q^* is that portion of the domain Ω^* bounded by the characteristics extending from the point Q to the curve C_Q. For example, in Fig. 28a, the domain Ω_Q^* takes the form of a triangle Q_1QQ_2. In formula (30), let us take for the domain of integration the domain Ω_Q^* and let us take for v the Riemann function $v(x, t, x', t')$. In brief, we set $v \equiv 1$. Then, we obtain

$$\iint\limits_{\Omega_Q^*} -f(x', t')\,dx'\,dt' = -\int\limits_{C_Q} \frac{\partial u}{\partial N}\,ds - \int\limits_{Q_1Q} \frac{\partial u}{\partial N}\,ds - \int\limits_{Q_2Q} \frac{\partial u}{\partial N}\,ds. \quad (30')$$

Furthermore, on the characteristics Q_1Q and Q_2Q, the direction of the conormal coincides with the directions of the vectors $\vec{Q_1Q}$ and $\vec{Q_2Q}$. Therefore,

$$\int\limits_{Q_1Q} \frac{\partial u}{\partial N}\,ds = u(Q) - u(Q_1), \qquad \int\limits_{Q_2Q} \frac{\partial u}{\partial N}\,ds = u(Q) - u(Q_2).$$

Substituting this into Eq. (30′), we obtain the following *fundamental integral formula*:

$$u(Q) = \frac{u(Q_1) + u(Q_2)}{2} - \frac{1}{2}\int_{C_Q} \frac{\partial u}{\partial N}\, ds + \frac{1}{2}\iint_{\Omega_Q^*} f(x', t')\, dx'\, dt'. \qquad (31)$$

Let C_1 denote the noncharacteristic part of the curve C, i.e., the part not consisting of segments of the characteristics, and let C_2 denote the characteristic part, i.e., the set of segments of characteristics entering into the composition of the curve C. For example, in Figs. 28a, b, and e, the curve C coincides with C_1; in Fig. 28d, the curve C coincides with C_2; in Fig. 28c, the curve C consists of C_1 and C_2. The integral formula (31) is remarkable in that it gives an explicit representation for the solution of the one-dimensional wave equation (27) at an arbitrary point of the domain Ω^* if we know the values of this solution and its conormal derivative on C_1 and if we know just its values on C_2. For example, if the curve C shown in Fig. 28a is assumed to coincide with the x-axis, then

$$\left.\frac{\partial u}{\partial N}\right|_{t=0} = -\left.\frac{\partial u}{\partial t}\right|_{t=0},$$

$$\iint_{\Omega_Q^*} f(x', t')\, dx'\, dt' = \int_0^t d\tau \int_{x-\tau}^{x+\tau} f(x', t-\tau)\, dx',$$

and formula (31) takes the form

$$u(x, t) = \frac{u(x-t) + u(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \left.\frac{\partial u(x', t')}{\partial t'}\right|_{t'=0} dx' +$$

$$+ \frac{1}{2}\int_0^t d\tau \int_{x-\tau}^{x+\tau} f(x', t-\tau)\, dx'.$$

This equation is the solution of the Cauchy problem for the one-dimensional wave equation (27′).

In the case of the domain Ω^* shown in Fig. 28d, we have

$$\int_{C_Q} \frac{\partial u}{\partial N}\, ds = \int_{Q_1 B} \frac{\partial u}{\partial s}\, ds + \int_{Q_2 B} \frac{\partial u}{\partial s}\, ds = u(B) - u(Q_1) + u(B) - u(Q_2),$$

and formula (31) takes the form

$$u(Q) = u(Q_1) + u(Q_2) - u(B) + \frac{1}{2}\iint_{\Omega_Q^*} f(x', t')\, dx'\, dt'. \qquad (31')$$

This formula is an example of the solution of the so-called *characteristic problem (or Goursat problem)*, i.e., the problem of finding a solution of the one-dimensional wave equation (27) in the angle formed by two characteristics issuing from a single point when the values of the solution are given on those characteristics.

In the case of the region Ω^* shown in Fig. 28d, the direction of the conormal $\vec{N}$ on the characteristics QQ_1 and QQ_2 coincides with the directions of the vectors $\vec{QQ_1}$ and $\vec{QQ_2}$. Therefore,

$$\int\limits_{Q_1Q} \frac{\partial u}{\partial N}\, ds = \int\limits_{QQ_1} \frac{\partial u}{\partial s}\, ds = u(Q_1) - u(Q),$$

$$\int\limits_{Q_2Q} \frac{\partial u}{\partial N}\, ds = \int\limits_{QQ_2} \frac{\partial u}{\partial s}\, ds = u(Q_2) - u(Q),$$

and formula (30′) takes the form

$$u(Q) = \frac{u(Q_1) + u(Q_2)}{2} + \frac{1}{2}\int\limits_{C_Q} \frac{\partial u}{\partial N}\, ds - \frac{1}{2}\iint\limits_{\Omega_Q^*} f(x', t')\, dx'\, dt'. \quad (31'')$$

For example, for $f(x, t) = 0$ with initial conditions

$$u\Big|_{x=\beta t} = 0, \quad \frac{\partial u}{\partial x}\Big|_{x=\beta t} = \cos \gamma t, \qquad \gamma, \beta = \text{const}, \ 0 < \beta < 1$$

we have

$$\frac{\partial u}{\partial x}\Big|_C = \frac{\partial u}{\partial n^*}\Big|_C \cos n^* x, \quad \frac{\partial u}{\partial N}\Big|_C = \frac{\partial u}{\partial n^*}\Big|_C \cos Nn^*.$$

But $\cos n^* x = \cos\alpha$, where α is the angle between the straight line $t = x/\beta$ and the x-axis, and

$$\cos Nn^* = \cos 2\left(\frac{\pi}{2} - \alpha\right) = -\cos 2\alpha.$$

Therefore,

$$\frac{\partial u}{\partial N}\Big|_C = \frac{\partial u}{\partial x}\Big|_C \frac{-\cos 2\alpha}{\sin\alpha} = -\frac{\partial u}{\partial x}\Big|_C \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} \frac{\sqrt{1 + \tan^2\alpha}}{\tan\alpha} =$$

$$= \frac{\partial u}{\partial x}\Big|_C \frac{1 - \beta^2}{\sqrt{1 + \beta^2}}.$$

If we denote by t_{Q_1} and t_{Q_2} the values of t corresponding to points of the intersection of the straight line $x = \beta t$ with the characteristics $x' - x - (t' - t) = 0$ and $x' - x + (t' - t) = 0$ passing through the point $Q(x, t)$, we obtain

$$t_{Q_1} = \frac{t-x}{1-\beta}, \quad t_{Q_2} = \frac{t+x}{1+\beta}.$$

If, in formula (31″), we shift from integration with respect to arclength to integration with respect to the variable t', we obtain

$$u(x,t) = \frac{1}{2}\int_{t_{Q_1}}^{t_{Q_2}} (1-\beta^2)\cos\gamma t'\,dt' =$$

$$= \frac{1-\beta^2}{2\gamma}\left[\sin\frac{\gamma(t+x)}{1+\beta} - \sin\frac{\gamma(t-x)}{1-\beta}\right]. \qquad (31''')$$

This example can be interpreted as the solution of the problem of propagation of sound waves from a point source with frequency $\gamma/2\pi$ when the source itself is being displaced in the positive direction of the x-axis with a constant velocity equal to β. Equation (31‴) shows that this sound source generates a forward and backward wave, both propagated with the same speed as in the case of a stationary source of sound, but that the frequency of the forward wave is increased in a ratio $1:(1-\beta)$ and the frequency of the backward wave is decreased in a ratio $1:(1+\beta)$. This phenomenon is known as the *Doppler effect*. The Doppler effect can be observed in the change in the pitch of the whistle of a moving locomotive.

3. The two-dimensional case. The fundamental integral formula. Solution of the characteristic problem. Let us now apply formula (30) to the two-dimensional wave equation (27). First of all, we recall that every circular cone in xyt-phase-space with axis parallel to the t-axis and with apex angle equal to $\pi/2$ and pointing in the direction of the positive t-axis is a real characteristic of the two-dimensional wave equation (27). Every such cone, as opposed to other real characteristics of the two-dimensional wave equation (27), will be referred to as a *characteristic cone*.

Suppose that Ω^* is a domain in xyt-space such that the characteristic cone with apex at some point $Q(x, y, t)$ and the surface S_Q cut by the characteristic cones on the surface S^* bound some simply connected domain Ω_Q. Here, the axis of the characteristic

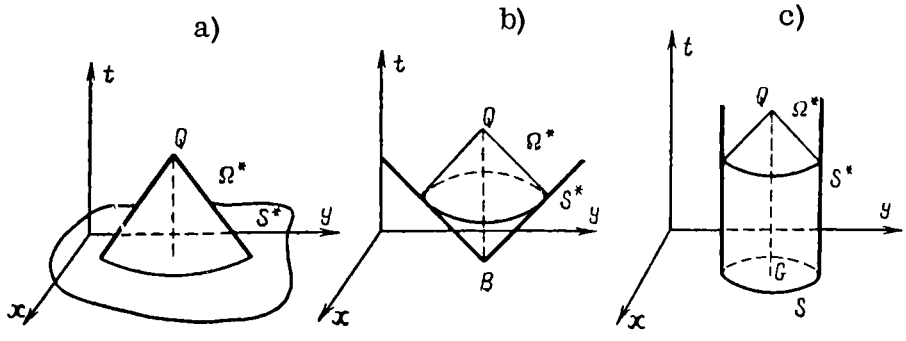

Fig. 29.

cone intersects S_Q at a single point. Examples of domains Ω^* of this construction are shown in Fig. 29.

It is easy to show by direction differentiation that in the region bounded by the characteristic cone but away from its axis, the function

$$v = v(Q, Q') = \frac{1}{2\pi} \ln \left[\frac{t-t'}{\rho} + \right.$$

$$\left. + \sqrt{\frac{(t-t')^2}{\rho^2} - 1} \right] (\rho = \sqrt{(x-x')^2 + (y-y')^2}), \tag{32}$$

as a function of the point $Q'(x', y', t')$ is a twice continuously differentiable solution of the homogeneous two-dimensional wave equation (27′).* On the axis of the characteristic cone ($\rho = 0$), the function v vanishes at infinity and, on the characteristic cone, satisfies the conditions

---

*It is interesting to note that the function

$$\omega(x, y, t, x', y', t') = \frac{\partial}{\partial t} v(Q, Q') = \frac{1}{2\pi} \frac{1}{\sqrt{(t-t')^2 - (x-x')^2 - (y-y')^2}}$$

is a unit impulse function due to an impulse at the point (x', y') at the instant t' for the two-dimensional wave equation (27). To see this, let us take for definiteness $t' = 0$ and let us suppose that the right-hand member of Eq. (27) is given in the form of a function $f_\varepsilon(x, y, t)$, which is equal to 0 everywhere except in a neighborhood of the point (x', y'):

$$r' = \sqrt{(x-x')^2 + (y-y')^2} < \varepsilon$$

for t between 0 and an arbitrarily small quantity Δt. Let us also suppose that the equation

$$\int_0^{\Delta t} dt \iint_{r' < \varepsilon} f_\varepsilon(x, y, t)\, dx\, dy = 1$$

holds for the impulse of a body force when the surface density of the membrane is equal to unity. Then, the retarded potential (25) yields the function

$$\omega_\varepsilon(x, y, t) = \frac{1}{2\pi} \int_0^t dr \iint_{\rho < r} \frac{f_\varepsilon(\alpha_1, \alpha_2, t-r)}{\sqrt{r^2 - \rho^2}}\, d\alpha_1\, d\alpha_2 \quad (\rho = \sqrt{(x-\alpha_1)^2 + (y-\alpha_2)^2}.$$

By setting $\rho' = \sqrt{(x'-\alpha_1)^2 + (y'-\alpha_2)^2}$, we obtain from this

$$\omega_\varepsilon(x, y, t) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{t-\Delta t}^{t} dr \iint_{\rho' < \varepsilon} \frac{f_\varepsilon(\alpha_1, \alpha_2, t-r)}{\sqrt{r^2 - \rho^2}}\, d\alpha_1\, d\alpha_2, & r' < t_1, \\ 0, & r' \geqslant t. \end{cases}$$

If we now take the limit as $\varepsilon \to 0$, we obtain the unit impulse function due to an impulse at the point (x', y') at the instant $t' = 0$:

$$\omega(x, y, t, x', t', 0) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1}{\sqrt{t^2 - (x-x')^2 - (y-y')^2}}, & r' < t, \\ 0, & r' > t, \end{cases}$$

$$v = 0, \quad \frac{\partial v}{\partial N} = 0. \tag{32'}$$

Let $(t-t')/\rho = \cot\varphi$ for $\varphi = \pi/2 - \eta$, where η is a sufficiently small positive number, be the equation of a cone with vertex at the point $Q(x, y, t)$ close to the characteristic cone. Let us denote by $\Omega_Q^* - e$ the domain obtained from Ω_Q^* by discarding (1) that part of it lying between the characteristic cone and the cone close to it referred to above and (2) the part cut from it by a thin circular cylinder with axis coinciding with the axis of the characteristic cone. The boundary of the region $\Omega_Q^* - e$ coincides with the cylindrical portion σ of that part Γ of the cone close to the characteristic cone and the part S_Q of the surface S_Q (see Fig. 30). Let us now apply Ostrogradskiy's

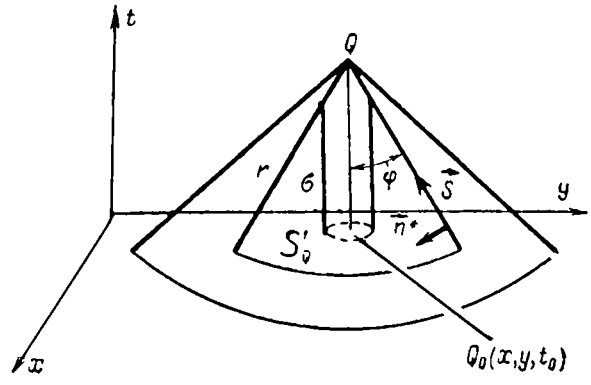

Fig. 30.

formula (30) to the domain $\Omega_Q^* - e$, taking as u a solution of the two-dimensional wave equation (27) that is twice continuously differentiable in the region Ω_Q^* and taking as v the function defined by Eq. (32). We obtain

$$\iiint\limits_{\Omega_Q^* - e} -vf\,dx'\,dy'\,dt' = -\iint\limits_{\sigma}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right)dS' -$$

$$-\iint\limits_{S_Q'}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right)dS' - \iint\limits_{\Gamma}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right)dS'. \tag{30''}$$

As Γ approaches the characteristic cone, v converges uniformly to 0. The same is true of the conormal derivative since, denoting by $\partial/\partial s$ the derivative along the generators of the cone and by l the distance from the vertex of the cone, we have

$$\frac{\partial v}{\partial s}\Big|_\Gamma = 0, \frac{\partial v}{\partial N}\Big|_\Gamma = \left(\frac{\partial v}{\partial s}\cos(N, s) + \frac{\partial v}{\partial n^*}\cos(N, n^*)\right)_\Gamma =$$

$$= -\frac{1}{2\pi l}\frac{\partial}{\partial \varphi}\ln(\cot\varphi + \sqrt{\cot^2\varphi - 1})\cos(N, n^*) =$$

$$= -\frac{1}{2\pi l}\frac{\cos(N, n^*)}{\sqrt{\cot^2\varphi - 1}}\frac{-1}{\sin^2\varphi} =$$

$$= \frac{1}{2\pi l}\frac{\cos 2\varphi}{\sqrt{\cot^2\varphi - 1}}\frac{1}{\sin^2\varphi} = \frac{\sqrt{\cos 2\varphi}}{2\pi l \sin\varphi}.$$

Therefore, assuming that Γ coincides with a portion of the characteristic cone, we can set the the last integral in the right-hand member of Eq. (30″) equal to 0. Furthermore, for $\rho = \varepsilon$, we have, on the cylindrical portion of the surface σ,

$$v = \frac{1}{2\pi}\ln\left[\frac{t-t'}{\varepsilon} + \sqrt{\frac{(t-t')^2}{\varepsilon^2} - 1}\right] =$$

$$= \frac{1}{2\pi}\ln\frac{1}{\varepsilon}\left[t - t' + \sqrt{(t-t')^2 + \varepsilon^2}\right] = O\left(\ln\frac{1}{\varepsilon}\right),$$

$$\frac{\partial v}{\partial N} = \frac{\partial v}{\partial \rho} = \frac{1}{\sqrt{\left(\frac{t-t'}{\rho}\right)^2 - 1}}\frac{-(t-t')}{2\pi\rho^2} = -\frac{t-t'}{2\pi\rho\sqrt{(t-t')^2 - \rho^2}}.$$

Therefore, shifting to polar coordinates, we get, for the first integral in the right-hand member of Eq. (30″),

$$-\lim_{\varepsilon\to 0}\iint_\sigma\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right)dS' =$$

$$= -\lim_{\varepsilon\to 0}\int_0^{2\pi} d\theta\int_{t_0}^{t-\varepsilon} u\frac{(t-t')\,dt'}{2\pi\sqrt{(t-t')^2 + \varepsilon^2}} = -\int_{t_0}^{t} u(x, y, t')\,dt'.$$

Thus, Eq. (30″) takes the form

$$\int_{t_0}^{t} u(x, y, t')\,dt' = \iint_{S_Q}\left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right)dS' +$$

$$+ \iiint_{\Omega_Q} v(Q, Q')f(P', t')\,dx'\,dy'\,dt'.$$

By differentiating this equation with respect to t, we obtain the following *fundamental integral formula*:

$$u(x, y, t) = \frac{\partial}{\partial t}\left\{\iint\limits_{S_Q}\int \left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right) dS' + \right.$$

$$\left. + \iiint\limits_{\Omega_Q^*} v(Q, Q') f(P', t')\, dx'\, dy'\, dt'\right\}. \tag{33}$$

This formula is completely analogous in its nature and notation to the fundamental integral formula (31) for the one-dimensional wave equation (27).*

Formula (33) gives an explicit representation for the solution of the so-called *characteristic problem for the two-dimensional*

*We can make the following comparison: The solution of Poisson's equation

$$u_{xx}+u_{yy}+u_{zz}=-f(x, y, z)$$

in the three-dimensional region bounded by the surface with inner normal $\vec{n}$ is given by the equation

$$u(x, y, z) = \iint\left(u\frac{\partial}{\partial n}\frac{1}{4\pi r} - \frac{1}{4\pi r}\frac{\partial u}{\partial n}\right) dS' + \iiint f(x', y', z')\frac{1}{4\pi r}\, d\tau' \tag{a}$$

$$(r = |P - P'|).$$

If this equation is integrated and then differentiated with respect to the same variable z, it takes the form

$$u(x, y, z) = \frac{1}{2}\frac{\partial}{\partial z}\left\{\iint\left(u\frac{\partial \varphi}{\partial n} - \varphi\frac{\partial u}{\partial n}\right) dS' + \iiint f(x', y', z')\,\varphi\, d\tau'\right\}, \tag{b}$$

where

$$\varphi = \int\frac{1}{2\pi r}\, dz = \int\frac{dz}{2\pi\sqrt{\rho^2 + (z - z')^2}} \quad (\rho = \sqrt{(x - x')^2 + (y - y')^2}\,. \tag{c}$$

If we now formally set $z = it$ in Poisson's equation and in Eq. (a), Poisson's equation becomes the two-dimensional wave equation

$$u_{xx}+u_{yy}-u_{tt}=-f(x, y, t),$$

and the function φ takes the form

$$\varphi = \frac{1}{2\pi}\ln\left[\frac{t - t'}{\rho} + \sqrt{\left(\frac{t - t'}{\rho}\right)^2 - 1}\right] = v(Q, Q').$$

Therefore, if we replace the inner normal $\vec{n}$ in the right-hand member of Eq. (b) with the conormal $\vec{N}$ and supply an extra factor equal to 2, this equation is transformed into formula (33):

$$u(x, y, t) = \frac{\partial}{\partial t}\left\{\iint\left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right) dS' + \iiint v(Q, Q') f(x', y', t')\, d\tau'\right\}.$$

wave equation (27). The characteristic problem here consists in finding the solution of the two-dimensional equation (27) in a domain Ω^* that is bounded by the circular cone S^* with axis parallel to the t-axis and with apex angle $\pi/2$ pointing in the direction of the negative t-axis (see Fig. 29b) under the condition that the values of the solution are given on the cone S^*. Specifically, if the values of the solution u are given on S^*, the values of its conormal derivative $\partial u/\partial N$ will also be given automatically on S^* since the direction of the conormal $\vec{N}$ on the cone S^* lies in its tangent plane. Therefore, the right-hand member of formula (33) is completely determined by the values of the solution on the cone S^* and hence it gives the desired solution of the characteristic problem.

If the surface S^* coincides with the plane $t=t_0$ (see Fig. 29a), then

$$\left.\frac{\partial u}{\partial N}\right|_{t'=t_0} = -\left.\frac{\partial u}{\partial t'}\right|_{t'=t_0}, \quad \left.\frac{\partial v}{\partial N}\right|_{t'=t_0} = -\left.\frac{\partial v}{\partial t'}\right|_{t'=t_0} = \frac{1}{2\pi\sqrt{(t-t_0)^2-\rho^2}},$$

$$\frac{\partial}{\partial t}\iint\limits_{S_Q} u\frac{\partial v}{\partial N}\,dS' = \frac{\partial}{\partial t}\iint\limits_{\rho<t-t_0}\frac{u(x', y', t_0)\,dx'\,dy'}{2\pi\sqrt{(t-t_0)^2-\rho^2}},$$

$$-\frac{\partial}{\partial t}\iint\limits_{S_Q} v\frac{\partial u}{\partial N}\,dS' = \frac{\partial}{\partial t}\iint\limits_{\rho<t-t_0} v\left.\frac{\partial u}{\partial t'}\right|_{t'=t_0} dx'\,dy' =$$

$$= \iint\limits_{\rho<t-t_0}\frac{u_{t_0}(x', y', t_0)\,dx'\,dy'}{2\pi\sqrt{(t-t_0)^2-\rho^2}} + \int\limits_{\rho=t-t_0} v\left.\frac{\partial u}{\partial t'}\right|_{t'=t_0} ds,$$

$$\frac{\partial}{\partial t}\iiint\limits_{\Omega_Q^*} vf\,dx'\,dy'\,dt' = \frac{\partial}{\partial t}\int\limits_{t_0}^{t} dt' \iint\limits_{\rho<t-t'} vf(x', y', t')\,dx'\,dy'$$

and, since, $v\big|_{\substack{t'=0\\ \rho=t-t_0}} = 0$, formula (33) takes the form

$$u(x, y, t) = \frac{\partial}{\partial t}\frac{1}{2\pi}\iint\limits_{\rho<t-t_0}\frac{u(x', y', t_0)\,dx'\,dy'}{\sqrt{(t-t_0)^2-\rho^2}} +$$

$$+\frac{1}{2\pi}\iint\limits_{\rho<t-t_0}\frac{u_{t_0}(x', y', t_0)\,dx'\,dy'}{\sqrt{(t-t_0)^2-\rho^2}} + \frac{1}{2\pi}\int\limits_{t_0}^{t} d\tau\iint\limits_{\rho<t-t_0}\frac{f(x', y', t+t_0-\tau)\,dx'\,dy'}{\sqrt{(\tau-t_0)^2-\rho^2}}.$$

For $t_0=0$, this formula yields the solution of the Cauchy problem for the two-dimensional wave equation (27).

4. The three-dimensional case. The basic integral formula. Solution of the characteristic problem. Let us now look at some applications of Ostrogradskiy's formula (30) to the integral

representations of twice continuously differentiable solutions of the three-dimensional wave equation (27).

Every circular hypercone in $x\,y\,z\,t$-phase-space with axis parallel to the t-axis and with apex angle $\pi/2$ pointing in the direction of the positive t-axis belongs to the set of real characteristics of the last equation. The equation of such a hypercone can be written in the form

$$(x-x')^2+(y-y')^2+(z-z')^2-(t-t')^2=0\ (t>t'),$$

where x, y, z, and t are the coordinates of the point Q, which is the apex of the hypercone and x', y', z', and t' are variable coordinates of the hypercone. We shall refer to such a hypercone, as distinct from the other real characteristics of the three-dimensional wave equation (27), as the *characteristic hypercone.*

Suppose that Ω^* is a domain in $x\,y\,z\,t$-space such that the characteristic hypercone with apex at any point $Q(x, y, z, t)$ of the region Ω^*, together with the portion F_Q cut out by the characteristic hypercone from the hypersurface S^*, bounds some simply connected domain Ω_Q. Here, the axis of the characteristic hypercone meets F_Q at a single point $Q_0(x, y, z, t_0)$. An example of a domain Ω^* of this construction is the half-space $-\infty<x, y, z<\infty$, $t>t_0$. Here, the hypersurface S^* is given by the equation $t-t_0=\text{const}$ and it coincides with $x\,y$ z-space. An example of a domain $\overset{*}{\Omega}_Q$ is the set of points $(x-x')^2+(y-y')^2+(z-z')^2-(t-t)^2<0$, $t_0<t'<t$, where x', y', z', and t' are the variable coordinates. An example of F_Q is the sphere $(x-x')^2+(y-y')^2+(z-z')^2-(t-t_0)^2<0$.

It is easy to see that

$$v=v(Q,Q')=\frac{1}{4\pi r}\left(\frac{t-t'}{r}-1\right), \tag{34}$$

$$r=\sqrt{(x-x')^2+(y-y')+(z-z')^2}$$

as a function of the point $Q'(x', y', z', t')$ is a twice continuously differentiable solution of the homogeneous three-dimensional wave equation (27′) everywhere off the axis of the characteristic hypercone ($r=0$). Here, the conditions

$$v=0,\ \frac{\partial v}{\partial N}=0 \tag{34′}$$

are satisfied on the characteristic hypercone.

Suppose that $\overset{*}{\Omega}_Q-e$ is the domain obtained from the domain $\overset{*}{\Omega}_Q$ by discarding those points in it that belong to the hypercylinder $r<\varepsilon$ in $x\,y\,z\,t$-space, where ε is a sufficiently small number. The boundary of the region $\overset{*}{\Omega}_Q-e$ consists of a portion σ of the boundary of the hypercylinder, a portion Γ of the characteristic hypercone, and a portion $\overset{*}{F}_Q$ of the hypersurface F_Q. If we apply

Ostrogradskiy's formula (30) to the region $\Omega_Q^{\bullet} - e$ (taking for u a solution of the three-dimensional wave equation (27) which is twice continuously differentiable in the domain $\Omega_Q^{\bullet}$ and taking as v the function defined by Eq. (34)), we obtain

$$\iiint\limits_{\Omega_Q^{\bullet}-e} -v(Q, Q')\, f(x', y', z', t')\, dx'\, dy'\, dz'\, dt' =$$

$$= -\iint\limits_{\sigma}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) dS' - \iiint\limits_{F_Q'}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) dS'. \quad (30''')$$

On σ, we have $\cos n^*t = 0$. Therefore, in spherical coordinates r, θ, φ with pole at the point (x, y, z), we have

$$\frac{\partial v}{\partial N} = \frac{\partial v}{\partial r} = -\frac{t-t'}{4\pi r^2},\ dS' = r^2\, d\theta\, d\varphi\, dt'.$$

In view of this, for the first integral in the right-hand member of Eq. (30''') we obtain

$$-\lim_{\varepsilon\to 0}\iiint\limits_{\sigma}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) dS' =$$

$$= \lim_{\varepsilon\to 0}\iiint\limits_{\sigma}\left[-\frac{1}{4\pi}\left(\frac{t-t'}{\varepsilon} - 1\right)\frac{\partial u}{\partial N} - u\frac{t-t'}{4\pi\varepsilon^2}\right]\varepsilon^2\, d\theta\, d\varphi\, dt' =$$

$$= -\int_{t_0}^{t}(t-t')\, u(x, y, z, t')\, dt'.$$

Thus, Eq. (30''') takes the form

$$\int_{t_0}^{t}(t-t')\, u(x, y, z, t')\, dt' = -\iiint\limits_{F_Q}\left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) dS' +$$

$$+\iiint\limits_{\Omega_Q^{\bullet}} v(Q, Q')\, f(x', y', z', t')\, dx'\, dy'\, dz'\, dt'.$$

If we differentiate this equation twice with respect to t, we obtain the following fundamental integral formula:

$$u(x, y, z, t) = \frac{\partial^2}{\partial t^2}\left\{\iiint_{F_Q}\left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right)dS' + \right.$$

$$\left. + \iiint_{\Omega_Q^*} v(Q, Q')\, f(x', y', z', t')\, dx'\, dy'\, dz'\, dt'\right\}. \tag{35}$$

This formula is completely analogous to the integral formulas (31) and (33). Recalling the expression for the function $v(Q, Q')$, we can write formula (35) in a more expanded form. Specifically, since $v(Q, Q')\big|_\Gamma = 0$ and $\partial v/\partial t = 1/4\pi r$, we have

$$\frac{\partial}{\partial t}\iiint_{F_Q} v\frac{\partial u}{\partial N}\,dS' = \iiint_{F_Q}\frac{1}{4\pi r}\frac{\partial u}{\partial N}\,dS',$$

$$\frac{\partial}{\partial t}\iiint_{\Omega_Q^*} v(Q, Q')\, f\, dx'\, dy'\, dz'\, dt' =$$

$$= \iiint_{\Omega_Q^*}\frac{f(x', y', z', t')}{4\pi r}\,dx'\, dy'\, dz'\, dt'.$$

Furthermore, on the boundary of the domain Ω^*, we have

$$\frac{\partial v}{\partial N} = -\frac{t-t'}{4\pi r^2}\left(\frac{x'-x}{r}\cos Nx + \frac{y'-y}{r}\cos Ny + \frac{z'-z}{r}\cos Nz\right) =$$

$$= -\frac{t-t'}{4\pi r^2}\frac{\partial r}{\partial n^*} + \frac{1}{4\pi r}\cos n^* t.$$

Therefore, formula (35) takes the form

$$u(x, y, z, t) = \frac{\partial^2}{\partial t^2}\iiint_{F_Q} u\left(-\frac{t-t'}{4\pi r^2}\frac{\partial r}{\partial n^*} + \frac{1}{4\pi r}\cos n^* t\right)dS' - \tag{35'}$$

$$-\frac{\partial}{\partial t}\iiint_{F_Q}\frac{1}{4\pi r}\frac{\partial u}{\partial N}\,dS' + \frac{\partial}{\partial t}\iiint_{\Omega_Q^*}\frac{f(x', y', z', t')}{4\pi r}\,dx'\, dy'\, dz'\, dt'.$$

Formula (35) or, what amounts to the same thing, formula (35′), provides an explicit representation for the solution of the three-

dimensional characteristic problem. This problem is posed just as in the two-dimensional case: We seek a solution of the three-dimensional equation (27) in a domain Ω^* which is bounded by a hypercone with axis parallel to the t-axis and with apex angle $\frac{\pi}{2}$ pointing in the direction of the negative t-axis under the condition that the values of the solution on this hypercone are given. This assertion is perfectly obvious since the direction of the conormal on the hypercone lies in its tangent plane.

If the domain Ω^* coincides with the half-space $-\infty < x, y, z < \infty, t > t_0$, i.e., if the equation of the hypersurface S^* takes the form $t - t_0 = 0$, then

$$\cos n^* t|_F = 1, \quad \left.\frac{\partial r}{\partial n^*}\right|_F = 0, \quad \left.\frac{\partial u}{\partial N}\right|_F = -\left.\frac{\partial u}{\partial t'}\right|_F.$$

In view of this, if we denote by $S_{t-t_0,P}$ the sphere of radius $t - t_0$ with center at the point (x, y, z), we may write formula (35′) in the form

$$u(x, y, z, t) = \frac{\partial}{\partial t}\iint\limits_{S_{t-t_0,P}} \frac{u(x', y', z', t_0)}{4\pi(t - t_0)}\, dS' + \iint\limits_{S_{t-t_0,P}} \frac{u_{t_0}(x', y', z', t_0)}{4\pi(t - t_0)}\, dS' +$$

$$+ \frac{\partial}{\partial t}\int\limits_{t_0}^{t} dt' \iiint\limits_{r<t-t'} \frac{f(x', y', z', t')}{4\pi r}\, dx'\, dy'\, dz'.$$

If we set $t - t' = \tau - t_0$, we obtain

$$\frac{\partial}{\partial t}\int\limits_{t_0}^{t} dt' \iiint\limits_{r<t-t'} \frac{f}{4\pi r}\, dx'\, dy'\, dz' = \int\limits_{t_0}^{t} dt' \iint\limits_{S_{t-t',P}} \frac{f\, dS'}{4\pi(t - t')} =$$

$$= \int\limits_{t_0}^{t} d\tau \iint\limits_{S_{\tau-t_0,P}} \frac{f(x', y', z', t + t_0 - \tau)}{4\pi(\tau - t_0)}\, dS'.$$

Thus, formula (35) as applied to the half-space $-\infty < x, y, z < \infty$, $t > t_0$ takes the form

$$u(x, y, z, t) = \frac{\partial}{\partial t}\iint\limits_{S_{t-t_0,P}} \frac{u(x', y', z', t_0)}{4\pi(t - t_0)}\, dS' + \iint\limits_{S_{t-t_0,P}} \frac{ut_0(x', y', z', t_0)}{4\pi(t - t_0)}\, dS' +$$

$$+ \iiint\limits_{r<t-t_0} \frac{f(x', y', z', t - r)}{4\pi r}\, dx'\, dy'\, dz'.$$

This formula gives the solution of the Cauchy problem for the three-dimensional wave equation (27) if the initial conditions are given for $t=t_0$.

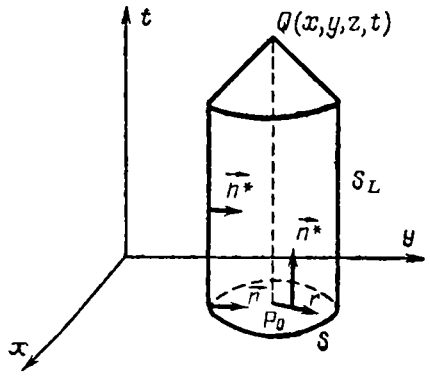

Fig. 31.

5. Kirchhoff's formula. Let us see what the basic integral formula (35) or, equivalently, (35′) looks like in the case when the domain Ω^* is a hypercylinder in the $xyzt$-space, the lateral surface of which has generators parallel to the t-axis. The boundary of the domain Ω^* in this case consists of the "base" of the given domain G in the three-dimensional xyz-space for $t=t_0$ and of its lateral surface S_L, consisting of the set of points of the boundary S of the domain G for $t>t_0$ (see Fig. 31).* For every value of t, let us agree to assume the constant t_0 so small (possibly negative) that the surface of the characteristic hypercone with apex at the point $Q(x, y, z, t)$ intersects the boundary of the hypercylinder Ω^* along the lateral surface. In this case, the hypersurface F_Q consists of the base G for $t=t_0$ and that portion of the lateral surface S_L, corresponding to the values of t' in the interval $t_0<t'<t-r$.

On the lateral surface S_L we have $\cos n^*t=0$ and, if $\vec{n}$ is the inner normal to S, then $\vec{N}=\vec{n}^*=\vec{n}$. Therefore, the last integral in the right-hand member of Eq. (35′) takes the form

$$\frac{\partial}{\partial t}\iiint\limits_G dS' \int\limits_{t_0}^{t-r} \frac{f(x', y', z', t')}{4\pi r}\, dt' =$$

$$= \frac{1}{4\pi}\iiint\limits_G \frac{f(x', y', z', t-r)}{r}\, dx'\, dy'\, dz'.$$

In evaluating the first and second integrals in the right-hand member of Eq. (35′), we need to consider only those portions corresponding to integration over the lateral surface of the hypercylinder since the terms corresponding to integration over the base of the cylinder are independent of t and, consequently, their derivatives with respect to t are equal to 0. Therefore, we see that the second integral in the right-hand member of Eq. (35′) coincides with the expression

*In this drawing, the three-dimensional xyt-space is shown, for greater ease and comprehension, instead of the four-dimensional $xyzt$-space.

$$-\frac{\partial}{\partial t}\iint_S dS' \int_{t_0}^{t-r} \frac{1}{4\pi r}\frac{\partial u}{\partial n}\,dt' =$$

$$= -\iint_S \frac{1}{4\pi r}\frac{\partial u(x', y', z', t')}{\partial n}\bigg|_{t'=t-r} dS'.$$

The first integral in the right-hand member of Eq. (35′) takes the form

$$-\frac{\partial}{\partial t^2}\iint_S dS' \int_{t_0}^{t-r} u\frac{t-t'}{4\pi r^2}\frac{\partial r}{\partial n}\,dt' =$$

$$= -\frac{\partial}{\partial t}\left\{\iint_S \frac{u}{4\pi r}\bigg|_{t'=t-r}\frac{\partial r}{\partial n}\,dS' + \iint_S dS' \int_{t_0}^{t-r} \frac{u}{4\pi r^2}\frac{\partial r}{\partial n}\,dt'\right\} =$$

$$= -\iint_S \frac{1}{4\pi r}\frac{\partial u}{\partial t'}\bigg|_{t'=t-r}\frac{\partial r}{\partial n}\,dS' - \iint_S \frac{u}{4\pi r^2}\bigg|_{t'=t-r}\frac{\partial r}{\partial n}\,dS'.$$

Thus, formula (35) or, what amounts to the same thing, (35′) takes the form

$$u(x, y, z, t) = \frac{1}{4\pi}\iint_S \left[u\frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} - \frac{\partial u}{\partial t'}\frac{1}{r}\frac{\partial r}{\partial n}\right]_{t'=t-r} dS' +$$

$$+ \frac{1}{4\pi}\iiint_G \frac{f(x', y', z', t-r)}{r}\,dx'\,dy'\,dz'. \qquad (36)$$

This important formula is known as *Kirchhoff's formula*. It follows from this formula that in order to find the value of the solution of the three-dimensional wave equation (27) at a point (x, y, z) of the domain G at the instant t, we need only know the values of

u, $\frac{\partial u}{\partial n}$, and $\frac{\partial u}{\partial t}$ at points of the surface S bounding G at the instant $t - r$ (where r is the distance between points of the surface S and the point (x, y, z)). In other words, the value of the solution at points of the domain G is expressed in terms of the "retarded" values of u, $\frac{\partial u}{\partial n}$ and $\frac{\partial u}{\partial t}$ on the surface S. In this respect, formula (36) differs in a fundamental way from its one-dimensional analog (31) as applied to the domain (a half-strip) shown in Fig. 23b and its two-dimensional analog (33) as applied to the cylinder shown in Fig. 29c. The significant point is that in the last two cases, as can be seen from formulas (31) and (33), the expressions for $u(x, t)$ and $u(x, y, t)$ contain at all instants of time terms defined directly in terms of the initial conditions given for $t=t_0$. In the three-dimensional case, on the other hand, i.e., in formula (36), there is no such term if $t-t_0>d$, where d is the diameter of the domain G. In particular, this means that the initial conditions of some boundary-value problem for the domain G, given for $t=t_0$, may exert an influence on the solution of the problem for $t - t_0>d$ only through the values of u, $\frac{\partial u}{\partial n}$ and $\frac{\partial u}{\partial t}$ on the surface S bounding G.

Formally, Kirchhoff's formula (36) reminds one of the integral formula for solutions of the three-dimensional Poisson equation

$$v(x, y, z) = \frac{1}{4\pi} \iint_S \left[v(x', y', z') \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial v(x', y', z')}{\partial n} \right] dS' - \\ - \frac{1}{4\pi} \iiint_G \frac{1}{r} \Delta v \, d\tau', \tag{37}$$

where (x, y, z) is a point of the region G and

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

(see Chapter III, Sect. 2, subsection 2).

In particular, if we assume the function u and the forcing term in the three-dimensional wave equation (27) to be independent of the time t, this equation degenerates into Poisson's equation and formula (36) becomes formula (37). Conversely, by means of an artificial device, Kirchhoff's formula (36) can be obtained quite simply from formula (37). Specifically, we set

$$v(x', y', z') = u(x', y', z', t')|_{t'=t-r} = u(x', y', z', t-r),$$

where $u(x', y', z', t')$ is a solution of the three-dimensional wave equation (27). Then, the left-hand member of Eq. (37) coincides with the left-hand member of Eq. (36). The integrand in the first integral on the right-hand side of Eq. (37) can be written in the form

$$v\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{r}\frac{\partial v}{\partial n}=\left[u(x', y', z', t')\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{r}\frac{\partial u(x', y', z', t')}{\partial n}+\right.$$
$$\left.+\frac{\partial u(x', y', z', t')}{\partial t'}\frac{1}{r}\frac{\partial r}{\partial n}\right]_{t'=t-r}.$$

Consequently, the first integral in the right-hand member of Eq. (37) is equal to the first integral in the right-hand member of Eq. (36) plus an integral of the form

$$A=\frac{1}{4\pi}\iint_S\frac{2}{r}u_{t'}(x', y', z', t')\Big|_{t'=t-r}\frac{\partial r}{\partial n}dS'=$$
$$=\frac{1}{4\pi}\iint_S\frac{2u_{t'}}{r}\left(\frac{\partial r}{\partial x'}\cos nx+\frac{\partial r}{\partial y'}\cos ny+\right.$$
$$\left.+\frac{\partial r}{\partial z'}\cos nz\right)_{t'=t-r}dS'=-\frac{1}{4\pi}\iiint_G\left[\frac{\partial}{\partial x'}\left(\frac{2u_{t'}}{r}\frac{\partial r}{\partial x'}\right)+\right.$$
$$\left.+\frac{\partial}{\partial y'}\left(\frac{2u_{t'}}{r}\frac{\partial r}{\partial y}\right)+\frac{\partial}{\partial z'}\left(\frac{2u_{t'}}{r}\frac{\partial r}{\partial z'}\right)\right]_{t'=t-r}d\tau'=$$
$$=-\frac{1}{4\pi}\iiint_G\left[\frac{2}{r}\left(u_{t'x'}\frac{\partial r}{\partial x'}+u_{t'y'}\frac{\partial r}{\partial y'}+v_{t'z'}\frac{\partial r}{\partial z'}\right)-\right.$$
$$\left.-\frac{2u_{t't'}}{r}+\frac{2u_{t'}}{r}\Delta r-\frac{2u_{t'}}{r}\right]_{t'=t-r}d\tau'.$$

Furthermore, we have

$$\frac{\partial v}{\partial x'}=\left[\frac{\partial u}{\partial x'}-\frac{\partial u}{\partial t'}\frac{\partial r}{\partial x'}\right]_{t'=t-r},$$
$$\frac{\partial^2 v}{\partial x'^2}=\left[\frac{\partial^2 u}{\partial x'^2}-2\frac{\partial^2 u}{\partial x'\partial y'}\frac{\partial r}{\partial x'}+\frac{\partial^2 u}{\partial t'^2}\left(\frac{\partial r}{\partial x'}\right)^2-\frac{\partial u}{\partial t'}\frac{\partial^2 r}{\partial x'^2}\right]_{t'=t-r}.$$

Since $u(x', y', z', t')$ is a solution of the wave equation, we obtain from the above expression

$$-\frac{1}{r}\Delta v = -\frac{1}{r}\left[\Delta u + u_{t't'} - 2\left(u_{t'x'}\frac{\partial r}{\partial x'} + u_{t'y'}\frac{\partial r}{\partial y'} + u_{t'z'}\frac{\partial r}{\partial z'}\right) - \right.$$

$$\left. - u_{t'}\Delta r\right]_{t'=t-r} = \frac{f(x', y', z', t-r)}{r} + \left[\frac{2}{r}\left(u_{t'x'}\frac{\partial r}{\partial x'} + u_{t'y'}\frac{\partial r}{\partial y'} + \right.\right.$$

$$\left.\left. + u_{t'z'}\frac{\partial r}{\partial z'}\right) - \frac{2u_{t't'}}{r} + \frac{u_{t'}}{r}\Delta r\right].$$

Therefore, the sum of the integral A and the second integral in the right-hand member of Eq. (37) is equal to

$$\frac{1}{4\pi}\iiint\limits_G \frac{f(x', y', z', t-r)}{r}\,d\tau' + \frac{1}{4\pi}\iiint\limits_G \left(-\frac{u_{t'}}{r}\Delta r + \frac{2u_{t'}}{r^2}\right)d\tau'.$$

Consequently, since $\Delta r = \frac{2}{r}$, this sum coincides with the second integral in the right-hand member of Eq. (36). Thus, we have obtained a new derivation of Kirchhoff's formula (36) based on the use of formula (37).

Kirchhoff's formula (36) is useful in showing a large number of problems. For example, if domain G is the sphere of radius $r=r_1$ with center at the point (x, y, z), we have

$$u\frac{\partial\frac{1}{r}}{\partial n} - \frac{1}{r}\frac{\partial u}{\partial n} - \frac{\partial u}{\partial t'}\frac{1}{r}\frac{\partial r}{\partial n} = \frac{1}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial t'} =$$

$$= \frac{1}{r^2}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial t'},$$

and Kirchhoff's formula (36) takes the form

$$u(x, y, z, t) = \frac{1}{4\pi}\iint\limits_{S_{r_1}, P}\left[\frac{1}{r_1^2}\frac{\partial (r_1 u)}{\partial r_1} + \frac{1}{r_1}\frac{\partial u}{\partial t'}\right]_{t'=t-r} dS' +$$

$$+ \frac{1}{4\pi}\iiint\limits_{r<r_1}\frac{f(x', y', z', t-r)}{r}\,dx'\,dy'\,dz',$$

where $S_{r_1, P}$ is the sphere of radium r_1 with center at the point $P(x, y, z)$. Since

$$\frac{1}{4\pi}\iint\limits_{S_{r_1}, P}\frac{1}{r_1^2}\frac{\partial r_1 u}{\partial r_1}\,dS' = \frac{\partial}{\partial r_1}\frac{1}{4\pi}\iint\limits_{S_{1_1}, P} r_1 u\,d\sigma,$$

by setting $r_1 = t$, we obtain the solution of the Cauchy problem for the three-dimensional wave equations (27) in the following form:

$$u(x, y, z, t) = \frac{\partial}{\partial t} \frac{1}{4\pi} \iint\limits_{S_{t,P}} \frac{u|_{t'=0}}{t} dS' + \frac{1}{4\pi} \iint\limits_{S_{t,P}} \frac{1}{t} \left(\frac{\partial u}{\partial t'} \right)_{t'=0} dS' +$$

$$+ \frac{1}{4\pi} \iiint\limits_{r<t} \frac{f(x', y', z', t-r)}{r} dx'\, dy'\, dz'.$$

7. Wave Potentials. Radiation Functions

1. Wave potentials. Functions of wave sources and dipoles. Let us make a comparison of Kirchhoff's formula (36) with formula (37). We introduce the differentiation symbols

$$\frac{\delta}{\delta n} \text{ and } \frac{\delta^*}{\delta n}, \text{ defined by}$$

$$\frac{\delta}{\delta n} \frac{u(x', y', z', t-r)}{r} = u(x', y', z', t-r) \frac{\partial \frac{1}{r}}{\partial n} -$$

$$- \frac{\partial u(x', y', z', t-r)}{\partial t} \frac{1}{r} \frac{\partial r}{\partial n}, \tag{38}$$

$$\frac{\delta^*}{\delta n} u(x', y', z', t-r) = \frac{\partial u(x', y', z', t')}{\partial n} \Big|_{t'=t-r}. \tag{38'}$$

This means that when we differentiate with respect to the inner normal $\vec{n}$ in Eq. (38), the dependence of the function on x', y', and z' is taken into account only through the distance

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2},$$

while in Eq. (38), the direct dependence on these variables is considered but the dependence on them through the distance r is not taken into account. In these notations, Kirchhoff's formula takes the form

$$u(x, y, z, t) = \frac{1}{4\pi} \iint\limits_S \left[\frac{\delta}{\delta n} \frac{u(x', y', z', t-r)}{r} - \right.$$

$$\left. - \frac{1}{r} \frac{\delta^*}{\delta n} u(x', y', z', t-r) \right] dS' +$$

$$+ \frac{1}{4\pi} \iiint\limits_G \frac{f(x', y', z', t-r)}{r} dx'\, dy'\, dz'. \tag{36'}$$

Formula (37), if we set $\Delta v = -f(x, y, z)$ takes the form

$$u(x, y, z) = \frac{1}{4\pi} \iint_S \left[\frac{\delta}{\delta n} \frac{u(x', y', z')}{r} - \frac{1}{r} \frac{\delta^*}{\delta n} u(x', y', z') \right] dS' +$$

$$+ \frac{1}{4\pi} \iiint_G \frac{f(x', y', z')}{r} dx' dy' dz'. \qquad (37')$$

These last equations underscore the great similarity of Kirchhoff's formula with the formula for solving the three-dimensional Poisson's equation (37).

In analogy with the potentials of a single layer, a double layer, and of a volume distribution, let us introduce the following integrals:

$$U(x, y, z, t) = \frac{1}{4\pi} \iint_S \frac{\sigma(x', y', z', t-r)}{r} dS'$$

$$(r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}), \qquad (39)$$

$$V(x, y, z, t) = \frac{1}{4\pi} \iint_S \frac{\delta}{\delta n} \frac{\vartheta(x', y', z', t-r)}{r} dS', \qquad (39')$$

$$I(x, y, z, t) = \frac{1}{4\pi} \iiint_G \frac{\rho(x', y', z', t-r)}{r} dx' dy' dz'. \qquad (39'')$$

These integrals are called wave potentials, and the functions

$$\sigma(x', y', z', t'), \quad \vartheta(x', y', z', t'), \quad \rho(x', y', z', t')$$

are called the *densities* of these potentials. These same integrals are also called *retarded potentials* since they are represented like ordinary potentials but their effect at certain points in space appears with a lag. The integral (39) is called the retarded potential of a single layer, the integral (39′) is called the retarded potential of a double layer, and the integral (39″) is called the retarded potential of a volume distribution. Kirchhoff's formula (36) or, equivalently, (36′) indicates that every twice continuously differentiable solution of the three-dimensional wave equation (27) can be represented as the sum of the retarded potentials of a single layer, a double layer, and of a volume.

The wave or retarded potentials (39), (39′), and (39″) possess a large number of properties completely analogous to the properties of ordinary potentials.

Each of the wave potentials (39), (39′), and (39″) is, outside the corresponding region of integration, a twice continuously differentiable solution of the homogeneous three-dimensional wave equation (27′) if the density corresponding to it is a continuous function of the variables x', y', z', and t' and in addition is twice continuously differentiable with respect to t.

Indeed, the three-dimensional Laplacian operator in spherical coordinates r, θ, φ takes the form

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$$

(see Chapter III, Sect. 3, subsection 1). Therefore, solutions of the homogeneous equation (27′) which depend only on r and t must be determined from the equation

$$\frac{\partial^2(ru)}{\partial r^2} - \frac{\partial^2(ru)}{\partial t^2} = 0$$

and they take the form

$$u = \frac{F(t-r)}{4\pi r},\quad u_1 = \frac{F_1(t+r)}{4\pi r}, \tag{40}$$

where $F(t)$ and $F_1(t)$ are arbitrary twice continuously differentiable functions of the independent variable t. The functions determined by Eqs. (40) are called *spherical waves*. The first of them is called an outgoing or *divergent wave*, and the second is called an incoming or convergent wave. It follows from this definition of spherical waves that the functions

$$\frac{F(x', y', z', t-r)}{4\pi r},\quad \frac{\delta}{\delta n}\frac{F(x', y', z', t-r)}{4\pi r}$$

$$\left(r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)$$

as functions of x, y, z, and t, are twice continuously differentiable solutions of the homogeneous wave equation (27′). Therefore, the integrals (39), (39′), and (39″) outside the corresponding regions of integration are twice continuously differentiable solutions of the homogeneous wave equation (27′) in three space coordinates.

The integrals (39) and (39′) converge uniformly at every point of $xyzt$-space and are therefore continuous functions with respect to the variables x, y, z, and t for values of all of the variables. To show this, we need only apply to the integrals (39) and (39″) the reasoning followed in examining the ordinary potentials of a single layer and of a volume (see Chapter III, Sect. 4, subsections 3 and 4).

Denoting differentiation with respect to t by a prime, we obtain for the retarded potential of a double layer (39′),

$$V(x, y, z, t) = \frac{1}{4\pi} \iint_S \left[\vartheta(x', y', z', t-r) \frac{\partial \frac{1}{r}}{\partial n} - \right.$$

$$\left. - \vartheta'(x', y', z', t-r) \frac{1}{r} \frac{\partial r}{\partial n} \right] dS' =$$

$$= \frac{1}{4\pi} \iint_S \left\{ \vartheta(x', y', z', t) \frac{\partial \frac{1}{r}}{\partial n} + \left[\vartheta'(x', y', z', t-\theta r) \frac{1}{r} \frac{\partial r}{\partial n} - \right. \right.$$

$$\left. \left. - \vartheta'(x', y', z', t-r) \frac{1}{r} \frac{\partial r}{\partial n} \right] \right\} dS'. \qquad (0 < \theta < 1)$$

From this, we obtain by using the mean-value theorem

$$V(x, y, z, t) = \frac{1}{4\pi} \iint_S \vartheta(x', y', z', t) \frac{\partial \frac{1}{r}}{\partial n} dS' +$$

$$+ \frac{1}{4\pi} \iint_S \vartheta''(x', y', z', t - \theta r + \theta_1 (r - \theta r))(1-\theta) \frac{\partial r}{\partial n} dS', \qquad (41)$$

where θ and θ_1 are constants ($0<\theta,\ \theta_1<1$). It is easy to see that the last integral on the right-hand side of this equation converges uniformly at every point of $xyzt$-space. Therefore, the limiting values of $V(x, y, z, t)$ as one approaches points of the surface S from the interior and the exterior (these two limits being denoted by $V^+(x, y, z, t)$ and $V^-(x, y, z, t)$, respectively) exist in those cases in which these limiting values exist for the potential of a double layer with density $\vartheta(x', y', z', t)$. If we denote this potential of a double layer [the first of the integrals on the right-hand side of Eq. (41)] by $V_1(x, y, z, t)$, we have at points of a Lyapunov surface S

$$V^+ - V_1^+ = V^- - V_1^- = V - V_1.$$

Therefore, from the theorem on the limiting values of the potential of a double layer (see Chapter III, Sect. 4, subsection 6), we obtain the following formulas:

$$V^+(x, y, z, t) = \frac{\vartheta(x, y, z, t)}{2} + V,$$
$$V^-(x, y, z, t) = -\frac{\vartheta(x, y, z, t)}{2} + V, \qquad (42)$$

where V is the integral (39′), with the condition that the point (x, y, z) lies on the surface S

$$V = \frac{1}{4\pi} \iint_S \left[\vartheta(x', y', z', t-r) \frac{\partial \frac{1}{r}}{\partial n} - \right.$$

$$\left. - \vartheta'(x', y', z', t-r) \frac{1}{r} \frac{\partial r}{\partial n} \right] dS'. \qquad (42')$$

In particular, it follows from formulas (42) that the retarded potential of a double layer (39′) has a jump

$$V^+(x, y, z, t) - V^-(x, y, z, t) = \nu(x, y, z, t)$$

on the surface S.

By analogy with the preceding, the normal derivative of a retarded potential of a single layer (39) can be represented in the form

$$\frac{\partial U(x, y, z, t)}{\partial n} = \frac{1}{4\pi} \iint_S \sigma(x', y', z', t) \frac{\partial \frac{1}{r}}{\partial n} dS' +$$

$$+ \frac{1}{4\pi} \iint_S \sigma''(x', y', z', t - \theta r + \theta_1(r - \theta r))(1 - \theta) \frac{\partial r}{\partial n} dS', \qquad (41')$$

where θ and θ_1 are constants $(0<\theta, \theta<1)$. The right-hand side of this equation differs from the right-hand side of Eq. (41) only in that it is the point (x, y, z) instead of the point (x', y', z') which is assumed to be variable in the differentiation with respect to the normal $\vec{n}$. Consequently, the behavior of the normal derivative of a retarded potential of a single layer at points of the surface S is completely determined by the behavior at these points of the first integral in the right-hand member of Eq. (41′), i.e., by the behavior of the normal derivative of the potential of a single layer with density $\sigma(x', y', z', t)$. Therefore, in those cases in which the limiting values of the normal derivative of the potential of a single layer exist, the limiting values of the normal derivative of the retarded potential of a single layer (39) will also exist. For these limiting values from within $\frac{\partial U^+(x, y, z, t)}{\partial n}$ and without $\frac{\partial U^-(y, y, z, t)}{\partial n}$, the following equations hold:

$$\frac{\partial U^+(x, y, z, t)}{\partial n} = -\frac{\sigma(x, y, z, t)}{2} + \frac{\partial U}{\partial n},$$

$$\frac{\partial U^-(x, y, z, t)}{\partial n} = \frac{\sigma(x, y, z, t)}{2} + \frac{\partial U}{\partial n}, \tag{43}$$

where $\frac{\partial U}{\partial n}$ is the result of formal substitution of the point (x, y, z) on the surface S into the expression of the normal derivative of the retarded potential of a single layer:

$$\frac{\partial U}{\partial n} = \frac{1}{4\pi}\iint_S \left[\sigma(x', y', z', t-r)\frac{\partial \frac{1}{r}}{\partial n} - \right.$$

$$\left. -\sigma'(x', y', z', t-r)\frac{1}{r}\frac{\partial r}{\partial n}\right] dS'. \tag{43'}$$

The retarded potential of a volume (39″) can be written in the form

$$I(x, y, z, t) = \iiint_{G-e} \frac{(\rho)_{t'=t-r}}{4\pi r}\, d\tau' + \iiint_e \frac{(\rho)_{t'=t-r} - (\rho)_{t'=t}}{4\pi r}\, d\tau' +$$

$$+ \iiint_e \frac{(\rho)_{t'=t}}{4\pi r}\, d\tau', \tag{41″}$$

where e is the interior of a sufficiently small sphere with center at the point (x, y, z). If we assume that the function $\rho(x', y', z', t')$ is continuous with respect to x', y', and z' and twice continuously differentiable with respect to t', then the first of the integrals in the right-hand member of Eq. (41″) has continuous first and second derivatives and satisfies the homogeneous equation (27′). Since the function $\rho(x', y', z', t-r)$ is twice continuously differentiable with respect to t, the second and third integrals in the right-hand member of Eq. (41″) are also twice continuously differentiable. Furthermore, the third integral in the right-hand member of Eq. (41″) satisfies the equation $\Delta u = -\rho(x, y, z, t)$. The derivatives of the third integral with respect to t, and all the derivatives of the first and second order of the second integral become arbitrarily small as the sphere e is contracted to a point. Thus, we conclude that the retarded potential of a volume (39″) at points of the region of integration has continuous first and second derivatives and satisfies the wave equation (27) with right-hand member equal to $-\rho(x, y, z, t)$.

Let us give a physical interpretation of the wave potentials (39), (39′), and (39″), interpreting the solution of the homogeneous

three-dimensional wave equation (27′) as the potential function of acoustic vibrations of air (see Chapter I, Sect. 2, subsection 4). From this point of view, let us look at the outgoing wave

$$u = \frac{F(t-r)}{4\pi r}.$$

Corresponding to this outgoing wave is the vector representing the velocity of particles of air $\vec{v} = -\nabla u$. Let us calculate the flow of air through some sphere σ. This flow is given by the equation

$$I = -\iint_{\sigma} v_n \, dS = \iint_{\sigma} \frac{\partial u}{\partial n} \, dS =$$

$$= -\iint_{\sigma} \left[\frac{F(t-r)}{4\pi r^2} + \frac{F'(t-r)}{4\pi r} \right] \frac{\partial r}{\partial n} \, dS$$

and it represents the amount of air flowing from the interior of the sphere in a unit of time. If we contract the sphere σ to a point, we always obtain zero for the flow except in the case when the point into which the sphere contracts coincides with the point (x', y', z'). In this last case, we obtain from the last equation $I = F(t)$ for the flow. The interpretation is this: At the point (x', y', z') there is a source of air or, as we say, a *wave source of intensity* $F(t)$. Therefore, the first of the functions (40), i.e., the outgoing wave, is also called the function of a wave source of intensity $F(t)$ at the point (x', y', z'). By virtue of what has been said, the wave or retarded potentials of a simple layer (39) and of a volume (39″) are wave fields caused by wave sources distributed on a surface S with density $\sigma(x', y', z', t)$, respectively. Let us now suppose that a wave source of intensity $-\frac{1}{h} F(x', y', z', t)$ is located at a point (x', y', z') on the surface S and that a wave source of intensity $\frac{1}{h} F(x', y', z', t)$, where h is the distance between the points (x', y', z') and (x_1, y_1, z_1), is located at the point (x_1, y_1, z_1) on the inner normal $\vec{n}$ to the surface S issuing from it. Then, the potential function corresponding to these two sources will be of the form

$$u = \frac{F(x', y', z', t-r_1)}{hr_1} - \frac{F(x', y', z', t-r)}{hr},$$

where

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2},$$
$$r_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}.$$

Taking the limit as the point (x_1, y_1, z_1) approaches the point (x', y', z'), we obtain a confluence at the latter point of the two wave sources of opposite intensity or, as we say, we obtain a wave dipole with axis $\vec{n}$. Corresponding to this wave dipole is the potential function

$$u(x, y, z, t) = \frac{\delta}{\delta n}\frac{F(x', y', z', t-r)}{r} = F(x', y', z', t-r)\frac{\partial \frac{1}{r}}{\partial n} - F'(x', y', z', t-r)\frac{1}{r}\frac{\partial r}{\partial n}.$$

A wave dipole can be represented visually in the form of a particle that moves back and forth causing compression in front of it and rarefaction behind it. By virtue of the last equation, we may say that the wave potential of a double layer (39′) is the potential caused by wave dipoles which are located on the surface S with density $\vartheta(x', y', z', t)$ and whose axes at the corresponding points coincide with the inner normal $\vec{n}$ to the surface S.

2. Radiation functions. From a physical point of view, a limiting case of the Cauchy problem for the wave equation, known as the *radiation problem*, is very important. This radiation problem can be formulated as follows:

Find the limit of the solution of the Cauchy problem

$$\Delta u - \frac{\partial^2 u}{\partial t^2} = -f, \quad \left(\Delta = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)$$
$$u\Big|_{t=0} = 0, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \tag{44}$$

as ε approaches 0 if

$$f = f_\varepsilon(x_i, t) = 0 \text{ for } r = \sqrt{x_1^2 + \ldots + x_n^2} > \varepsilon,$$
$$\lim_{\varepsilon\to 0}\int\cdots\int_{r<\varepsilon} f_\varepsilon(x_i, t)\,dx_1 \ldots dx_n = F(t), \tag{44′}$$

where $F(t)$ is a given function which vanishes for $t<0$ and is twice continuously differentiable for $t\geqslant 0$.

In a number of physical problems, the function f is proportional to a force density. Therefore, the conditions of this problem may

be understood as meaning that a state of rest existed throughout all "x-space" at the initial instant of time and that a concentrated force proportional to $F(t)$ began to act at that instant at the coordinate origin in the x-space. The preferred position of the coordinate origin as opposed to other points in the x-space is of no significance in the statement of the problem since the wave equation is invariant under translation of the coordinate origin.

The solution of this radiation problem can be obtained quite easily. Specifically, before we take the limit, the solution of the Cauchy problem (44), (44′) may, in accordance with formulas (23), (25), and (12), be written in the following forms, corresponding to the three-, two-, and one-dimensional cases, respectively:

$$u(x, y, z, t) = \frac{1}{4\pi} \iiint\limits_{r'<\varepsilon} \frac{f(\alpha_1, \alpha_2, \alpha_3, t-r')}{r'} d\alpha_1 d\alpha_2 d\alpha_3$$

$$(r' = \sqrt{(x-\alpha_1)^2 + (y-\alpha_2)^2 + (z-\alpha_3)^2}), \tag{45}$$

$$u(x, y, t) = \frac{1}{2\pi} \int_0^t dr' \iint\limits_{\rho<r'} \frac{f(\alpha_1, \alpha_2, t-r')}{\sqrt{r'^2-\rho^2}} d\alpha_1 d\alpha_2 \tag{45′}$$

$$(\rho = \sqrt{(x-\alpha_1)^2 + (y-\alpha_2)^2}),$$

$$u(x, t) = \frac{1}{2} \int_0^t dr' \int_{x-r'}^{x+r'} f(\alpha_1, t-r')\, d\alpha_1. \tag{45″}$$

In the three-dimensional case, by taking into account the δ-like nature of the function $f(\alpha_1, \alpha_2, \alpha_3, t-r')$, we obtain, after passage to the limit, the solution of the radiation problem in the following form:

$$u(x, y, z, t) = \frac{F(t-r)}{4\pi r}, \qquad (r = \sqrt{x^2+y^2+z^2}). \tag{46}$$

In the two-dimensional case, the inner integral in the right-hand member of Eq. (45′) vanishes in the limit for $r'<r$, and the solution of the radiation problem is obtained in the form

$$u(x, y, t) = \frac{1}{2\pi} \int_r^t \frac{F(t-\tau)}{\sqrt{\tau^2-r^2}} d\tau \quad (r = \sqrt{x^2+y^2}). \tag{46′}$$

In the one-dimensional case, the inner integral in the right-hand member of Eq. (45″) vanishes for $r'<r<|x|$, and the solution of the radiation problem takes the form

$$u(x, t) = \frac{1}{2}\int_r^t F(t-\tau)\,d\tau = \frac{1}{2}\int_0^{t-r} F(\lambda)\,d\lambda \quad (r = |x|). \tag{46''}$$

In each of these formulas (46), (46′), and (46″), we need to take the right-hand member equal to 0 if $r>t$.

Let us look at certain properties of the *radiation functions* (46), (46′), and (46″). Each of these functions satisfies the homogeneous wave equation (27′) (with the appropriate number of space variables) for $t>0$ throughout the entire x-space away from the coordinate origin. This is easily verified by differentiation.

Each of these three functions satisfies the homogeneous initial conditions

$$u\Big|_{t=0} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0$$

at $t=0$ away from the coordinate origin in the x-space. This follows the fact that the radiation functions were constructed so as to vanish for $r>t$.

Let us show that, at the coordinate origin in x-space, the functions (46), (46′), and (46″) satisfy for $t>0$ the condition which characterizes a wave source of intensity $F(t)$:

$$I = \lim_{\varepsilon\to 0} \overbrace{\int\cdots\int}^{n-1}_{S_\varepsilon} \frac{\partial u(x_i, t)}{\partial n}\,dS = F(t), \tag{47}$$

where $F(t)$ is a given function of t for $-\infty<t<\infty$, S_ε is a sphere of radius ε in x-space with center at the coordinate origin, and n is the inner normal to S_ε. In the one-dimensional case, the left-hand member of Eq. (47) degenerates into the sum

$$I = \lim_{\varepsilon\to 0}\left\{\frac{\partial u(x, t)}{\partial x}\Big|_{x=-\varepsilon} - \frac{\partial u(x, t)}{\partial x}\Big|_{x=\varepsilon}\right\}.$$

Indeed, in the one-dimensional case we have, from Eq. (46″)

$$I = \lim_{\varepsilon\to 0}\left\{-\frac{F(t-r)}{2}\frac{\partial r}{\partial x}\Big|_{x=-\varepsilon} + \frac{F(t-r)}{2}\frac{dr}{dx}\Big|_{x=\varepsilon}\right\} =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2}[F(t-\varepsilon) + F(t+\varepsilon)] = F(t).$$

Consequently, Eq. (47) is satisfied. In the three-dimensional case, the function $u = \frac{F(t-r)}{4\pi r}$ also satisfies condition (47) since

$$\left.\frac{\partial u}{\partial n}\right|_{S_\varepsilon} = -\left.\frac{\partial u}{\partial r}\right|_{r=\varepsilon} = \left.\frac{F(t-r)}{4\pi r^2}\right|_{r=\varepsilon} + \left.\frac{1}{4\pi r} F'(t-r)\right|_{r=\varepsilon},$$

and the area of the sphere S_ε is equal to $4\pi\varepsilon^2$. To show that condition (47) is satisfied in the two-dimensional case, let us first transform the right-hand member of (46′). We have

$$u(x, y, t) = \frac{1}{2\pi}\int_r^t \frac{F(t-\tau)}{\sqrt{\tau^2-r^2}}\,d\tau = \frac{1}{2\pi} F(t-\tau)\ln\left(\tau+\sqrt{\tau^2-r^2}\right)\Big|_r^t +$$

$$+\frac{1}{2\pi}\int_r^t F'(t-\tau)\ln\left(\tau+\sqrt{\tau^2-r^2}\right)d\tau = -\frac{1}{2\pi} F(t-r)\ln r +$$

$$+\frac{1}{2\pi} F(0)\ln\left(t+\sqrt{t^2-r^2}\right) + \frac{1}{2\pi}\int_r^t F'(t-\tau)\ln\left(\tau+\sqrt{\tau^2-r^2}\right)d\tau.$$

From this we obtain

$$\left.\frac{\partial u}{\partial n}\right|_{S_\varepsilon} = -\left.\frac{\partial u}{\partial r}\right|_{S_\varepsilon} = \left[\frac{F(t-r)}{2\pi r} + o\left(\frac{1}{r}\right)\right]_{r=\varepsilon}.$$

If we substitute this expression into Eq. (47) and remember that the circumference of a circle of radius ε is $2\pi\varepsilon$, we see that Eq. (47) is satisfied in the two-dimensonal case.

We can give the radiation functions (46), (46′), and (46″) a simple physical interpretation, beginning with a consideration of acoustic vibrations of air. Indeed, if u is a solution of the acoustic equation $\Delta u - u_{tt} = 0$, then $\vec{v} = -\nabla u$ represents the velocity vector of the air particles. Therefore, in the three-dimensional case, the relation $I = F(t)$ denotes an air source at the coordinate origin of intensity $F(t)$. In the two-dimensional case, the same equation $I = F(t)$ denotes the amount of air liberated in a unit of time by a unit length of an infinite straight line passing through the coordinate origin perpendicularly to the xy-plane. In the one-dimensional case, the equation $I = F(t)$ denotes the amount of air liberated in a unit of time by a unit area of the plane passing through the point $x=0$ and perpendicularly to the x-axis.

It is natural to call every solution of the homogeneous wave equation (27′) that satisfies condition (47) for arbitrary values of the time t $(-\infty < t < \infty)$ a wave source of intensity $F(t)$ at the coordinate origin, as we have done (see subsection 1) in the three-dimensional case. From this point of view, we can say that the radiation functions (46), (46′), and (46″) are wave sources of intensity $F(t)$ at the coordinate origin in three-dimensional,

two-dimensional, and one-dimensional spaces, respectively, but with the condition that these wave sources act not at all instants but only beginning at the instant $t=0$.

Let us note certain qualitative differences among the radiation functions in the three-, two-, and one-dimensional spaces. Suppose that we are observing radiation at a distance r from the coordinate origin, i.e., from a wave source that began to radiate at the instant $t=0$. From formulas (46), (46′), and (46″), it is obvious that the leading wavefront will be quite distinct in all three cases and that it will reach the point of observation at the instant $t=\frac{r}{a}$, where a is the velocity assumed to be equal to one. At subsequent instants of time the pictures at the point of observation will be quite different. In the three-dimensional case, the value of the function u at the point of observation will depend only on the value of the function $F(t)$ at the instant $t-\frac{r}{a}$, i.e., on the impulse of the force that existed at the coordinate origin at that instant of time. In particular, if the action of the wave source at the coordinate origin ceased at the instant t, the disturbance at the point of observation at the instant $t+\frac{r}{a}$ will disappear. Here, as in the case of the Cauchy problem in three-dimensional space, the trailing wavefront will be sharply delineated or, as we say, *Huygens' principle* holds. The qualitative picture is quite different in two- and one-dimensional spaces. As we can see from formulas (46′) and (46″), a disturbance at a distance r from the coordinate origin depends not only on the values of the function $F(t)$ at the instant t, but also on the values of this function $F(t)$ in the entire interval of variation of its argument from 0 to $t-\frac{r}{a}$. This means that disturbances that arrive at the point of observation do not pass through that point without aftereffects. These leave certain "traces" at that point and the superposition of these traces on each other causes a resulting disturbance. Thus, in contrast with the three-dimensional case, a disturbance at a point of observation depends not only on the impulse at the coordinate origin that occurred at the instant $t-\frac{r}{a}$ but also on the entire history of the radiation beginning at the instant $t=0$ up to the instant $t-\frac{r}{a}$. In particular, if at some instant the action of the wave source is cut off, the disturbance at the point of observation will not stop. Here, just as in the cases of the Cauchy problem for two- and one-dimensional spaces, there will not be a sharply defined trailing wavefront. The action of a wave source that is sharply delineated in time will not be so sharply delineated in time at the point of observation. The beginning will be

sharp, but the end will be diffuse. Thus, in the one- and two-dimensional spaces, Huygens' principle does not hold.

8. Solution of the Cauchy Problem for the Telegraph Equation and for the n-Dimensional Equation (for $n>3$). The Mean-Value Theorem

1. Solution of the Cauchy problem for the one- and two-dimensional telegraph equations by the method of auxiliary variables. The general form of the telegraph equation in n space variables

$$\Delta w - \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2} + b\frac{\partial w}{\partial t} + cw = -F, \tag{48}$$

becomes, when we introduce the new unknown function $v = we^{-\frac{a^2 b}{2}t}$

$$\Delta v - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + \left(\frac{a^2 b^2}{4} + c\right)v = -Fe^{-\frac{a^2 b}{2}t}. \tag{48'}$$

In those cases in which the variable $\frac{a^2 b^2}{4} + c$ is positive, the last equation may be written in the form

$$\Delta v - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + k^2 v = -\frac{f}{a^2}, \tag{49}$$

where

$$k = \sqrt{\frac{a^2 b^2}{4} + c} \text{ and } f = a^2 F e^{-\frac{a^2 b}{2}t}.$$

It turns out that the solution of the Cauchy problem for the telegraph equation (49) can be obtained immediately by using formulas giving the solution of the Cauchy problem for the wave equation but with the number of space coordinates greater by 1 than the number of space coordinates in the telegraph equation.

Let us first look at the Cauchy problem for the one-dimensional telegraph equation

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + k^2 v = -\frac{f(x, t)}{a^2},$$

$$v\Big|_{t=0} = \psi(x), \quad \frac{\partial v}{\partial t}\Big|_{t=0} = \psi_1(x), \tag{50}$$

where $\psi(x)$ and $\psi_1(x)$ are functions defined for $-\infty<x<\infty$ and where the first of these is three times continuously differentiable and the second twice continuously differentiable. If we introduce the function $u=v(x, y)\,e^{ky}$, where $v(x, y)$ is the solution of the problem (50) and y is an auxiliary variable, this function will be a solution of the Cauchy problem of the following form:

$$u_{xx}+u_{yy}-\frac{1}{a^2}u_{tt}=-\frac{f}{a^2}e^{ky},$$

$$u\Big|_{t=0}=\psi^*(x, y)=\psi(x)\,e^{ky},\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\psi_1^*(x, y)=\psi_1(x)\,e^{ky}. \tag{50'}$$

This is easily verified by differentiation. But the solution of the Cauchy problem (50′) is unique and is given by

$$u(x, y, t)=\frac{\partial U_{\psi^*}}{\partial t}+U_{\psi_1^*}+v^*(x, y, t),$$

where U_{ψ^*} is a function defined by Eq. (24) and $v^*(x, y, t)$ is the retarded potential (25) [see Sect. 5]. If we substitute into this the initial functions $\psi^*(x, y)=\psi(x)\,e^{ky}$, $\psi_1^*(x, y)=\psi_1(x)\,e^{ky}$, and the forcing term in Eq. (50′) and then divide both members of the equation by e^{ky}, we obtain the solution of the Cauchy problem for the one-dimensional telegraph equation (50) in the form

$$v(x, t)=\frac{\partial}{\partial t}\frac{1}{2\pi a}\iint\limits_{\rho<at}\frac{\psi(x+\xi)\,e^{k\eta}}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta+$$

$$+\frac{1}{2\pi a}\iint\limits_{\rho<at}\frac{\psi_1(x+\xi)\,e^{k\eta}}{\sqrt{a^2t^2-\rho^2}}\,d\xi\,d\eta+$$

$$+\frac{1}{2\pi a^2}\int_0^{at}dr\iint\limits_{\rho<r}\frac{f\left(x+\xi, t-\frac{r}{a}\right)e^{k\eta}}{\sqrt{r^2-\rho^2}}\,d\xi\,d\eta\quad(\rho=\sqrt{\xi^2+\eta^2}). \tag{51}$$

In the case of the Cauchy problem for the two-dimensional telegraph equation

$$v_{xx}+v_{yy}-\frac{1}{a^2}v_{tt}+k^2v=-\frac{f(x, y, t)}{a^2},$$

$$v|_{t=0}=\psi(x, y),\quad \frac{\partial v}{\partial t}\Big|_{t=0}=\psi_1(x, y), \tag{52}$$

where $\psi(x, y)$ and $\psi_1(x, y)$ are functions defined for $-\infty<x, y<\infty$, with ψ three times continuously differentiable and ψ_1 twice continuously differentiable just as in the case of the one-dimensional

telegraph equation, we can also use the *auxiliary variable method.* Specifically, it is easy to verify that if v is a solution of the problem (52), the function $u=v(x, y, t)\,e^{kz}$ must be a solution of the Cauchy problem for the three-dimensional wave equation

$$u_{xx}+u_{yy}+u_{zz}-\frac{1}{a^2}u_{tt}=-\frac{f}{a^2}e^{kz},$$

$$u|_{t=0}=\psi^*(x, y, z)=\psi(x, y)e^{kz},\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi_1(x, y, z)=\psi_1(x, y)e^{kz}. \tag{52'}$$

But the solution of this equation is unique and, on the basis of formulas (21) and (23) [see Sect. 5], it may be written in the form

$$u(x, y, z, t)=\frac{\partial}{\partial t}\frac{1}{4\pi a^2}\iint\limits_{r=at}\frac{\psi(x+\xi, y+\eta)}{t}e^{k(z+\zeta)}\,d\sigma+$$

$$+\frac{1}{4\pi a^2}\iint\limits_{r=at}\frac{\psi_1(x+\xi, y+\eta)}{t}e^{k(z+\zeta)}\,d\sigma+ \tag{53'}$$

$$+\frac{1}{4\pi a^2}\iiint\limits_{r<at}\frac{f\left(x+\xi, y+\eta, t-\frac{r}{a}\right)}{r}e^{k(z+\zeta)}\,d\tau\quad (r=\sqrt{\xi^2+\eta^2+\zeta^2}).$$

If we divide both members of this equation by e^{kz}, we obtain the solution of the Cauchy problem for the two-dimensional telegraph equation (52)

$$v(x, y, t)=\frac{\partial}{\partial t}\frac{1}{4\pi a^2}\iint\limits_{r=at}\frac{\psi(x+\xi, y+\eta)}{t}e^{k\zeta}\,d\sigma+$$

$$+\frac{1}{4\pi a^2}\iint\limits_{r=at}\frac{\psi_1(x+\xi, y+\eta)}{t}e^{k\zeta}\,d\sigma+$$

$$+\frac{1}{4\pi a^2}\iiint\limits_{r<at}\frac{f\left(x+\xi, y+\eta, t-\frac{r}{a}\right)}{r}e^{k\zeta}\,d\tau. \tag{53}$$

2. Solution of the Cauchy problem for the three-dimensional telegraph equation and for the n-dimensional wave equation (for $n>3$). Let us find explicit formulas giving the solution of the Cauchy problem for the n-dimensional wave equation, when $n>3$

$$\Delta u-\frac{1}{a^2}u_{tt}=\sum_{i=1}^{n}u_{x_ix_i}-\frac{1}{a^2}u_{tt}=-\frac{f(x_1, \ldots x_n, t)}{a^2}, \tag{54}$$

$$u|_{t=0} = \psi(x_1, \ldots, x_n), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x_1, \ldots, x_n). \tag{54'}$$

To obtain the solution of the Cauchy problem (54), (54'), let us use the mean values of the functions just as in the case of the three-dimensional wave equation.

Suppose $u(x_1, \ldots, x_n, \tau) = u(x_i, \tau)$ is a twice continuously differentiable function of the variables $x_1, x_2, \ldots, x_n$, and depends on a parameter τ. Then, the mean value of this function over a sphere S_{r, x_i} of radius r with center at the point $x_1, \ldots, x_n$ is defined as an expression of the form

$$Q_u(x_i, r, \tau) = \frac{1}{\omega_n} \overbrace{\int \cdots \int}^{n-1}_{S_{r,x_i}} u(x_i + \beta_i r, \tau)\, d\omega_n =$$

$$= \frac{1}{\sigma_n} \overbrace{\int \cdots \int}^{n-1}_{S_{1,x_i}} u(x_i + \beta_i r, \tau)\, d\sigma_n, \tag{55}$$

where ω_n is the area of the sphere S_{r,x_i}; and $d\omega_n$ is an element* of

*An element of area of a hypersurface $\varphi(x_1, \ldots, x_n) = 0$ in n-dimensional space is defined by the equation

$$d\omega_n = \frac{\sqrt{\varphi_{x_1}^2 + \varphi_{x_2}^2 + \cdots \varphi_{x_n}^2}}{|\varphi_{x_n}|}\, dx_1 \ldots dx_{n-1}.$$

In particular, in the case of the sphere $\sum_{i=1}^{n} x_i^2 = r^2$ it will be

$$d\omega_n = \frac{r}{|x_n|}\, dx_1 \ldots dx_{n-1} = \frac{r\, dx_1 \ldots dx_{n-1}}{\sqrt{r^2 - x_1^2 - x_2^2 - \ldots - x_{n-1}^2}}.$$

If we set

$$x_1 = r\alpha, \quad x_2 = r\sqrt{1-\alpha^2}\, y_2, \ldots, x_n = r\sqrt{1-\alpha^2}\, y_n$$

in this equation, so that

$$y_2^2 + y_3^2 + \ldots + y_n^2 = \frac{x_2^2 + x_3^2 + \ldots + x_n^2}{r^2(1-\alpha^2)} = 1,$$

we obtain

$$d\omega_n = \frac{r^{n-1}\, dy_2\, dy_3 \ldots dy_{n-1}}{\sqrt{1 - y_2^2 - \ldots - y_{n-1}^2}} (1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha = r^{n-1} d\sigma_{n-1} (1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha,$$

where $d\sigma_{n-1}$ is an element of area σ_{n-1} of the sphere of unit radius in $(n-1)$-dimensional space. For example, for $n = 4$, we have

that area; $\beta_i = \dfrac{x_i' - x_i}{r}$ are the direction cosines; the x_i' are the coordinates of the points of the sphere S_{r,x_i}; S_{1,x_i} is the sphere

---

$$\sigma_3 = 4\pi, \int_{-1}^{1} (1-\alpha^2)^{\frac{1}{2}}\, d\alpha = \frac{\pi}{2}$$

so that $\omega_4 = 2\pi^2 r^3$. For arbitrary n, we have

$$\sigma_n = \sigma_{n-1} \int_{-1}^{1} (1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha = 2\sigma_{n-1} \int_{0}^{1} (1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha =$$

$$= \sigma_{n-1} \int_{0}^{1} t^{-\frac{1}{2}} (1-t)^{\frac{n-3}{2}}\, dt = \sigma_{n-1} B\left(\frac{1}{2}, \frac{n-1}{2}\right) =$$

$$= \sigma_{n-1} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right) = \sigma_{n-1} \sqrt{\pi} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \Gamma\left(\frac{n-1}{2}\right)$$

and, hence,

$$\sigma_n = \sigma_2 \pi^{\frac{n-2}{2}} \Gamma(1) \frac{1}{\Gamma\left(\frac{n}{2}\right)} = 2\pi^{\frac{n}{2}} \frac{1}{\Gamma\left(\frac{n}{2}\right)}.$$

For the volume $v_n(r)$ of an n-dimensional sphere $\sum_{i=1}^{n} x_i^2 < r$ we have

$$v_n(r) = \int_0^r \omega_n(\rho)\, d\rho = 2\pi^{\frac{n}{2}} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^r \rho^{n-1}\, d\rho =$$

$$= \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} (\sqrt{\pi} r)^n = \frac{r}{n} \omega_n = \frac{r^n}{n} \sigma_n.$$

For the integral of the function $u(x_1, \ldots, x_n)$ over the sphere $\sum_{i=1}^{n} x_i^2 = r^2$, we have

$$\overbrace{\int \cdots \int}^{n-1} u(x_1, \ldots, x_n)\, d\omega_n =$$

$$= r^{n-1} \int_{-1}^{1} (1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha \overbrace{\int \cdots \int}^{n-2} u(r\alpha, r\sqrt{1-\alpha^2} y_2), \ldots, r\sqrt{1-\alpha^2} y_n)\, d\sigma_{n-1}.$$

In particular, if the integrand u depends on a single variable x_1, the following equation is valid:

$$\overbrace{\int \cdots \int}^{n-1} u(x_1)\, d\omega_n = r^{n-1} \sigma_{n-1} \int_{-1}^{1} (1-\alpha^2)^{\frac{n-3}{2}} u(r\alpha)\, d\alpha.$$

of unit radius with center at the point $x_1, \ldots, x_n$, σ_n is the area of the unit sphere S_{1,x_i}; and $d\sigma_n$ is an element of that area.

In the first place, let us show that $Q = Q_u(x_i, r, \tau)$, as a function of the variables r and $x_1, x_2, \ldots, x_n$, is just as in the case of $n=3$ [see Sect. 5], a twice continuously differentiable solution of the Darboux equation

$$\Delta Q - \frac{\partial^2 Q}{\partial r^2} - \frac{n-1}{r}\frac{\partial Q}{\partial r} = 0 \quad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right) \tag{56}$$

and satisfies the initial conditions

$$Q|_{r=0} = u(x_i, \tau), \quad \left.\frac{\partial Q}{\partial r}\right|_{r=0} = 0. \tag{56'}$$

By applying the mean-value theorem to the integral (55), we see that the first of conditions (56′) is satisfied. Furthermore,

$$\frac{\partial Q}{\partial r} = \frac{1}{\sigma_n}\overbrace{\int\cdots\int}^{n-1}_{S_{1,x_i}} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\beta_i\, d\sigma_n = -\frac{1}{r^{n-1}\sigma_n}\overbrace{\int\cdots\int}^{n-1}_{S_{r,x_i}} \frac{\partial u(x_i+\beta_i r)}{\partial n}\, d\omega_n,$$

where $\vec{n}$ the inner normal to the sphere S_{r,x_i}. If we apply Ostrogradskiy's formula to the last integral, we obtain

$$\frac{\partial Q}{\partial r} = \frac{1}{r^{n-1}\sigma_n}\int\cdots\int_{v_{r,x_i}} \Delta u\, d\omega,$$

where $d\omega$ is an element of volume of the sphere v_{r,x_i} bounded by the sphere S_{r,x_i}. It follows from this last equation that the second of conditions (56′) is satisfied since the volume of an n-dimensional sphere is proportional to the nth power of its radius (cf. footnote to page 295). Furthermore,

$$\frac{\partial^2 Q}{\partial r^2} = -\frac{n-1}{r^n\sigma_n}\overbrace{\int\cdots\int}^{n}_{v_{r,x_i}} \Delta u\, d\tau + \frac{1}{r^{n-1}\sigma_n}\overbrace{\int\cdots\int}^{n-1}_{S_{r,x_i}} \Delta u\, d\omega_n =$$

$$= -\frac{n-1}{r}\frac{\partial Q}{\partial r} + \frac{1}{\sigma_n}\int\cdots\int_{S_{1,x_i}} \Delta u\, d\sigma_n = -\frac{n-1}{r}\frac{\partial Q}{\partial r} + \Delta Q,$$

i.e., Eq. (56) is satisfied.

For our purposes, the significant thing is that the mean value (55) is the only solution of the problem (56), (56′), at least in the class of twice continuously differentiable functions.

To see that this is the case, let us suppose that $w(x_1, \ldots, x_n, r)$ is a solution of the Darboux equation, satisfying the homogeneous initial conditions

$$L(w) = \Delta w - \frac{\partial^2 w}{\partial r^2} - \frac{\lambda}{r}\frac{\partial w}{\partial r} = 0 \quad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},\ \lambda > 0\right)$$
$$w|_{r=0} = 0, \quad \left.\frac{\partial w}{\partial r}\right|_{r=0} = 0. \tag{56''}$$

We have

$$0 = -2w_r L(w) = -2\sum_{i=1}^{n}(w_r w_{x_i})_{x_i} + \left(\sum_{i=1}^{n} w_{x_i}^2 + w_r^2\right)_r + \frac{2\lambda}{r} w_r^2.$$

If we integrate both sides of this equation over the domain G bounded by the characteristic hypercone and the plane $r=0$, we obtain, by virtue of Ostrogradskiy's formula and consideration of the homogeneous initial conditions,

$$0 = \int_G \cdots \int \frac{2\lambda}{r} w_r^2 \, d\omega - \int_M \cdots \int \left[-2w_r \sum_{i=1}^{n} w_{x_i}\frac{\partial x_i}{\partial n} + \right.$$
$$\left. + \left(w_r^2 + \sum_{i=1}^{n} w_{x_i}^2\right)\frac{\partial r}{\partial n}\right] d\sigma, \tag{56'''}$$

where $d\omega$ is an element of volume of the domain G, M is the lateral surface of the characteristic hypercone, $d\sigma$ is an element of this surface, and $\vec{n}$ is the inner normal to the surface M. But the equation of the characteristic hypercone is written in the form

$$\varphi_x^2 + \varphi_{x_1}^2 + \ldots + \varphi_{x_n}^2 - \varphi_r^2 = 0, \quad \varphi(x_1, x_2, \ldots, x_n, r) = 0$$

(see Chapter II, Sect. 2). Therefore, on the characteristic hypercone, we have the equation

$$\left(\frac{\partial x_1}{\partial n}\right)^2 + \left(\frac{\partial x_2}{\partial n}\right)^2 + \ldots + \left(\frac{\partial x_n}{\partial n}\right)^2 - \left(\frac{\partial r}{\partial n}\right)^2 = 0.$$

From this it follows that the integrand in the second integral on the right-hand side of Eq. (56''') can be written in the form

$$\left(\frac{\partial r}{\partial n}\right)^{-1} \sum_{i=1}^{n}\left(w_{x_i}\frac{\partial r}{\partial n} - w_r\frac{\partial x_i}{\partial n}\right)^2.$$

Since $\left.\frac{\partial r}{\partial n}\right|_M < 0$ and $\lambda > 0$, we conclude from (56‴) that $w_r \equiv 0$ and, from the homogeneous initial conditions, that $w \equiv 0$.

Let us note one further auxiliary proposition. If

$$\widetilde{\varphi}(r) = \int_{-1}^{1} \varphi(r\alpha)(1-\alpha^2)^{\frac{n-3}{2}} d\alpha, \tag{57}$$

where $\varphi(r)$ is an arbitrary twice continuously differentiable function of its argument, then

$$\widetilde{\varphi}''(r) + \frac{n-1}{r}\widetilde{\varphi}'(r) = \int_{-1}^{1} \varphi''(r\alpha)(1-\alpha^2)^{\frac{n-3}{2}} d\alpha. \tag{57'}$$

This means that corresponding to the operation $\frac{d^2}{dr^2} - \frac{n-1}{r}\frac{d}{dr}$ in the class of functions $\widetilde{\varphi}(r)$ is the operation $\frac{d^2}{dr^2}$ in the class of functions $\varphi(r)$. In fact, we have

$$\widetilde{\varphi}_r = \int_{-1}^{1} \varphi'(r\alpha)\alpha(1-\alpha^2)^{\frac{n-3}{2}} d\alpha = -\frac{1}{n-1}(1-\alpha^2)^{\frac{n-1}{2}}\varphi'(r\alpha)\Big|_{-1}^{1} +$$

$$+ \int_{-1}^{1} \frac{\varphi''(r\alpha)r}{n-1}(1-\alpha^2)^{\frac{n-1}{2}} d\alpha,$$

$$\widetilde{\varphi}_{rr} = \int_{-1}^{1} \varphi''(r\alpha)\alpha^2(1-\alpha^2)^{\frac{n-3}{2}} d\alpha.$$

If we multiply the first of these equations by $\frac{n-1}{r}$ and add the result to the second, we obtain (57').

Suppose now that $u = u(x_1, \ldots, x_n, t)$ is a twice continuously differentiable solution of the Cauchy problem for the homogeneous n-dimensional wave equation

$$\Delta u - \frac{1}{a^2} u_{tt} = 0 \quad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right), \tag{3}$$

$$u|_{t=0} = \psi(x_1, \ldots, x_n), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \tag{58}$$

Let us extend this function to negative values of t as an even function and let us look at its "image" in the sense of Eq. (57):

$$\tilde{u}\left(x_1, \ldots, x_n, \frac{r}{a}\right) = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} u\left(x_1, \ldots, x_n, \frac{r}{a}\alpha\right)(1-\alpha^2)^{\frac{n-3}{2}} d\alpha. \quad (58')$$

If we apply the operator $\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$ to this function, we obtain on the basis of (57′)

$$a^2\left(\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{n-1}{r}\frac{\partial \tilde{u}}{\partial r}\right) =$$

$$= \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} u''\left(x_1, \ldots, x_n, \frac{r}{a}\alpha\right)(1-\alpha^2)^{\frac{n-3}{2}} d\alpha,$$

where the primes denote differentiation with respect to the variable $\frac{r}{a}\alpha$. Remembering that $u(x_1, \ldots, x_n, t)$ satisfies the homogeneous wave equation (3), we obtain

$$\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{n-1}{r}\frac{\partial \tilde{u}}{\partial r} = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} \Delta u\,(1-\alpha^2)^{\frac{n-3}{2}} d\alpha = \Delta \tilde{u}.$$

Furthermore, it is easy to see that the function (58′) satisfies the initial conditions* $\tilde{u}|_{r=0} = \psi(x_1, \ldots, x_n)$ and

$$\left.\frac{\partial \tilde{u}}{\partial r}\right|_{r=0} = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} u'(x, \ldots, x_n, 0)\frac{\alpha}{a}(1-\alpha^2)^{\frac{n-3}{2}} d\alpha = 0.$$

Thus, the function $\tilde{u}\left(x_1, \ldots, x_n, \frac{r}{a}\right)$ is a solution of the problem (56), (56′). But the solution of this problem is unique and, in accordance with (55), is given by the equation

$$Q_\psi(x_i, r, 0) = \frac{1}{\sigma_n} \int \cdots \int_{S_{1,x_i}} \psi(x_1 + \beta_1 r, \ldots, x_n + \beta_n r)\, d\sigma_n$$

*Since the area σ_n of a sphere of unit radius in n-dimensional space and the area σ_{n-1} of a sphere of unit radius in $(n-1)$-dimensional space are related by

$$\sigma_n = \sigma_{n-1} \int_{-1}^{1} (1-\alpha^2)^{\frac{n-3}{2}} d\alpha$$

(cf. footnote to page 295).

Therefore,

$$\widetilde{u}\left(x_i,\ \frac{r}{a}\right) = Q_{\psi}(x_i, r, 0).$$

This means that

$$2\frac{\sigma_{n-1}}{\sigma_n}\int_0^1 u(x_1, \ldots, x_n, r\alpha)(1-\alpha^2)^{\frac{n-3}{2}}\,d\alpha = \frac{1}{\sigma_n}\int\cdots\int_{S_{1,x_i}} \psi(x_i + \beta_i a r)\,d\sigma_n.$$

We have arrived at the conclusion that if a twice continuously differentiable solution $u=u(x_1, \ldots, x_n, t)$ of the Cauchy problem (3), (58) exists, it must be a solution of the integral equation (59). Let us find a solution of this integral equation.

First, let us consider the problem of inverting the integral

$$\widetilde{\varphi}(r) = \int_0^1 \varphi(r\alpha)(1-\alpha^2)^{\frac{n-3}{2}}\,d\alpha. \tag{60}$$

If we set $r\alpha = \sqrt{\sigma}$, $r = \sqrt{s}$, this last equation takes the form

$$\widetilde{\varphi}(\sqrt{s}) = \int_0^s \varphi(\sqrt{\sigma})\left(1-\frac{\sigma}{s}\right)^{\frac{n-3}{2}}\frac{d\sigma}{2\sqrt{s\sigma}}.$$

If we introduce the notations

$$w(s) = 2\widetilde{\varphi}(\sqrt{s})\,s^{\frac{n-3}{2}}, \quad \varkappa(s) = \varphi(\sqrt{s})\frac{1}{\sqrt{s}} = \varphi(r)\frac{1}{r},$$

Eq. (60) may be written in the form

$$w(s) = \int_0^s \varkappa(\sigma)(s-\sigma)^{\frac{n-3}{2}}\,d\sigma. \tag{61}$$

In solving the integral equation (61), we need to treat the cases of even and odd n separately. In the case of odd n, if we differentiate both members of Eq. (61) $\frac{n-1}{2}$ times, we obtain

$$\varkappa(s) = \frac{1}{\left(\frac{n-3}{2}\right)!} \frac{d^{\frac{n-1}{2}}}{ds^{\frac{n-1}{2}}} w(s). \tag{61'}$$

Therefore, for odd n, the solution of the integral equation (60) is written in the form

$$\varphi(r) = \frac{2r}{\left(\frac{n-3}{2}\right)!} \frac{d^{\frac{n-1}{2}}}{(dr^2)^{\frac{n-1}{2}}} [r^{n-3}\widetilde{\varphi}(r)]. \tag{60'}$$

Furthermore, for even n, if we differentiate both members of Eq. (61) $\frac{n-2}{2}$ times, we arrive at Abel's integral equation

$$\omega(s) = \int_0^s \frac{\varkappa(\sigma)\, d\sigma}{\sqrt{s-\sigma}}, \tag{62}$$

where

$$\omega(s) = \left(\frac{n-3}{2} \cdot \frac{n-5}{2} \cdots \frac{1}{2}\right)^{-1} \frac{d^{\frac{n-2}{2}}}{ds^{\frac{n-2}{2}}} w(s) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2}\right)} w^{\left(\frac{n-2}{2}\right)}(s).$$

But the solution of the integral equation (62) is known and is given by the formula*

$$\varkappa(s) = \frac{1}{\pi} \frac{d}{ds} \int_0^s \frac{\omega(\sigma)}{\sqrt{s-\sigma}}\, d\sigma. \tag{62'}$$

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*To see this, we reverse the order of integration in the integrate over the triangular region $0<\sigma<s<\lambda$,

$$\int_0^\lambda \frac{\omega(s)\, ds}{\sqrt{\lambda-s}} = \int_0^\lambda \frac{ds}{\sqrt{\lambda-s}} \int_0^s \frac{\varkappa(\sigma)\, d\sigma}{\sqrt{s-\sigma}} = \int_0^\lambda \varkappa(\sigma)\, d\sigma \int_\sigma^\lambda \frac{ds}{\sqrt{(\lambda-s)(s-\sigma)}}.$$

But the inner integral in the right-hand member of this equation is easily evaluated by making the substitution

$$s = \frac{\lambda+\sigma}{2} + \frac{\lambda-\sigma}{2} \sin\theta$$

and is shown to be equal to π. Therefore, by differentiating the right-hand member of the last equation with respect to λ, we obtain (62').

Consequently, for even n, the solution of the integral equation (61) is of the form

$$\varkappa(s) = \frac{1}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \frac{d}{ds} \int_0^s \frac{w^{\left(\frac{n-2}{2}\right)}(\sigma)}{\sqrt{s-\sigma}}\, d\sigma, \tag{61''}$$

and the solution of the integral equation (60) is given by the equation

$$\varphi(r) = \frac{4r}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \frac{d}{dr^2} \int_0^r \frac{\rho}{\sqrt{r^2-\rho^2}} \frac{d^{\frac{[n-2}{2}}}{(d\rho^2)^{\frac{n-2}{2}}} [\rho^{n-2}\widetilde{\varphi}(\rho)]\, d\rho. \tag{60''}$$

Now, if we write the integral equation (59) in the form

$$\int_0^1 u(x_1, \ldots, x_n, r\alpha)(1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha = \frac{\sigma_n}{2\sigma_{n-1}} Q_\psi(x_i, ar, 0),$$

we obtain, on the basis of (60′) and (60″), the solution of this integral equation for odd and even n in the forms

$$u(x_1, \ldots, x_n, t) = \frac{1}{\left(\frac{n-3}{2}\right)!} \frac{\sigma_n}{\sigma_{n-1}} t \frac{\partial^{\frac{n-1}{2}}}{(\partial t^2)^{\frac{n-1}{2}}} [t^{n-2} Q_\psi(x_i, ta, 0)], \tag{59'}$$

$$u(x_1, \ldots, x_n, t) =$$

$$= \frac{1}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \frac{\sigma_n}{\sigma_{n-1}} \frac{\partial}{\partial t} \int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}} \frac{\partial^{\frac{[n-2}{2}}}{(\partial\rho^2)^{\frac{n-2}{2}}} [\rho^{n-2} Q_\psi(x_i, \rho a, 0)]\, d\rho, \tag{59''}$$

respectively.

Thus, we have shown that if a twice continuously differentiable solution of the Cauchy problem (3), (58) exists, it is given by Eq. (59′) for odd n and by Eq. (59″) for even n.

In all that follows, we shall assume for definiteness that $n=4$. Next to the cases of $n=1, 2, 3$, for which we have already solved the Cauchy problem, this case is the most important one. Formula (59″) for the solution of the Cauchy problem (3), (58) yields

$$u(x_1, \ldots, x_4, t) = \frac{\partial}{\partial t} \int_0^t \frac{\rho}{\sqrt{t^2-\rho^2}} \frac{\partial}{\partial \rho^2} [\rho^2 Q_\psi(x_i, \rho a, 0)]\, d\rho, \tag{63}$$

where, in accordance with (55),

$$Q_\psi(x_i, \rho a, 0) = \frac{1}{2\pi^2} \iiint_{S_{1,x_i}} \psi(x_i + \beta_i \rho a)\, d\sigma_4 =$$

$$= \frac{1}{2\pi^2 \rho^3 a^3} \iiint_{S_{\rho a, x_i}} \psi(x_i + \beta_i \rho a)\, d\omega_4, \tag{64}$$

since the area σ_4 of a unit sphere in four-dimensional space is equal to $2\pi^2$ and the area ω_4 of a sphere of radius ρa is equal to $2\pi^2 \rho^3 a^3$ in the same space.

Let us examine the right-hand member of Eq. (63). We begin by looking at the integral

$$U_\psi = \int_0^t \frac{\rho}{\sqrt{t^2 - \rho^2}} \frac{\partial}{\partial \rho^2} [\rho^2 Q_\psi(x_i, \rho a, 0)]\, d\rho. \tag{65}$$

The derivative of this integral with respect to t coincides with the right-hand member of Eq. (63). If we make the change of variable $\xi = \left(\frac{\rho}{t}\right)^2$, we obtain

$$U_\psi = \frac{t}{2} \int_0^1 \frac{1}{\sqrt{1-\xi}} \frac{\partial}{\partial \xi} [\xi Q_\psi(x_i, ta\sqrt{\xi}, 0)]\, d\xi =$$

$$= \frac{t}{4\pi^2} \int_0^1 \frac{1}{\sqrt{1-\xi}} \frac{\partial}{\partial \xi} \left[\xi \iiint_{S_{1,x_i}} \psi(x_i + \beta_i ta\sqrt{\xi})\, d\sigma_4\right] d\xi \tag{65'}$$

and, after differentiating the inner integral with respect to ξ,

$$U_\psi = \frac{t}{2} \int_0^1 \frac{1}{\sqrt{1-\xi}} \left\{ Q_\psi(x_i, ta\sqrt{\xi}, 0) + \right.$$

$$\left. + \sqrt{\xi}\, \frac{ta}{4\pi^2} \iiint_{S_{1,x_i}} \sum_{k=1}^{4} \psi_{x_k}(x_i + \beta_i ta\sqrt{\xi})\, \beta_k\, d\sigma_4 \right\} d\xi.$$

It is obvious from this equation that if the function $\psi(x_1, \ldots, x_4)$ is $k+1$ times continuously differentiable, the integral U_ψ will have continuous derivatives with respect to $x_1, \ldots, x_4$, and t up through the kth order, since, in this case, the integral U_ψ can be differentiated under the integral sign.

Let us show that the integral U_ψ given by Eq. (65) under the condition that the function $\psi(x_1, \ldots, x_4)$ is three times continuously differentiable, satisfies the homogeneous four-dimensional wave equation (3) and the initial conditions

$$U_\psi|_{t=0} = 0, \quad \frac{\partial U_\psi}{\partial t}\Big|_{t=0} = \psi(x_1, \ldots, x_4). \tag{66}$$

It follows immediately from (65′) that the first of conditions (66) is satisfied. Furthermore, from (65′), we obtain

$$\Delta U_\psi = \frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial(\xi A)}{\partial \xi}\,d\xi, \quad A = \iiint\limits_{S_{1,x_i}} \Delta\psi(x_i + \beta_i at\sqrt{\xi})\,d\sigma_4,$$

where Δ is the four-dimensional Laplacian operator with respect to the variables $x_1, \ldots, x_4$. Differentiating (65′) with respect to t, we have

$$\frac{\partial U_\psi}{\partial t} = \frac{U_\psi}{t} + \frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\left[\xi a\sqrt{\xi}\iiint\limits_{S_{1,x_i}}\sum_{k=1}^{4}\psi_{x_k}(x_i + \right.$$

$$\left. + \beta_i ta\sqrt{\xi})\,\beta_k\,d\sigma_4\right]d\xi.$$

From this and from (65′), remembering that $Q_\psi(x_i, r, 0)|_{r=0} = \psi$, we obtain

$$\frac{\partial U_\psi}{\partial t}\Big|_{t=0} = \psi,$$

so that the second of conditions (66) is satisfied. Now before evaluating $\frac{\partial^2 U_\psi}{\partial t^2}$, let us make a modification in the expression for $\frac{\partial U_\psi}{\partial t}$. By shifting to an integral over a sphere of radius $ta\sqrt{\xi}$ and using Ostrogradskiy's formula, we obtain

$$\frac{\partial U_\psi}{\partial t} = \frac{U_\psi}{t} + \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\iiint\limits_{S_{ta\sqrt{\xi}\,x_i}}\sum_{k=1}^{4}\psi_{x_k}(x_i + \right.$$

$$\left. + \beta_i\, ta\sqrt{\xi})\,\beta_k\,d\omega_4\,d\xi = \frac{U_\psi}{t} + \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial I}{\partial \xi}\,d\xi,$$

$$I = \iiint\limits_{v_{ta\sqrt{\xi}},\,x_i} \Delta\psi(x_i + \beta_i ta\sqrt{\xi})\,d\tau,$$

where $v_{ta\sqrt{\xi},x_i}$ is in the interior of sphere $S_{ta\sqrt{\xi},x_i}$. Furthermore, we have

$$\frac{\partial^2 U}{\partial t^2} = -\frac{U_\psi}{t^2} + \frac{1}{t}\left[\frac{U_\psi}{t} + \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial I}{\partial \xi}\,d\xi\right] -$$

$$-\frac{1}{2\pi^2 a^2 t^3}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial I}{\partial \xi}\,d\xi + \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial}{\partial \xi}\,\frac{\partial I}{\partial t}\,d\xi$$

or

$$\frac{\partial^2 U}{\partial t^2} = \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial}{\partial \xi}\,\frac{\partial I}{\partial t}\,d\xi - \frac{1}{4\pi^2 a^2 t^3}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial I}{\partial \xi}\,d\xi.$$

If we calculate $\frac{\partial I}{\partial t}$ in the first integral in the right-hand member of this equation and $\frac{\partial I}{\partial \xi}$ in the second and shift to integration over the unit sphere, we obtain

$$\frac{\partial^2 U}{\partial t^2} = \frac{1}{4\pi^2 a^2 t^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial}{\partial \xi}[Aa\sqrt{\xi}\,(ta\sqrt{\xi})^3]\,d\xi -$$

$$-\frac{1}{4\pi^2 a^2 t^3}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,A\frac{ta}{2\sqrt{\xi}}\,(ta\sqrt{\xi})^3\,d\xi =$$

$$= \frac{a^2 t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\left[\frac{\partial}{\partial \xi}(\xi^2 A) - \frac{1}{2}\,\xi A\right]d\xi =$$

$$= \frac{a^2 t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\,\frac{\partial(\xi A)}{\partial \xi}\,d\xi +$$

$$+\frac{a^2 t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\left[\frac{\partial}{\partial \xi}(\xi^2 A) - \frac{\partial(\xi A)}{\partial \xi} - \frac{\xi A}{2}\right]d\xi. \qquad (65'')$$

But simple calculation yields

$$\int_0^1 \frac{1}{\sqrt{1-\xi}}\left[\frac{\partial}{\partial\xi}(\xi^2 A)-\frac{\partial}{\partial\xi}(\xi A)\right]d\xi=\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial\xi}[\xi A(\xi-1)]\,d\xi=$$

$$=\int_0^1 \frac{1}{\sqrt{1-\xi}}\left[\frac{\partial(\xi A)}{\partial\xi}(\xi-1)+\xi A\right]d\xi=$$

$$=-\int_0^1 \frac{\partial(\xi A)}{\partial\xi}\sqrt{1-\xi}\,d\xi+\int_0^1 \frac{\xi A}{\sqrt{1-\xi}}d\xi=-\sqrt{1-\xi}\,\xi A\Big|_0^1-$$

$$-\int_0^1 \frac{\xi A\,d\xi}{2\sqrt{1-\xi}}+\int_0^1 \frac{\xi A\,d\xi}{\sqrt{1-\xi}}=\frac{1}{2}\int_0^1 \frac{\xi A\,d\xi}{\sqrt{1-\xi}}.$$

From this it follows that the second integral in the right-hand member of Eq. (65″) is equal to 0. If we now equate the results of the calculations for ΔU_ψ and $\frac{\partial^2 U_\psi}{\partial t^2}$, we obtain

$$\frac{1}{a^2}\frac{\partial^2 U_\psi}{\partial t^2}=\Delta U_\psi.$$

This means that the integral U_ψ under the condition that the function $\psi(x_1, \ldots, x_4)$ is three times continuously differentiable, is a twice continuously differentiable solution of the homogeneous four-dimensional wave equation (3) with initial conditions (66). If the function $\psi(x_1, \ldots, x_4)$ is four times continuously differentiable, the function defined by Eq. (63) will be a twice continuously differentiable solution of the Cauchy problem (3), (58) since, on the basis of (66)

$$u\Big|_{t=0}=\frac{\partial U_\psi}{\partial t}\Big|_{t=0}=\psi,\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\frac{\partial^2 U_\psi}{\partial t^2}\Big|_{t=0}=a^2\,\Delta U_\psi\Big|_{t=0}=0.$$

Thus, we arrive at the following conclusion:

If the function $\psi(x_1, \ldots, x_4)$ is four times continuously differentiable and the function $\psi_1(x_1, \ldots, x_4)$ is three times continuously differentiable, a solution of the Cauchy problem for the homogeneous four-dimensional wave equation

$$\Delta u-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=0\quad\left(\Delta=\sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2}\right),$$

$$u\Big|_{t=0}=\psi(x_1, \ldots, x_4),\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\psi_1(x_1, \ldots, x_4) \tag{67}$$

is unique in the class of twice continuously differentiable functions and is given by the equation

$$u(x_1, \ldots, x_4, t) = \frac{\partial U_\psi}{\partial t} + U_{\psi_1}, \tag{67'}$$

where U_ψ is the integral defined by Eq. (65) or, what amounts to the same thing, by Eq. (65').

To find the solution of the Cauchy problem for the nonhomogeneous four-dimensional wave equation (54), (54') in the most general case, we need only find a solution of the nonhomogeneous equation with homogeneous initial conditions

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f}{a^2}\left(\Delta = \sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2}\right)$$
$$u\Big|_{t=0} = 0 \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \tag{68}$$

By using the method of impulses (see Sect. 3), let us seek a solution of this problem in the form

$$v(x_1, \ldots, x_4, t) = \int_0^t \varphi(x_1, \ldots, x_4, t, t_0)\, dt_0, \tag{68'}$$

where*

$$\varphi(x_1, \ldots, x_4, t, t_0) = \int_0^{t-t_0} \frac{\rho}{\sqrt{(t-t_0)^2+\rho^2}}\frac{\partial}{\partial \rho^2}[\rho^2 Q_f(x_i, \rho a, t_0)]\, d\rho. \tag{68''}$$

Let us show that, if the function $f(x_1, \ldots, x_4, t)$ is three times continuously differentiable, the integral (68') will be a twice continuously differentiable solution of the problem (68). From (68'), it is immediately clear that the first of the initial conditions of problem (68) is satisfied. Since the function (68'') is a function of the type of U_ψ defined by Eq. (65), this function will be a twice continuously differentiable solution of the boundary-value problem

*If we make the change of variables $\tau = t - t_0$ and $\xi = \frac{\rho^2}{\tau^2}$, Eq. (68') may be written in the form

$$v(x_1, \ldots, x_4, t) = \int_0^t \frac{\tau}{4\pi^2}\, d\tau \int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\left[\xi \iiint_{S_1, x_i} f(x_i + \beta_i a\tau\sqrt{\xi}, t-\tau)\, d\sigma_4\right] d\xi.$$

$$\Delta\varphi - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} = 0,\ \varphi\Big|_{t=0} = 0,\ \frac{\partial\varphi}{\partial t}\Big|_{t=t_0} = f(x_1, \ldots, x_4, t_0).$$

It follows from this that, in particular, the integral (68′) will be a twice continuously differentiable function. Furthermore, we have

$$\frac{\partial v}{\partial t} = \int_0^t \frac{\partial\varphi}{\partial t}\,dt_0 + \lim_{t_0\to t-0}\varphi = \int_0^t \frac{\partial\varphi}{\partial t}\,dt_0.$$

From this it follows that the second of the initial conditions of the problem (68) is satisfied. If we differentiate the last equation with respect to t, we obtain

$$\frac{\partial^2 v}{\partial t^2} = \int_0^t \frac{\partial^2\varphi}{\partial t^2}\,dt_0 + \lim_{t_0\to t-0}\frac{\partial\varphi}{\partial t} = \int_0^t \frac{\partial^2\varphi}{\partial t^2}\,dt_0 + f(x_1, \ldots, x_4, t)$$

and, consequently,

$$\Delta v - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} = \int_0^t \left[\Delta\varphi - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2}\right] dt_0 - \frac{1}{a^2} f(x_i, t) = -\frac{f(x_i, t)}{a^2}.$$

This means that the integral (68′) is indeed a twice continuously differentiable solution of the problem (68).

The solution of the Cauchy problem (54), (54′) for the most general case with $n=4$ may be written as the sum of the right-hand member of formula (67′) and the retarded potential (68′).

Now, in analogy with the formulas giving the solution as the Cauchy problem for the one- and two-dimensional telegraph Eqs. (51) and (53), we can find a formula giving the solution of the Cauchy problem for the three-dimensional telegraph equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_3^2} - \frac{1}{a^2}\frac{\partial^2 v}{\partial t^2} + k^2 v = -\frac{f(x_1, x_2, x_3, t)}{a^2},$$
$$v\Big|_{t=0} = \psi(x_1, x_2, x_3),\ \frac{\partial v}{\partial t}\Big|_{t=0} = \psi_1(x_1, x_2, x_3). \tag{69}$$

Consider the new function $u = v(x_1, x_2, x_3, t)e^{kx_4}$, where v is a solution of the problem (69) and x_4 is an auxiliary variable. We can show by simple differentiation that the function u is a solution of the Cauchy problem for the four-dimensional wave equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f e^{kx_4}}{a^2}\left(\Delta = \sum_{i=1}^{4}\frac{\partial^2}{\partial x_i^2}\right),$$

$$u\Big|_{t=0} = \psi(x_1, x_2, x_3)\, e^{kx_4}, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x_1, x_2, x_3)\, e^{kx_4}.$$

But the solution of this last problem may, in accordance with (67′) and (68′), be written in the form

$$u(x_1, x_2, x_3, x_4, t) =$$

$$= \frac{\partial}{\partial t}\frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\left[\xi \iiint_{S_{1,x_i}} \psi(x_i + \beta_i at\sqrt{\xi}) \times\right.$$

$$\left.\times e^{k(x_4+\beta_i ta\sqrt{\xi})}\, d\sigma_4\right] d\xi + \frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}} \times$$

$$\times \frac{\partial}{\partial \xi}\left[\xi \iiint_{S_{1,x_i}} \psi_1(x_i + \beta_i ta\sqrt{\xi})\, e^{k(x_4+\beta_i ta\sqrt{\xi})}\, d\sigma_4\right] d\xi +$$

$$+ \int_0^t d\tau \frac{\tau}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\left[\xi \iiint_{S_{1,x_i}} f(x_i + \beta_i \tau a\sqrt{\xi}, t-\tau) \times\right.$$

$$\left.\times e^{k(x_4+\beta_i \tau a\sqrt{\xi})}\, d\sigma_4\right] d\xi.$$

If we divide both numbers of this equation by the common factor e^{kx_4}, we conclude that, if the functions $\psi_1(x_i)$ and $f(x_i, t)$ are three times continuously differentiable and the function $\psi(x_i)$ is four times continuously differentiable, then there is a unique twice continuously differentiable solution of the Cauchy problem for the three-dimensional telegraph equation (69) and that it is given explicitly by the equation

$$v(x_1, x_2, x_3, t) = \frac{\partial}{\partial t}\frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial \xi}\left[\xi \iiint_{S_{1,x_i}} (\psi(x_i + \beta_i ta\sqrt{\xi}) \times\right.$$

$$\left.\times e^{k\beta_i ta\sqrt{\xi}}\, d\sigma_4\right] d\xi + \frac{t}{4\pi^2}\int_0^1 \frac{1}{\sqrt{1-\xi}} \times$$

$$\times \frac{\partial}{\partial \xi}\left[\xi \iiint_{S_{1,x_i}} \psi_1(x_i + \beta_i ta\sqrt{\xi})\, e^{k\beta_i ta\sqrt{\xi}}\, d\sigma_4\right] d\xi +$$

$$+\int_0^t d\tau \frac{\tau}{4\pi^2}\int_0^\tau \frac{1}{\sqrt{1-\xi}}\frac{\partial}{\partial\xi}\left[\xi\int\int_{S_{1,x_i}}\int f(x_i+\beta_i\tau a\sqrt{\xi}, t-\tau)\times\right.$$

$$\left.\times e^{k\beta_i\tau a\sqrt{\xi}}\, d\sigma_4\right] d\xi, \qquad (69')$$

where the notations are the same as in formulas (65) and (65′). It is only necessary to understand that here the functions $\psi(x_i)$, $\psi_1(x_i)$, and $f(x_i, \tau)$ do not depend on the variable x_4.

3. The mean-value theorem. We shall now give another important consequence of the uniqueness of the solution of the Cauchy problem for the Darboux equation.

Suppose that we are given the so-called *ultrahyperbolic equation*

$$v_{x_1x_1}+v_{x_2x_2}+\ldots+v_{x_kx_k}=v_{y_1y_1}+v_{y_2y_2}+\ldots+v_{y_my_m}-cv, \qquad (70)$$

where c is a constant and $x_1, x_2, \ldots, x_k, y_1, \ldots, y_m$ are independent variables.* Obviously, we may assume without loss of generality that $c>0$. If we introduce a new auxiliary independent variable x_{n+1} and a new independent function $u=ve^{\sqrt{c}x_{k+1}}$, Eq. (70) may be written in the form

$$u_{x_1x_1}+\ldots+u_{x_{k+1}x_{k+1}}=u_{y_1y_1}+\ldots u_{y_my_m}$$

or

$$\Delta_x u=\Delta_y u,$$

where

$$\Delta_x=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} \text{ and } \Delta_y=\sum_{i=1}^{n}\frac{\partial^2}{\partial y_i^2},$$

n denoting $\max(k+1, m)$. Here, it is assumed that the function u may be independent of some of the variables x_i or y_i.

Suppose that $u(x_i, y_i)$ is a twice continuously differentiable solution of Eq. (71). Let us construct the mean values of this equation over the sphere $S_{r,\,x_i}$ of radius r in the $x_1, x_2, \ldots, x_n$ space with center at the point $(x_1, \ldots, x_n)$ and over the sphere

*Any second-order linear partial differential equation with constant coefficients can be reduced to Eq. (70) if the characteristic determinant of that equation is a non-degenerate quadratic form. For this, it is sufficient to reduce the equation to canonical form and then introduce a new unknown function by multiplying the original function by e^{λ}, where λ is a linear function of the independent variables.

S_{r, y_i} in $y_1, \ldots, y_n$-space with center at the point $(y_1, \ldots, y_n)$. These mean values take the forms

$$\mu = Q_u(x_i, r, y_i) = \frac{1}{\omega_n} \overbrace{\int \cdots \int}^{n-1}_{S_{r, x_i}} u(x_i + \beta_i r, y_i)\, d\omega_n, \tag{72}$$

$$\nu = Q_u(y_i, r, x_i) = \frac{1}{\omega_n} \overbrace{\int \cdots \int}^{n-1}_{S_{r, y_i}} u(x_i, y_i + \beta_i r)\, d\omega_n, \tag{72'}$$

respectively.

Let us show that the equation

$$Q_u(x_i, r, y_i) = Q_u(y, r, x_i) \tag{72''}$$

known as the *mean-value theorem*,* holds. By virtue of the properties of mean values that we established in subsection 2, the function (72) is a solution of the problem

$$\Delta_x \mu - \frac{\partial^2 \mu}{\partial r^2} - \frac{n-1}{r} \frac{\partial \mu}{dr} = 0, \quad \mu\Big|_{r=0} = u(x_i, y_i), \quad \frac{\partial \mu}{\partial r}\Big|_{r=0} = 0, \tag{72'''}$$

and the function (72′) is a solution of the following problem:

$$\Delta_y \nu - \frac{\partial^2 \nu}{\partial r^2} - \frac{n-1}{r} \frac{\partial \nu}{\delta r} = 0, \quad \nu\Big|_{r=0} = u(x_i, y_i), \quad \frac{\partial \nu}{\partial r}\Big|_{r=0} = 0.$$

We obtain $\Delta_x \nu = \Delta_y \nu$ by differentiating Eq. (72′) under the integral sign and taking account of Eq. (71). Therefore, the function ν is also a solution of the Cauchy problem (72‴). But the solution of this problem for the Darboux equation is unique and, hence, Eq. (72″) is valid.

This mean-value theorem has quite a number of applications. Suppose, for example, that u does not depend on the variables y_i. Then, Eq. (71) becomes Laplace's equation

$$Q_u(y_i, r, x_i) = u(x_1, \ldots, x_n)$$

and Eq. (72″) coincides with the theorem on the arithmetic mean for harmonic functions. Suppose, for example, that the function u does not depend on any of the y_i except y_1. Then,** and

*Due to L. Asgeirsson. (Ed.)

**See footnote to page 295.

$$Q_u(y_i, r, x_i) = \frac{1}{\sigma_n} \int\cdots\int_{S_{1,y_i}} u(x_i, y_1 + \beta_i r)\, d\sigma_n =$$

$$= \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} (1-\alpha^2)^{\frac{n-3}{2}}\, u(x_i, y_1 + r\alpha)\, d\alpha$$

Eq. (72″), for $y_1 = 0$, takes the form

$$\frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} u(x_1, \ldots, x_n, r\alpha)\,(1-\alpha^2)^{\frac{n-3}{2}}\, d\alpha =$$

$$= \frac{1}{\sigma_n} \int\cdots\int_{S_{1,x_i}} u(x_i + \beta_i r, 0)\, d\sigma_n.$$

If the function $u(x_1, \ldots, x_n, y_1)$ is an even function of the variable y_1, we see that the last equation coincides with the integral equation (59) for $a=1$, from which we obtained the solution of the Cauchy problem for the wave equation when the number of space coordinates exceeds 3.

9. General Linear Hyperbolic Equations With Two Independent Variables. The Unit Impulse Function

1. **Adjoint differential operators.** Every linear hyperbolic second-order equation with two independent variables can, after being reduced to canonical form, be written in the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} + cu = -f(x, t), \tag{73}$$

where a, b, c and f are given functions of the independent variables x and t.

In the study of problems associated with Eq. (73) as in the study of problems associated with linear second-order differential equations, in the most general case

$$\sum_{i,k=1}^{n+1} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n+1} b_i \frac{\partial u}{\partial x_i} + cu = -f, \tag{74}$$

where the $a_{ik} = a_{ki}$, the b_i, and f are functions of the independent variables, a very important tool is provided by integral formulas establishing the relationship between integrals over a domain

and over a boundary of that domain. The left-hand member of Eq. (74)

$$L(u) = \sum_{i,k=1}^{n+1} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n+1} b_i \frac{\partial u}{\partial x_i} + cu \tag{75}$$

is usually called a second-order linear differential operator, and a differential operator of the following form:

$$M(v) = \sum_{i,k=1}^{n+1} \frac{\partial^2}{\partial x_i \partial x_k}(a_{ik}v) - \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}(b_i v) + cv \tag{75'}$$

is called the *adjoint of the operator* $L(u)$ *(in the sense of Lagrange).*

It is easy to see that, for arbitrary twice continuously differentiable functions u and v,

$$vL(u) - uM(v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(v \sum_{k=1}^{n+1} a_{ik} \frac{\partial u}{\partial x_k} - u \sum_{k=1}^{n+1} \frac{\partial}{\partial x_k}(a_{ik}v) + b_i uv\right), \tag{76}$$

that is, the expression $vL(u) - uM(v)$ is the divergence of some well-defined vector. This property of adjoint operators is the defining property in the introduction of the very concept of adjoint operators. By direct differentiation, we can check that the property of adjointness of operators is symmetric; i.e., if $M(v)$ is the adjoint of $L(u)$, then $L(u)$ is the adjoint of $M(v)$. If $L(u) \equiv M(u)$, the operator $L(u)$ is said to be *self-adjoint.* For example, the Laplace operator $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$ is self-adjoint. The same is true of the wave operator $\Box u = \sum_{i=1}^{n} u_{x_i x_i} - \frac{1}{a^2} u_{tt}$. However, the heat equation operator $\sum_{i=1}^{n} u_{x_i x_i} - \frac{1}{a^2} u_t$ is not self-adjoint since the adjoint of this operator is an operator of the form $\sum_{i=1}^{n} u_{x_i x_i} + \frac{1}{a^2} u_t$.

The identity (76) makes it possible to apply Ostrogradskiy's formula to the expression $vL(u) - uM(v)$. If Ω^* is an $(n+1)$-dimensional domain in $x_1, x_2, \ldots, x_n, t$-space bounded by the surface S^*, and $\vec{n}^*$ is the inner normal to S^*, then application of Ostrogradskiy's formula to the identity (76) yields the following integral formula:

$$\int\cdots\int_{\Omega^*} (vL(u) - uM(v))\, dx_1 \ldots dx_{n+1} = -\overbrace{\int\cdots\int}^{n}_{S^*} \sum_{i=1}^{n+1} \left[v \sum_{k=1}^{n+1} a_{ik} \frac{\partial u}{\partial x_k} - u \sum_{k=1}^{n+1} \frac{\partial}{\partial x_k}(a_{ik}v) + b_i uv \right] \cos n^* x_i \, dS, \tag{77}$$

where dS is an element of area of the boundary S^* of the domain Ω^*.

Equation (77) is the general form of Ostrogradskiy's formula for second-order adjoint differential operators.

2. The fundamental integral formula. The Riemann function. If

$$L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} + cu, \tag{78}$$

$$M(v) = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial t}(bv) + cv, \tag{78'}$$

formula (77) takes the form*

$$\iint\limits_{\Omega^*} (vL(u) - uM(v))\,dx\,dt = -\int\limits_{S^*} \left\{\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} + auv\right)\cos n^*x + \right.$$
$$\left. + \left(-v\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial t} + buv\right)\cos n^*t\right\} ds, \tag{77'}$$

where Ω^* is a plane domain, S^* is the curve bounding it, ds is an element of arclength of the curve S^*, and $\vec{n}^*$ is the inner normal to S^*. If we introduce the direction of the conormal $\vec{N}$, i.e., the direction symmetric with $\vec{n}^*$ about the plane $t=0$, so that

$$\cos n^*x = \cos Nx, \quad -\cos n^*t = \cos Nt,$$

then formula (77') takes the form

$$\iint\limits_{\Omega^*} (vL(u) - uM(v))\,dx\,dt = -\int\limits_{S^*} \left[v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} + \right.$$
$$\left. + uv(a\cos n^*x + b\cos n^*t)\right] ds \tag{79}$$

It is easy to see that formula (79) coincides with formula (30) (see Sect. 6, subsection 2) with two independent variables when

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*This formula can also be established directly from Green's formula

$$\iint\limits_{\Omega^*} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dt = -\int\limits_{S^*} (P\cos n^*x + Q\cos n^*t)\,ds.$$

by setting

$$P = v\left[\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} + auv,\ Q = -v\frac{\partial u}{\partial t} + u\frac{\partial v}{\partial t} + buv.\right.$$

$a=b=0$, as we would expect. Therefore, it is natural to try to apply formula (79) to the solution of boundary-value problems for Eq. (73) analogously to the way in which formula (30) was applied to the solution of the Cauchy problem and the characteristic problem for the one-dimensional wave equation.

Suppose that Ω^* is a domain in the xt-plane satisfying condition α (see Sect. 6, subsection 2). Suppose that Ω_Q^* is that portion of the domain Ω^* bounded by the characteristics QQ_1 and QQ_2 issuing from the point $Q \in \Omega^*$ and by the curve C_Q which forms part of the boundary of the domain Ω^* with end points Q_1 and Q_2. For example, in Fig. 28a, the domain Ω_Q^* is represented in the form of the triangle Q_1QQ_2 with curvilinear base Q_1Q_2; in Fig. 28d, it is represented in the form of the rectangle Q_1QQ_2B.

Keeping in mind the property of the conormal $\vec{N}$, we can rewrite formula (79) as applied to the domain Ω_Q^* shown in Fig. 28a, b, c, and d (here, we do not consider the case shown in Fig. 28e) in the form

$$\iint\limits_{\Omega_Q^*} (vL(u) - uM(v))\,dx\,dt = -\int\limits_{Q_1}^{Q} \left[v\frac{\partial u}{\partial s} - u\frac{\partial v}{\partial s} + \frac{uv}{\sqrt{2}}(a-b)\right] ds -$$

$$-\int\limits_{Q_2}^{Q} \left[v\frac{\partial u}{\partial s} - u\frac{\partial v}{\partial s} - \frac{uv}{\sqrt{2}}(a+b)\right] ds - \int\limits_{C_Q} \Big(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} + \tag{79'}$$

$$+ uv\,(a\cos n^*x + b\cos n^*t)\Big]\,ds.$$

Here, if we integrate $v\frac{\partial u}{\partial s}$ by parts and transpose the value of uv at the point Q to the left-hand member of the equation, we obtain

$$2(uv)_Q = \int\limits_{C_Q} \left[u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N} - uv\,(a\cos n^*x + b\cos n^*t)\right] ds -$$

$$-\iint\limits_{\Omega_Q^*} (vL(u) - uM(v))\,dx\,dt + (uv)_{Q_1} + (uv)_{Q_2} +$$

$$+\int\limits_{Q_1}^{Q} u\left[2\frac{\partial v}{\partial s} - v\frac{a-b}{\sqrt{2}}\right] ds + \int\limits_{Q_2}^{Q} u\left[2\frac{\partial v}{\partial s} + v\frac{a+b}{\sqrt{2}}\right] ds. \tag{79''}$$

In what follows, we will say that characteristics belong to the first family if they make an angle $\frac{\pi}{4}$ with the x-axis, and to the

second family if they make an angle $\frac{3\pi}{4}$ with the x-axis. By characteristic angle with vertex at the point $Q(x, t)$, we mean the region bounded by the characteristics issuing from the point $Q(x, t)$ in the direction of decreasing t.

Let us now introduce an auxiliary function.

We shall refer to the function $v(x, t, x', t')$, where $(t>t')$, which depends on the point $Q(x, t)$ as a parameter, as the *Riemann function* of the operator $L(u)$ if, as a function of x' and t', it possesses the following three properties:

(a) in the characteristic angle with vertex at the point $Q(x, t)$, it is a twice continuously differentiable solution of the adjoint homogeneous equation

$$M_{x't'}(v) = \frac{\partial^2 v}{\partial x'^2} - \frac{\partial^2 v}{\partial t'^2} - (av)_{x'} - (bv)_{t'} + cv = 0;$$

(b) on the boundary of this characteristic angle, we have $\frac{\partial v}{\partial s} = \frac{a-b}{2\sqrt{2}} v$ on the characteristic of the first family and $\frac{\partial v}{\partial s} = -\frac{a+b}{2\sqrt{2}} v$ on the characteristic of the second family, where $\frac{\partial}{\partial s}$ denotes differentiation along the characteristics in the upward direction;

(c) $v(x, t, x, t) = 1$.

Now, let us replace the variables of integration x and t in formula (79″) with x' and t' and let us denote the coordinates of the fixed point Q by x and t. Then, if we take the Riemann function just introduced $v(x, t, x', t')$ as v, formula (79″) becomes

$$u(x, t) = \frac{(uv)_{Q_1} + (uv)_{Q_2}}{2} + \frac{1}{2} \int_{C_Q} \left[u \frac{\partial v}{\partial N} - v \frac{\partial u}{\partial N} - uv (a \cos n^* x + b \cos n^* t) \right] ds - \frac{1}{2} \iint_{\Omega_Q} v(x, t, x', t') L_{x't'}(u) \, dx' \, dt'. \tag{80}$$

This formula is known as *Riemann's formula*. It is the fundamental integral formula for solutions of linear hyperbolic equations with two independent variables.

Let us observe what this integral formula implies when applied to twice continuously differentiable solutions of Eq. (73) when the Riemann function of the operator $L(u)$ is known. As in Sect. 6, subsection 2, let us suppose that C_1 is the noncharacteristic portion of the curve C, i.e., the portion not containing segments of the characteristics, and that C_2 is the characteristic

portion of the curve C, i.e., the portion consisting of segments of the characteristics. If a solution u is given on C_2, this gives the conormal derivative $\frac{\partial u}{\partial N}$ on C_2. Thus, we draw the following conclusion.

Formula (80) gives an explicit representation for the solution of Eq. (73) in the domain Ω^* if its values are given on C_2 and if its values and the values of its conormal derivatives are given on C_1.

For example, suppose that the curve C shown in Fig. 28a is such that it is intersected by each characteristic at only one point (for this it is necessary and sufficient that $\left|\frac{dt(x)}{dx}\right|$ be less than 1, where $t=t(x)$ is the equation of the curve C). Then, formula (80) yields the solution of Eq. (73) explicitly in the domain Ω^*, which lies above the curve C, if the values of the solution and the values of its conormal or normal derivative are given on C. Sometimes one also says that in this case formula (8) yields the solution of the *generalized Cauchy problem* for Eq. (73).

If the curve C consists of two characteristis issuing from a single point B (see Fig. 28d), then formula (80) yields an explicit form for the solution $u(x, t)$ of Eq. (73) in the domain Ω^* bounded by these characteristics when the values of the solution are known on these characteristics. In other words, we can say that formula (80) yields the solution of the characteristic problem for Eq. (73) for the domain Ω^* described above.

One can easily note, that in the case in which Eq. (73) degenerates into the string equation, i.e., when $a=b=c=0$, in the characteristic angle with vertex at the point $Q(x, t)$ we have $v(x, t, x', t') \equiv 1$, and all our conclusions drawn from formula (80) coincide with the results that were obtained for the wave equation from formula (31).

Above, we defined the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x' and t' (there $t>t'$). To make clear the behavior of $v(x, t, x', t')$ as a function of x and t, we introduce first the Riemann function $\tilde{v}(x', t', x, t)$ (where $t>t'$) of the operator $M(v)$, which is the adjoint of the operator $L(u)$. By definition $\tilde{v}(x', t', x, t)$, as a function of x and t, must possess the following properties:

(a′) $L(\tilde{v})=0$ in the upper characteristic angle, i.e., in the angle bounded by the characteristics issuing from the point $Q'(x', t')$ in the direction of increasing t;

(b′) on the boundary of this angle $\frac{\partial \tilde{v}}{\partial s} = -\frac{a-b}{2\sqrt{2}}\tilde{v}$ on the characteristic of the first family and $\frac{\partial \tilde{v}}{\partial s} = \frac{a+b}{2\sqrt{2}}\tilde{v}$ on the

characteristic of the second family, where $\frac{\partial}{\partial s}$ denotes the differentiation along the characteristics upward.

(c′) $\tilde{v}(x', t', x', t') = 1$.

Let us establish a connection between the functions $v(x, t, x', t')$ and $\tilde{v}(x', t', x, t)$. For this we set

$$u(x^\circ, t^\circ) = \tilde{v}(x', t', x^\circ, t^\circ)$$

and apply formula (80) to the evaluation of

$$u(x, t) = \tilde{v}(x', t', x^\circ, t^\circ)\Big|_{\substack{x^\circ = x \\ t^\circ = t}},$$

where x' and t' are the coordinates of the point B shown in Fig. 28d. Recalling the properties of the conormal, we can write formula (80) in the form

$$u(x, t) = \frac{(uv)_{Q_1} + (uv)_{Q_2}}{2} + \frac{1}{2}\int_B^{Q_1}\left[-u\frac{\partial v}{\partial s} + v\frac{\partial u}{\partial s} - \right.$$

$$\left. - \frac{uv}{\sqrt{2}}(a+b)\right]ds + \frac{1}{2}\int_B^{Q_2}\left[-u\frac{\partial v}{\partial s} + v\frac{\partial u}{\partial s} - \frac{uv}{\sqrt{2}}(-a+b)\right]ds -$$

$$- \frac{1}{2}\iint_{\Omega_Q^*} v(x, t, x^\circ, t^\circ) L_{x^\circ t^\circ}(u)\, dx^\circ\, dt^\circ. \qquad (80')$$

If we now integrate the expression for $u\frac{\partial u}{\partial s}$ by parts and make the substitution $u = \tilde{v}(x', t', x^\circ, t^\circ)$, we obtain

$$\tilde{v}(x', t', x, t) = [\tilde{v}(x, t, x^\circ, t^\circ)\tilde{v}(x', t', x^\circ, t^\circ)\Big|_{\substack{x^\circ = x' \\ t^\circ = t'}} +$$

$$+ \frac{1}{2}\int_B^{Q_1} v\left[2\frac{\partial\tilde{v}}{\partial s} - v\frac{a+b}{\sqrt{2}}\right]ds + \frac{1}{2}\int_B^{Q_2} v\left[2\frac{\partial\tilde{v}}{\partial s} + \tilde{v}\frac{a-b}{\sqrt{2}}\right]ds.$$

But the last two integrals in the right-hand member of this equation vanish by virtue of the definition of the function $\tilde{v}(x', t', x^\circ, t^\circ)$. Thus, we arrive at the so-called *law of symmetry* of the Riemann functions of two adjoint operators:

$$v(x, t, x', t') = \tilde{v}(x', t', x, t) \qquad (t > t'). \qquad (81)$$

It follows from this that the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x and t possesses the same properties as does $\tilde{v}(x', t', x, t)$, i.e., the properties of (a'), (b'), and (c').

We can give a simple physical interpretation of the Riemann function in the general case. By virtue of formula (80), the solution of the Cauchy problem for Eq. (73) with homogeneous initial conditions given for $t=0$ may be written in the form

$$u(x, t) = \frac{1}{2} \iint\limits_{\Omega_Q^*} v(x, t, x^*, t^*) f(x^*, t^*)\, dx^*\, dt^*,$$

where Ω_Q^* is a domain having the form of a rectilinear triangle Q_1QQ_2 (see Fig. 28a). Let us suppose that the function $f(x, t)$ is different from zero only in some neighborhood S_ε of the point (x', t') of the form $|x-x'|<\varepsilon, 0<t-t'<\varepsilon)$, that it is positive in this neighborhood, and that it satisfies the condition

$$\frac{1}{2} \iint\limits_{S_\varepsilon} f(x, t)\, dx\, dt = 1.$$

Then, the preceding equation takes the form

$$u_\varepsilon(x, t) = \iint\limits_{S_\varepsilon} v(x, t, x^*, t^*) \frac{f(x^*, t^*)}{2}\, dx^*\, dt^* =$$

$$= v(x, t, x^\circ, t^\circ) \frac{1}{2} \iint\limits_{S_\varepsilon} f(x^*, t^*)\, dx^*\, dt^*,$$

where (x°, t°) is a point lying in the neighborhood of the point (x', t'). If we take the limit as we shrink the neighborhood S_ε to the point (x', t'), we obtain

$$u(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t) = v(x, t, x', t').$$

In a number of physical problems, the function $f(x, t)/2$ represents, as we saw earlier, the density of a body force up to some constant of proportionality. Therefore, the expression

$$\frac{1}{2} \iint\limits_{S_\varepsilon} f(x^*, t^*)\, dx^*\, dt^* = \frac{1}{2} \int\limits_{t'}^{t'+\varepsilon} \int\limits_{x'-\varepsilon}^{x'+\varepsilon} f(x^*, t^*)\, dx^*\, dt^*$$

represents, up to a constant factor, the impulse of a force applied to the segment $[x'-\varepsilon, x'+\varepsilon]$ at the instant t'. Thus, the

Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x and t represents the disturbance at the point x at the instant t that is caused by a concentrated impulse of a force applied at the point x' at an instant $t'<t$. For example, if

$$L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2},$$

then the Riemann function $v(x, t, x', t')$ as a function of x and t is equal to unity in the upper characteristic angle with vertex at the point $Q'(x', t')$ and is equal to 0 outside that angle. We have previously interpreted this solution as the result of a concentrated impulse equal to $2a\rho$, applied to a string at rest at the point x' at the instant t' (see Sect. 4).

Thus, in the general case, we can call the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ the function due to a concentrated impulse of the operator $L(u)$ at the point x' at the instant t' or, up to a constant factor, the unit impulse function of the operator $L(u)$ due to a unit impulse at the point x' at the instant t'.

3. The existence and uniqueness of solutions of the Cauchy problem. Solutions of the characteristic problem and Riemann functions. Let us turn now to the questions of existence and uniqueness of solutions of the Cauchy problem and of the characteristic problem for Eq. (73). In the process we shall also answer the question of the existence and uniqueness of the Riemann function $v(x, t, x', t')$ since, by definition, as a function of x and t it constitutes the solution of a well-determined characteristic problem for Eq. (73) for the upper characteristic angle with vertex at the point $Q'(x', t')$ and as a function of x' and t' it constitutes the solution of the characteristics problem for the equation adjoint to Eq. (73) for the characteristics angle with vertex at the point $Q(x, t)$.

As a preliminary, let us transform Eq. (73) to an equation in new variables ξ and η, taking for the direction of the coordinate axes ξ and η the directions of the characteristics of the first and second family, respectively. Obviously, to do this we only need to set

$$\xi + i\eta = (x + it)\, e^{-i\frac{\pi}{4}}, \quad \xi = \frac{x+t}{\sqrt{2}}, \quad \eta = \frac{t-x}{\sqrt{2}}. \tag{73'}$$

Equation (73) then takes the form

$$L(u) = -2\frac{\partial^2 u}{\partial \xi\, \partial \eta} + \frac{a+b}{\sqrt{2}}\frac{\partial u}{\partial \xi} + \frac{b-a}{\sqrt{2}}\frac{\partial u}{\partial \eta} + cu = -f. \tag{73''}$$

The curve C for which the Cauchy problem was posed in the xt-plane (see Fig. 28a), occupies, after a rotation by an amount

$-\pi/4$, the position in the $\xi\eta$-plane shown in Fig. 32a. Every straight line parallel to either of the coordinate axes ξ or η intersects the curve C at one point. The Cauchy problem posed above for Eq. (73) is now transformed into the problem of finding a twice continuously differentiable solution of Eq. (73″) in the region lying to the right of and above the curve C with the condition that the values of the solution and its normal derivative are given on C. The characteristic problem for Eq. (73) for the upper characteristic angle and the definition of a unit impulse function $v(x, t, x', t')$ as a function of x and t reduces to solving the characteristic problem for Eq. (73″) for the "upper right" characteristic angle in the $\xi\eta$-plane, which is the region bounded by two rectilinear rays issuing from a single point in the positive direction of the ξ-axis and the positive direction of the η-axis, respectively (see Fig. 32b). Thus, the questions of the existence

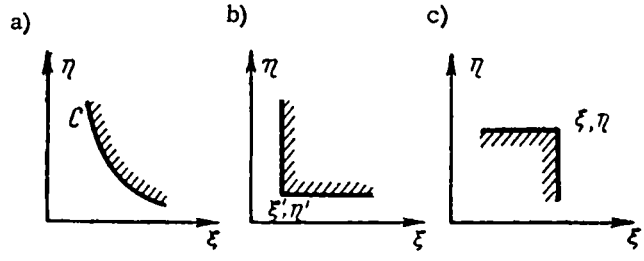

Fig. 32.

and uniqueness of a solution of the Cauchy problem, the solutions of the characteristic problem, and the Riemann functions for Eq. (73) reduce fully and completely to these same problems for Eq. (73″). Therefore, we shall look at these problems immediately in connection with Eq. (73″). If we denote the variables ξ and η by x and y, respectively, for convenience, we may write Eq. (73″) in the form

$$L^*(u) = \frac{\partial^2 u}{\partial x\,\partial y} + A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = -F(x, y), \tag{73‴}$$

where $A(x, y,)$ $B(x, y)$, $C(x, y)$, and $F(x, y)$ are given functions of x and y, which we shall assume in what follows to be continuous. Under these hypotheses regarding the coefficients, let us look at the Cauchy problem for Eq. (73‴) for the region lying to the right of the curve C (see Fig. 33a) and the characteristic problem for the upper right characteristic angle (Fig. 33b). Without loss of generality, we may assume that the initial values of u and $\partial u/\partial n^*$ given on the curve C in the case of the Cauchy problem and the initial values of u given on the characteristic in the case of the characteristic problem are homogeneous.

(In the opposite case, we would only need to subtract from the desired solution some twice continuously differentiable function satisfying the given nonhomogeneous initial conditions, as a result of which we would arrive at the corresponding problems with homogeneous initial conditions and possibly with a different right-hand member of the equation.)

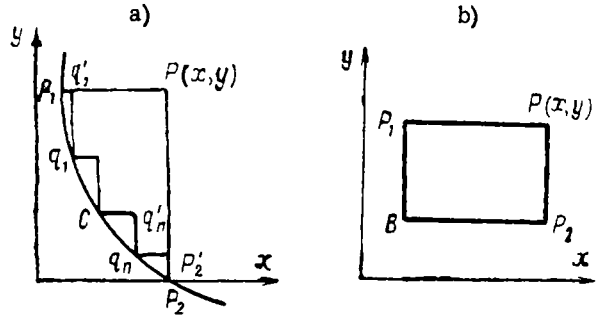

Fig. 33.

Let us now suppose that we have determined the functions $u(x, y)$, $\lambda(x, y)$, and $\mu(x, y)$ from the system of integral equations

$$\lambda(x, y) = -\int_{P_2}^{P} [F(x, \eta) + A(x, \eta)\lambda(x, \eta) + B(x, \eta)\mu(x, \eta) + C(x, \eta)u(x, \eta)]\, d\eta, \tag{82}$$

$$\mu(x, y) = -\int_{P_1}^{P} [F(\xi, y) + A(\xi, y)\lambda(\xi, y) + B(\xi, y)\mu(\xi, y) + C(\xi, y)u(\xi, y)]\, d\xi, \tag{82'}$$

$$u(x, y) = -\iint_{\Omega_P^*} [F(\xi, \eta) + A(\xi, \eta)\lambda(\xi, \eta) + B(\xi, \eta)\mu(\xi, \eta) + C(\xi, \eta)u(\xi, \eta)]\, d\xi\, d\eta, \tag{82''}$$

where P is the point with coordinates x and y in the region in which we are seeking the solution of Eq. (73‴) and where Ω_P^* is the curvilinear triangle PP_1P_2 in the case of the Cauchy problem (see Fig. 33a) and the rectangle PP_1BP_2 in the case of the characteristic problem (see Fig. 33b), and where P_1 and P_2 are the points shown in Figs. 33a and 33b, respectively. If we differentiate Eq. (82″) with respect to x and then with respect to y, we obtain

$$\frac{\partial u(x, y)}{\partial x} = -\int_{P_2}^{P} [F(x, \eta) + A(x, \eta)\lambda(x, \eta) + B(x, \eta)\mu(x, \eta) + C(x, \eta)u(x, \eta)]\,d\eta = \lambda(x, y),$$

$$\frac{\partial u(x, y)}{\partial y} = -\int_{P_1}^{P} [F(\xi, y) + A(\xi, y)\lambda(\xi, y) + B(\xi, y)\mu(\xi, y) + C(\xi, y)u(\xi, y)]\,d\xi = \mu(x, y),$$

$$\frac{\partial^2 u(x, y)}{\partial x\,\partial y} = -\{F(x, y) + A(x, y)\lambda(x, y) + B(x, y)\mu(x, y) + C(x, y)u(x, y)\}.$$

This means that if the functions $u(x, y)$, $\lambda(x, y)$ and $\mu(x, y)$ constitute a solution of the system of integral equations (82), (82′), (82″), then the function $u(x, y)$ is a solution of Eq. (73‴). Here, as we can see from Eq. (82″), this solution vanishes on the curve C in the case of the Cauchy problem (see Fig. 33a) and on the characteristics issuing from the point B in the case of the characterisitic problem (see Fig. 33b). Furthermore, on the basis of the equations

$$\frac{\partial u(x, y)}{\partial x} = \lambda(x, y), \quad \frac{\partial u(x, y)}{\partial y} = \mu(x, y)$$

in the case of the Cauchy problem, the derivatives $\partial u/\partial x$ and $\partial u/\partial y$ vanish on the curve C because, when the point P lies on the curve C, the points P_1 and P_2 merge with it. Thus, we conclude that, if the function $u(x, y)$ is determined from the system of integral equations (82), (82′), (82″), it will be the desired twice continuously differentiable solution of the Cauchy problem for Eq. (73‴) if the region of integration is the curvilinear triangle PP_1P_2 (see Fig. 33a) and will be a twice continuously differentiable solution of the characteristic problem if the region of integration is the rectangle PP_1BP_2 (see Fig. 33b). The converse also holds: if $u(x, y)$ is a twice continuously differentiable solution of the Cauchy problem or the characteristic problem for Eq. (73‴) with homogeneous initial conditions, then the functions

$$u(x, y), \quad \lambda(x, y) = \frac{\partial u(x, y)}{\partial x}, \quad \mu(x, y) = \frac{\partial u(x, y)}{\partial y}$$

satisfy the system of integral equations (82), (82′), (82″). To see this, one need only transpose all the terms except the first in Eq. (73‴) to the right side of the equation and then, remembering the homogeneous initial conditions, integrate both sides of the equation with respect to y [to obtain (82)], with respect to x

[to obtain (82′)], and finally with respect to x and y [to obtain (82″)].

Thus, we have shown that the solution of the Cauchy problem and the solution of the characteristic problem for Eq. (73‴) can be reduced to solving the system of integral equations (82), (82′), (82″). To solve this system of integral equations, we use the method of successive approximations. Let us take three continuous functions $u_0(x, y)$, $\lambda_0(x, y)$, and $\mu_0(x, y)$ and let us substitute them into the right-hand members of Eqs. (82), (82′), (82″), respectively, for $u(x, y)$, $\lambda(x, y)$, and $\mu(x, y)$. In the left-hand members of these equations, we shall then have not $u(x, y)$, $\lambda(x, y)$, and $\mu(x, y)$ but three continuous functions $u_1(x, y)$, $\lambda_1(x, y)$ and $\mu_1(x, y)$. If we do the same with these functions as we did with u_0, λ_0, and μ_0, we obtain three new functions $u_2(x, y)$, $\lambda_2(x, y)$, and $\mu_2(x, y)$. If we continue this process indefinitely, we obtain a sequence of sets of functions

$$u_n(x, y),\ \lambda_n(x, y),\ \mu_n(x, y), \qquad (n = 0, 1, \ldots).$$

All these functions are continuous. Since the coefficients, A, B, and C are continuous, there exists a number M such that $|A| + |B| + |C| < M$. Clearly, for $n = 1$, we have the inequalities

$$\begin{gathered} |u_n(x, y) - u_{n-1}(x, y)|, \quad |\lambda_n(x, y) - \lambda_{n-1}(x, y)|, \\ |\mu_n(x, y) - \mu_{n-1}(x, y)| < D(MA)^{n-1} \frac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!}, \end{gathered} \tag{82‴}$$

where D is some sufficiently large positive constant, x_0 and y_0 are the coordinates of a point lying to the left of and below points in the domain Ω_P, and A is a constant equal to the greater of the two numbers $x - x_0$, $y - y_0$. Let us show that inequalities (82‴) hold for arbitrary n. Assuming that they hold for n, we obtain

$$|u_{n+1}(x, y) - u_n(x, y)| < \iint\limits_{\Omega_P} (|A| + |B| + |C|) D(AM)^{n-1} \times$$

$$\times \frac{(\xi + \eta - x_0 - y_0)^{n-1}}{(n-1)!} d\xi\, d\eta \leqslant D(MA)^n \int\limits_{x_0}^{x} \frac{(\xi + \eta - x_0 - y_0)^{n-1}}{(n-1)!} d\xi <$$

$$< D(MA)^n \frac{(x + y - x_0 - y_0)^n}{n!},$$

$$|\lambda_{n+1}(x, y) - \lambda_n(x, y)| < \int\limits_{y_0}^{y} MD(MA)^{n-1} \frac{(x + \eta - x_0 - y_0)^{n-1}}{(n-1)!} d\eta <$$

$$< D(MA)^n \frac{(x + y - x_0 - y_0)^n}{n!},$$

$$|\mu_{n+1}(x, y) - \mu_n(x, y)| < \int_{x_0}^{x} MD(MA)^{n-1} \frac{(\xi + y - x_0 - y_0)^{n-1}}{(n-1)!} d\xi <$$

$$< D(MA)^n \frac{(x + y - x_0 - y_0)^n}{n!}.$$

This means that inequalities (82‴) hold for $(n + 1)$ and, hence, are valid in general. Now we may assert that the series

$$u_0 + \sum_1^\infty (u_n - u_{n-1}), \quad \lambda_0 + \sum_1^\infty (\lambda_n - \lambda_{n-1}),$$
$$\mu_0 + \sum_1^\infty (\mu_n - \mu_{n-1}) \tag{83}$$

are majorized by the convergent series

$$E + \sum_{n=1}^{\infty} D(MA)^{n-1} \frac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!},$$

where E is some constant such that $|u_0|, |\lambda_0|, |\mu_0| < E$. Consequently, the series (83), and therefore u_n, λ_n, and μ_n also, converge uniformly to certain continuous functions, which we shall note, respectively, by $u(x, y)$, $\lambda(x, y)$, and $\mu(x, y)$. Since the convergence is uniform, in the recursion relations defining u_n, λ_n, and μ_n in terms of u_{n-1}, λ_{n-1}, and u_{n-1}, we may take the limit under the various integral signs.

Thus, we conclude that the functions $u(x, y)$, $\lambda(x, y)$, and $\mu(x, y)$ to which the successive approximations u_n, λ_n, and μ_n converge uniformly constitute a continuous solution of the system of integral equations (82), (82′), (82″).

Let us show that this solution is unique. Suppose that $u^*(x, y)$, $\lambda^*(x, y)$, and $\mu^*(x, y)$ constitutes a second solution of the system (82), 82′), (82″). Let us denote by $u^0(x, y)$, $\lambda^0(x, y)$, and $\mu^0(x, y)$ the differences $u(x, y) - u^*(x, y)$, $\lambda(x, y) - \lambda^*(x, y)$, and $\mu(x, y) - \mu^*(x, y)$. Then, u^0, λ^0, and μ^0 will be a solution of the system (82), (82′), (82″) but only for $F \equiv 0$. Let us take u^0, λ^0, and μ^0 as the initial approximation in the solution of our system of equations by the method of successive approximations. For arbitrary n, we obtain $u_n = u^0$, $\lambda_n = \lambda^0$, and $\mu_n = \mu^0$. On the other hand, by virtue of the calculations made above, the following inequalities hold for u_n, λ_n, and μ_n:

$$|u_n|, |\lambda_n|, |\mu_n| < D(MA)^{n-1} \frac{(x + y - x_0 - y_0)^{n-1}}{(n-1)!}.$$

Since n may be taken arbitrarily large, it follows from these inequalities that $u^0=\lambda^0=\mu^0\equiv 0$, so that the solution of the system of equations (82), (82′), (82″) is unique. This proves the existence and uniqueness of the solution of the Cauchy problem and the solution of the characteristic problem for Eq. (73‴). In particular, it proves the existence and uniqueness of the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x and t. All this is done under the assumption that the coefficients of Eq. (73‴) or, what amounts to the same thing, of Eq. (73) are continuous. It should be noted that, for the existence and uniqueness of the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x' and t' (but not as a function of x and t), it is sufficient to require, on the basis of the results just given, that the coefficients a and b in the expression for the operator $L(u)$ be continuously differentiable and that the coefficient c be continuous. This is quite obvious since this Riemann function as a function of x' and t' is defined as the solution of the characteristic problem for the equation $M(v)=0$ and the coefficients of the operator $M(v)$, which is adjoint to the operator $L(u)$ are, in accordance with (79′), expressed in terms of the derivatives of the coefficients of the operator $L(u)$.

4. Construction of the Riemann functions for the telegraph equation and for the Euler-Poisson equation. Let us now look at some examples of the construction of the Riemann function for the operator $L(u)$ defined by Eq. (78). Here, as in the investigation of the question of the existence and uniqueness of a solution to the Cauchy problem and the characteristic problem, a transformation of the operator $L(u)$ and its adjoint operator $M(v)$ is often useful. We make the change of independent variables

$$\xi+i\eta=(x+it)\,e^{-i\frac{\pi}{4}},\quad \xi=\frac{x+t}{\sqrt{2}},\quad \eta=\frac{t-x}{\sqrt{2}}. \tag{73′}$$

In these variables, the operator $L(u)$ becomes

$$L_{\xi\eta}(u)=-2\frac{\partial^2 u}{\partial\xi\,\partial\eta}+\frac{a+b}{\sqrt{2}}\frac{\partial u}{\partial\xi}+\frac{b-a}{\sqrt{2}}\frac{\partial u}{\partial\eta}+cu, \tag{84}$$

and, by virtue of (78′), the operator $M(v)$, after some simple calculations, becomes

$$M_{\xi\eta}(v)=-2\frac{\partial^2 v}{\partial\xi\,\partial\eta}-\frac{\partial}{\partial\xi}\left(\frac{a+b}{\sqrt{2}}v\right)-\frac{\partial}{\partial\eta}\left(\frac{b-a}{\sqrt{2}}v\right)+cv. \tag{84′}$$

From this it is clear that the operator (84′) will be adjoint to the operator (84); i.e., the property of adjointness of the operators $L(u)$ and $M(v)$ is invariant under the change of variables (73′). As a result of this change of variables, the characteristic angle with

vertex at the point $x + iy$ is moved into the "lower left" characteristic angle in the $\xi\eta$-plane at the point $\xi+i\eta=(x+iy)e^{-i\frac{\pi}{4}}$, i.e., into the region bounded by the rectilinear rays issuing from the last-mentioned point in the negative direction of the ξ-axis and in the negative direction of the η-axis, respectively (see Fig. 32c). The upper characteristic angle with vertex at the point $x'+iy'$ is moved into the upper right characteristic angle with vertex at the point $\xi'+i\eta'=(x'+iy')e^{-i\frac{\pi}{4}}$ in the $\xi\eta$-plane (see Fig. 32b).

Therefore, to determine the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x' and t', we need only to require that it satisfy the following conditions:

(A) $M_{\xi'\eta'}(v)=0$ in the lower left characteristic angle with vertex at the point $\xi+i\eta=(x+iy)e^{-i\frac{\pi}{4}}$;

(B) $\frac{\partial v}{\partial\xi'}=\frac{a-b}{2\sqrt{2}}v$ for $\eta'=\eta$ and $\frac{\partial v}{\partial\eta'}=-\frac{a+b}{2\sqrt{2}}v$ for $\xi'=\xi$;

(C) $v=1$ for $\xi'=\xi$, $\eta'=\eta$.

To determine the Riemann function $v(x, t, x', t')$ of the operator $L(u)$ as a function of x and t, it is sufficient to require that it satisfy the following conditions:

(A') $L_{\xi\eta}(v)=0$ in the upper right characteristic angle with vertex at the point $\xi'+i\eta'=(x'+iy')e^{-i\frac{\pi}{4}}$;

(B′) $\frac{\partial v}{\partial\xi}=-\frac{a-b}{2\sqrt{2}}v$ for $\eta=\eta'$ and $\frac{\partial v}{\partial\eta}=\frac{a+b}{2\sqrt{2}}v$ for $\xi=\xi'$;

(C′) $v=1$ for $\xi=\xi'$, $\eta=\eta'$.

Suppose, for example, that we are given the telegraph equation*

$$L(u)=\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u}{\partial t^2}+cu=-f(x, t). \qquad (c>0). \tag{85}$$

Here, the operator $L(u)$ is self-adjoint. Let us seek $v(x, t, x', t')$ as a function of x' and t' in the characteristic angle in the form of a function $v=v(\lambda)$ that depends only on $\lambda=-(\xi'-\xi)(\eta'-\eta)$. The condition (A) can now be written in the form

$$M_{\xi'\eta'}(v)=-2\frac{\partial^2v}{\partial\xi'\,\partial\eta'}+cv=0.$$

If in this equation we make the substitutions

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*Any equation of the form

$$u_{xx}-u_{tt}+au_x+bu_t+cu=-f$$

with constant coefficients can be reduced to this equation. We need only make the substitution $v=ue^{\frac{a}{2}x-\frac{b}{2}t}$.

$$\frac{\partial v}{\partial \xi'} = -\frac{dv}{d\lambda}(\eta' - \eta), \quad \frac{\partial^2 v}{\partial \xi' \partial \eta'} = -\frac{d^2 v}{d\lambda^2}\lambda - \frac{\partial v}{d\lambda},$$

we obtain the equation

$$2\lambda \frac{d^2 v}{d\lambda^2} + 2\frac{dv}{d\lambda} + cv = 0.$$

If we set $z = \sqrt{2c\lambda}$, then

$$\frac{dv}{d\lambda} = \frac{dv}{dz}\frac{\sqrt{2c}}{2\sqrt{\lambda}} = \frac{dv}{dz}\frac{c}{z}, \quad \frac{d^2 v}{d\lambda^2} = \frac{d^2 v}{dz^2}\frac{c^2}{z^2} + \frac{dv}{dz}\left(-\frac{c}{z^2}\right)\frac{c}{z}$$

and the last equation takes the form

$$v'' + \frac{1}{z}v' + v = 0, \tag{85'}$$

where the prime denotes differentiation with respect to z. This is the special case of Bessel's equation

$$v'' + \frac{1}{z}v' + \left(1 - \frac{\nu^2}{z^2}\right)v = 0 \qquad (\nu = \text{const}) \tag{85''}$$

with $\nu = 0$. One of the solutions of equation (85″) is the so-called Bessel's function of the first kind of order ν

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+\nu)}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{86}$$

In particular, Bessel's function of the first kind of order zero

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k}, \quad z = \sqrt{2c\lambda} = \sqrt{-2c(\xi' - \xi)(\eta' - \eta)}, \tag{86'}$$

is a solution of Eq. (85′).

Let us know consider conditions (B) and (C). It follows from these conditions that the Riemann function $v(x, t, x', t')$ must be equal to unity on the characteristics $\xi' = \xi$ and $\eta' = \eta$. But the function (86′) satisfies these conditions. Therefore, the Riemann function for the telegraph equation (83) can be written in the form

$$v(x, t, x', t') = J_0(\sqrt{c}\sqrt{(x - x')^2 - (t - t')^2}). \tag{86''}$$

By using this Riemann function, we can immediately write the solution of the Cauchy problem for the telegraph equation

$$u_{xx} - u_{tt} + cu = -f(x, t),$$
$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x) \tag{87}$$

in an explicit form. Riemann's formula (80) yields

$$u(x, t) = \frac{\psi(x-t) + \psi(x+t)}{2} +$$

$$+ \frac{1}{2}\int_{x-t}^{x+t} \left\{ -\psi(x') \frac{\partial}{\partial t'} J_0\left(\sqrt{c}\sqrt{(x-x')^2 - (t-t')^2}\right) + \right.$$

$$\left. + \psi_1(x') J_0\left(\sqrt{c}\sqrt{(x-x')^2 - (t-t')^2}\right) \right\}_{t'=0} dx' + \tag{87'}$$

$$+ \frac{1}{2}\int_0^t dt' \int_{x-t'}^{x+t'} J_0\left(\sqrt{c}\sqrt{(x-x')^2 - (t-t')^2}\right) f(x', t')\, dx'.$$

Furthermore, by differentiating, we easily verify that

$$\frac{d}{dz}\left(\frac{J_\nu(z)}{z^\nu}\right) = -\frac{J_{\nu+1}(z)}{z^\nu}, \quad \frac{d}{dz}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z). \tag{88}$$

In particular, for $\nu = 0$, it follows from (88) that $J_0(z) = -J_1(z)$. If we substitute this into Eq. (87'), we obtain the solution of the Cauchy problem for the telegraph equation (87) in the following form:

$$u(x, t) = \frac{\psi(x-t) + \psi(x+t)}{2} +$$

$$+ \frac{1}{2}\int_{x-t}^{x+t} \psi_1(x') J_0\left(\sqrt{c}\sqrt{(x-x')^2 - t^2}\right) dx' -$$

$$- \frac{\sqrt{c}\, t}{2}\int_{x-t}^{x+t} \psi(x') \frac{J_1\left(\sqrt{c}\sqrt{(x-x')^2 - t^2}\right)}{\sqrt{(x-x')^2 - t^2}}\, dx' + \tag{89}$$

$$+ \frac{1}{2}\int_0^t dt' \int_{x-t'}^{x+t'} J_0\left(\sqrt{c}\sqrt{(x-x')^2 - (t-t')^2}\right) f(x', t')\, dx'.$$

In particular, for $c=0$ and $f=0$, we obtain from this relation d'Alembert's integral corresponding to the case of $a=1$ (see Sect. 3).

As another important example, we cite the construction of the Riemann function $v(x, t, x', t')$ for the so-called Euler-Poisson operator

$$L(u)=\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}+\frac{\beta+\beta'}{x}\frac{\partial u}{\partial x}+\frac{\beta'-\beta}{x}\frac{\partial u}{\partial t}, \tag{90}$$

where β and β' are constants. To avoid going into the matter of the adjoint operator, let us seek $v(x, t, x', t')$ as a function of x and t in the upper characteristic angle with vertex at the point x', t'. The operator (90) is written in the variables ξ and η as follows:

$$L_{\xi\eta}(u)=-2\frac{\partial^2 u}{\partial\xi\,\partial\eta}+\frac{2\beta'}{\xi-\eta}\frac{\partial u}{\partial\xi}-\frac{2\beta}{\xi-\eta}\frac{\partial u}{\partial\eta} \tag{90''}$$

and condition (A') takes the form

$$\frac{\partial^2 v}{\partial\xi\,\partial\eta}-\frac{\beta'}{\xi-\eta}\frac{\partial v}{\partial\xi}+\frac{\beta}{\xi-\eta}\frac{\partial v}{\partial\eta}=0 \tag{90''}$$

in the upper right characteristic angle with vertex at the point $\xi'+i\eta'=(x'+iy')\,e^{-i\frac{\pi}{4}}$. Conditions (B') and (C') take the form

$$\frac{1}{v}\frac{\partial v}{\partial\xi}\Big|_{\eta=\eta'}=-\frac{\beta}{\xi-\eta'},\quad \frac{1}{v}\frac{\partial v}{\partial\eta}\Big|_{\xi=\xi'}=\frac{\beta'}{\xi'-\eta},\quad v\Big|_{\substack{\xi=\xi'\\ \eta=\eta'}}=1$$

or, after integration,

$$v|_{\eta=\eta'}=\left(\frac{\xi'-\eta'}{\xi-\eta'}\right)^{\beta},\quad v|_{\xi=\xi'}=\left(\frac{\xi'-\eta'}{\xi'-\eta}\right)^{\beta'}. \tag{90'''}$$

Let us show that

$$v(\xi, \eta, \xi', \eta')=\frac{(\xi'-\eta')^{\beta+\beta'}}{(\xi-\eta')^{\beta}(\xi'-\eta)^{\beta'}}F\left(\beta, \beta', 1, \frac{(\xi-\xi')(\eta-\eta')}{(\xi-\eta')(\eta-\xi')}\right), \tag{91}$$

where $F(a, b, c, z)$ is the hypergeometric series

$$F(a, b, c, z)=1+\frac{ab}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\ldots. \tag{91'}$$

is a solution of Eq. (90'') under the conditions (90'''); that is, it is the Riemann function for the operator (90). Obviously, the function (91) satisfies the boundary conditions (90'''). Let us replace v in (90'') with $u=\xi^{-\beta}\varphi(z)$, where $z=\frac{\eta}{\xi}$. We have

$$\frac{\partial u}{\partial\xi}=-\beta\xi^{-\beta-1}\varphi-\xi^{-\beta-2}\eta\varphi',\quad \frac{\partial u}{\partial\eta}=\xi^{-\beta-1}\varphi',$$

$$\frac{\partial^2 u}{\partial\xi\,\partial\eta}=-(\beta+1)\xi^{-\beta-2}\varphi'-\xi^{-\beta-3}\eta\varphi''.$$

If we multiply $\frac{\partial u}{\partial \xi}$ by $\beta' \xi^{\beta+1}$, $\frac{\partial u}{\partial \eta}$ by $-\beta \xi^{\beta+1}$, and $\frac{\partial^2 u}{\partial \xi \partial \eta}$ by $-(\xi-\eta)\xi^{\beta+1}$ and add, we obtain

$$\varphi'' \frac{\eta(\xi-\eta)}{\xi^2} + \varphi' \left[(\beta+1)\frac{\xi-\eta}{\xi} - \beta - \beta' \frac{\eta}{\xi}\right] - \beta\beta'\varphi = 0.$$

Since $\frac{\eta}{\xi} = z$, we have

$$z(1-z)\varphi'' + [1-(\beta+\beta'+1)z]\varphi' - \beta\beta'\varphi = 0.$$

But this is the hypergeometric equation with parameters $a=\beta$, $b=\beta'$, $c=1$. Consequently, the solution of Eq. (90'') will be a function of the following form:

$$u(\xi, \eta) = \xi^{-\beta} F\left(\beta, \beta', 1, \frac{\eta}{\xi}\right). \tag{92}$$

Equation (90'') has the remarkable property that if $\Phi(\xi, \eta)$ is a solution of this equation, a function of the form

$$\Phi^*(\xi, \eta) = (\xi-\xi')^{-\beta}(\eta-\xi')^{-\beta'}\Phi\left(\frac{\xi-\eta'}{\xi-\xi'}, \frac{\eta-\eta'}{\eta-\xi'}\right). \tag{92'}$$

where ξ' and η' are arbitrary constants, will also be a solution of it. To see this, set

$$\frac{\xi-\eta'}{\xi-\xi'} = \xi^*, \quad \frac{\eta-\eta'}{\eta-\xi'} = \eta^*.$$

Then,

$$\frac{\partial \Phi^*}{\partial \xi} = -\beta(\xi-\xi')^{-\beta-1}(\eta-\xi')^{-\beta'}\Phi +$$
$$+ \frac{\partial \Phi}{\partial \xi^*}(\xi-\xi')^{-\beta-2}(\eta-\xi')^{-\beta'}[(\eta'-\xi'),$$
$$\frac{\partial \Phi^*}{\partial \eta} = -\beta'(\xi-\xi')^{-\beta}(\eta-\xi')^{-\beta'-1}\Phi +$$
$$+ \frac{\partial \Phi}{\partial \eta^*}(\xi-\xi')^{-\beta}(\eta-\xi')^{-\beta'-2}(\eta'-\xi'),$$
$$\frac{\partial^2 \Phi^*}{\partial \xi \partial \eta} = \beta\beta'(\xi-\xi')^{-\beta-1}(\eta-\xi')^{-\beta'-1}\Phi -$$
$$- \frac{\partial \Phi}{\partial \eta^*}\beta(\xi-\xi')^{-\beta-1}(\eta-\xi')^{-\beta-2}(\eta'-\xi') -$$
$$- \frac{\partial \Phi}{\partial \xi^*}\beta'(\xi-\xi')^{-\beta-2}(\eta-\xi')^{-\beta'-1}(\eta'-\xi') +$$
$$+ \frac{\partial^2 \Phi}{\partial \xi^* \partial \eta^*}(\xi-\xi')^{-\beta-2}(\eta-\xi')^{-\beta'-2}(\eta'-\xi')^2.$$

If we multiply $\frac{\partial \Phi^*}{\partial \xi}$ by $-\beta'$, $\frac{\partial \Phi^*}{\partial \eta}$ by β, and $\frac{\partial^2 \Phi}{\partial \xi \partial \eta}$ by $\xi-\eta$ and add the results, we obtain

$$(\xi-\eta)\frac{\partial^2\Phi^*}{\partial\xi\,\partial\eta}-\beta'\frac{\partial\Phi^*}{\partial\xi}+\beta\frac{\partial\Phi^*}{\partial\eta}=\Phi\beta\beta'(\xi-\xi')^{-\beta-1}(\eta-\xi')^{-\beta'-1}[\eta-\xi'-$$
$$-(\xi-\xi')+\xi-\eta]+(\xi-\xi')^{-\beta-1}(\eta-\xi')^{-\beta'-1}(\eta'-\xi')\times$$
$$\times\left\{\frac{\partial^2\Phi}{\partial\xi^*\,\partial\eta^*}\frac{(\eta'-\xi')(\xi-\eta)}{(\xi-\xi')(\eta-\xi')}+\frac{\partial\Phi}{\partial\xi^*}\left[-\beta'\frac{\xi-\eta}{\xi-\xi'}-\beta'\frac{\eta-\xi'}{\xi-\xi'}\right]+\right. \tag{93}$$
$$\left.+\frac{\partial\Phi}{\partial\eta^*}\left[-\beta\frac{\xi-\eta}{\eta-\xi'}+\beta\frac{\xi-\xi'}{\eta-\xi'}\right]\right\}.$$

Obviously,

$$\xi^*-\eta^*=\frac{\xi-\eta'}{\xi-\xi'}-\frac{\eta-\eta'}{\eta-\xi'}=\frac{-\xi\xi'-\eta\eta'+\xi\eta'+\xi'\eta}{(\xi-\xi')(\eta-\xi')}=$$
$$=\frac{(\eta'-\xi')(\xi-\eta)}{(\xi-\xi')(\eta-\xi')}.$$

If we substitute this into the right-hand member of Eq. (93), we see that it is identically equal to 0. If, in Eq. (92'), we take for $\Phi(\xi, \eta)$ the function defined by Eq (92), we obtain the following solution of Eq. (90''):

$$\Phi^*(\xi,\eta)=(\xi-\eta')^{-\beta}(\eta-\xi')^{-\beta'}F\left(\beta,\beta',1,\frac{(\xi-\xi')(\eta-\eta')}{(\xi-\eta')(\eta-\xi')}\right).$$

If we multiply both members of this equation by $((\xi'-\eta')^{\beta+\beta'}(-1)^{-\beta'}$, we see that the function (91) satisfies Eq. (90'') and, consequently, is the Riemann function for the operator $L(u)$ defined by Eq. (90).

In accordance with (73'), we have

$$\xi=\frac{x+t}{\sqrt{2}},\quad \eta=\frac{t-x}{\sqrt{2}},\quad \xi'=\frac{x'+t'}{\sqrt{2}},\quad \eta'=\frac{t'-x'}{\sqrt{2}}.$$

Therefore,

$$\frac{(\xi'-\eta')^{\beta+\beta'}}{(\xi-\eta')^{\beta}(\xi'-\eta)^{\beta'}}=\frac{(2x')^{\beta+\beta'}}{(x+x'+t-t')^{\beta}(x+x'-(t-t'))^{\beta'}},$$
$$\frac{(\xi-\xi')(\eta-\eta')}{(\xi-\eta')(\eta-\xi')}=\frac{x-x'+t-t'}{x+x'+t-t'}\cdot\frac{t-t'-(x-x')}{t-t'-(x+x')}.$$

If we substitute this into (91), we conclude that the Riemann function for the Euler-Poisson operator (90) in the variables x and t takes the form

$$v(x,t,x',t')=\frac{(2x')^{\beta+\beta'}}{(x+x'+t-t')^{\beta}(x+x'-(t-t'))^{\beta'}}\times$$
$$\times F\left(\beta,\beta',1,\frac{(x-x')^2-(t-t')^2}{(x+x')^2-(t-t')^2}\right). \tag{91'''}$$

The method of solving the Cauchy problem expounded in this section is based primarily on the use of Ostrogradskiy's formula for adjoint differential operators (77) and on the Riemann function, which is a solution of the corresponding partial differential equation with a specified singularity. Therefore, it is only natural that this method can be applied not only to equations with two independent variables but to general hyperbolic second-order linear equations. This was shown by Hadamard in a series of articles.*

*See R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II, pp. 740-744, New York, Interscience, 1962.

10. Boundary-Value Problems for Steady-State Oscillations and Problems Without Initial Conditions. Radiation Conditions

1. **Boundary-value problems on steady-state oscillations for the wave equation.** We shall say that an oscillatory process in some region or in all space is steady-state if it is periodic with respect to the time t.

Among the solutions of the wave equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f}{a^2}, \tag{1}$$

that correspond to steady-state oscillatory processes (acoustic, mechanical, electromagnetic, etc.), of special importance are solutions of the form

$$u = v_1 \cos \omega t - v_2 \sin \omega t, \tag{94}$$

where v_1 and v_2 are time-independent functions and ω is the number, called the frequency of the oscillations, equal to the number of complete cycles in 2π units of time. If τ is the period of the oscillations, then $\omega = 2\pi/\tau$. In what follows, we shall refer to solutions of the wave equation (1) that are of the form (94) as *steady-state oscillations with given frequency* ω.

We can arrive at boundary-value problems of finding steady-state oscillations by reasoning as follows: Let G denote a bounded region in x-space bounded by a surface S. Let us suppose that the right-hand member of Eq. (1) in the region G and the linear operator $\varkappa u$ of the boundary conditions on the surface S are of the form

$$f = f_1 \cos \omega t - f_2 \sin \omega t, \tag{94'}$$

$$\varkappa u|_S = \Phi_1 \cos \omega t - \Phi_2 \sin \omega t, \tag{94''}$$

where f_1, f_2, Φ_1 and Φ_2 are time-independent functions. Then, it is natural to expect that, as a result of a protracted application of a periodic force (94′) and a periodic boundary regime (94″), the effect of the initial conditions will decay to zero with the passage of time and that the solution of Eq. (1) in the region of G will take the form of a steady-state oscillation (94). Thus, we arrive at the first, second, and third interior boundary-value problems for steady-state oscillations. Each of these problems consists in finding a twice continuously differentiable solution in the domain G of Eq. (1) with right-hand member (94′) that is of the form (94) and that satisfies the boundary conditions (94″) on the surface S.

Here, if the boundary-condition operator is of the form $\varkappa u = u$, we can speak of the first boundary-value problem on steady-state oscillations; if $\varkappa u = \partial u / \partial n$ or $\varkappa u = \partial u / \partial n - \alpha u$, where α is a positive continuous time-independent function on the surface S, we can speak respectively of the second and third boundary-value problems for steady-state oscillations for the domain G. The exterior first, second, and third boundary-value problems for steady-state oscillations are posed analogously. We only need to replace the region interior to the surface S with the region exterior to it; also, here, in analogy with the regularity conditions for harmonic functions at infinity, we need to impose quite definite restrictions on the behavior of steady-state oscillations as we approach the point at infinity in x-space. These restrictions are made in such a way as to ensure uniqueness of the solution of the boundary-value problems dealing with steady-state oscillation, and they are usually called *radiation conditions*. As an example of such a problem, we mention the problem of finding twice continuously differentiable solutions of (1) in all x-spaces with right-hand member (94′) that are of the form (94) and that satisfy specific restrictions, the radiation conditions, as we approach infinity. This problem may be called the problem on *steady-state oscillations in space*.

Let us note some general properties of steady-state oscillations (94). If we substitute the expression for u given by Eq. (94) and the expression for f given by (94′) into the left- and right-hand members, respectively, of (1), we obtain

$$\left(\Delta v_1 + \frac{\omega^2}{a^2} v_1\right) \cos \omega t - \left(\Delta v_2 + \frac{\omega^2}{a^2} v_2\right) \sin \omega t = -\frac{f_1}{a^2} \cos \omega t + \\ + \frac{f_2}{a^2} \sin \omega t.$$

Since the functions $\cos \omega t$ and $\sin \omega t$ are linearly independent, we have

$$\Delta v_1 + k^2 v_1 = -\frac{f_1}{a^2}, \quad \Delta v_2 + k^2 v_2 = -\frac{f_2}{a^2} \quad \left(k = \frac{\omega}{a}\right). \tag{95}$$

This means that if we set $F = f_1 + i f_2$, the function $v = v_1 + i v_2$ must be a solution of the equation

$$\Delta v + k^2 v = -\frac{F}{a^2} \quad \left(k = \frac{\omega}{a}\right). \tag{96}$$

Conversely, if $v = v_1 + i v_2$ is a solution of Eq. (96), by multiplying both members of this equation by $e^{i\omega t}$, we obtain

$$\Delta (v e^{i\omega t}) - \frac{1}{a^2} \frac{\partial}{\partial t^2} (v e^{i\omega t}) = -\frac{F}{a^2} e^{i\omega t}.$$

Consequently, the real part of the function $ve^{i\omega t}$, which is equal to $v_1 \cos \omega t - v_2 \sin \omega t$, satisfies Eq. (1) with right-hand member of the form (94′). Let us agree to call the function $v = v_1 + iv_2$ the *complex amplitude* and the function $F = f_1 + if_2$ the *complex exciting force*. Thus, we conclude that the set of all steady-state oscillations (94) is completely defined by the set of complex amplitudes $v = v_1 + iv_2$ by means of the equation

$$u = \mathrm{Re}\,[ve^{i\omega t}] \tag{97}$$

with the condition that $v = v_1 + iv_2$ is a solution of Eq. (96). Therefore, Eq. (96) is one of the most important equations in mathematical physics and is often encountered under the special name *Helmholtz's equation* or the *oscillation equation.** In what follows, we shall also call it the *amplitude equation*, remembering that, in accordance with the general classification, this equation is of elliptic type, and that among second-order differential equations, oscillatory processes are described in the most direct form by hyperbolic equations.

Each of the boundary-value problems on steady-state oscillations is immediately reduced to the corresponding boundary-value problem for the amplitude equation. For example, the solution of the problem for steady-state oscillations for all space reduces to finding a solution of the amplitude equation (96) that is twice continuously differentiable in all space and that satisfies the radiation conditions at infinity. Similarly, every interior boundary-value problem for steady-state oscillations with boundary conditions (94″) reduces to finding in the corresponding region a twice continuously differentiable solution of the amplitude equation (96) that satisfies the boundary conditions

$$\varkappa v|_S = \varkappa (v_1 + iv_2)|_S = \Phi_1 + i\Phi_2. \tag{98}$$

Analogously, every exterior boundary-value problem for steady-state oscillations with boundary conditions (94″) reduces to the exterior boundary-value problem for the amplitude equation (96) with boundary conditions (98). Here, by "exterior boundary conditions for the amplitude equation (96)," we mean problems on finding

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*The origin of this name is evidently connected with the fact that in the case of a single degree of freedom of motion, the oscillatory process is described by the so-called oscillation equation

$$d^2x/dt_2 + k^2x = -F(t),$$

where t is the time and x is the displacement from the equilibrium position. The study of oscillatory processes with one or several degrees of freedom of motion constitutes the subject of a special discipline, namely, the theory of oscillations. See, for example, J. Haag, *Oscillatory Motions*, Belmont, California, Wadsworth Publishing Co., 1962.

solutions of this equation that are twice continuously differentiable in the region exterior to the surface S, that satisfy the boundary conditions (98), and that satisfy the radiation conditions at infinity.

We shall confine ourselves to a consideration of interior boundary-value problems for the amplitude equation (96) and to a consideration of the boundary-value problem for this equation as applied to all space.

In considering the interior boundary-value problems, we can assume without loss of generality that the boundary function $\Phi=\Phi_1+i\Phi_2$ is identically zero since, in the opposite case, we need only subtract from the desired solution some twice continuously differentiable function that satisfies the given nonhomogeneous boundary conditions. Thus, the problem reduces to solving the interior boundary-value problem with zero boundary function

$$\Delta v + k^2 v = -\frac{F}{a^2}, \quad \varkappa v|_S = 0 \quad \left(k^2 = \frac{\omega^2}{a^2}\right). \tag{99}$$

For definiteness, let us first consider the case in which the number of independent variables at n is three. Let $G(P_0, P)$ be the Green's function of the interior boundary-value problem for Laplace's equation*

$$\Delta v = 0, \quad \varkappa v|_S = 0. \tag{100}$$

Then, in accordance with formulas (16), (17), and (18) (see Chapter III, Sect. 2, subsection 3), the solution $v(P)$ of the boundary-value problem (99) must satisfy the integral equation

$$v(P_0) = \widetilde{F}(P_0) + k^2 \iiint_G G(P_0, P)\, v(P)\, d\tau, \tag{101}$$

where

$$\widetilde{F}(P_0) = \frac{1}{a^2} \iiint_G G(P_0, P)\, F(P)\, d\tau. \tag{101'}$$

Conversely, it follows from potential theory (see Chapter III, Sect. 4, subsection 3) that if $v(P)$ is a solution of the integral equation (101), then $v(P)$ is also a solution of the boundary-value problem (99). Thus, the solution of boundary-value problem (99)

*In the case of the second boundary-value problem, by "Green's function" we mean the generalized Green's function (see Chapter III, Sect. 2, subsection 3).

reduces to solving the integral equation (101). The Green's function $G(P_0, P)$ is, as we know, symmetric. Therefore, the integral equation (101), since it is an equation with symmetrical kernel, has a discrete real spectrum of eigenvalues. If k^2 does not coincide with any of the eigenvalues of the integral equation (101), this equation has a unique solution for every function $F(P)$. The same is true of the boundary-value problem (99). Suppose that $k^2 = \omega^2/a^2$ coincides with one of the eigenvalues of the integral equation (101). Then, the given integral equation, and hence the problem (99), have nonzero solutions for $\widetilde{F}(P) \equiv 0$ or, what amounts to the same thing, for $F(P) \equiv 0$.* Here, if the function $F(P)$ is not identically equal to zero, the integral equation (101) and, consequently, the problem (99), may fail to have a solution. They may have more than one solution only when the function $\widetilde{F}(P)$ is orthogonal to the eigenfunctions of the integral equation corresponding to the given eigenvalue. On the other hand, if the function $\widetilde{F}(P)$ does not satisfy these orthogonality conditions, the integral equation (101) and problem (99) will not have a solution.

Obviously, all these considerations remain valid if there are two instead of three independent variables. In this case, we need only work with the Green's function $G(z_0, z)$ of the interior boundary-value problems for Laplace's equation in two independent variables. In the case of a single independent variable, all the results given with regard to the problem (99) still remain valid. These results are obtained by reducing the problem to an integral equation analogous to Eq. (101) if we define in an appropriate manner the Green's function of the interior boundary-value problems for the operator $L(v) = d^2v/dx^2$ or for the operator $L(v) = d^2v/dx^2 - \gamma^2 v$, where γ^2 is a numerical parameter. Let us show this, for example, by examining the first boundary-value problem

$$\frac{d^2v}{dx^2} + k^2 v = -\frac{F(x)}{a^2}, \quad v|_{x=0} = 0, \quad v|_{x=l} = 0. \tag{102}$$

Here, the Green's function $G(x_0, x)$ of the operator $L(v) = d^2v/dx^2$ for the first boundary-value problem

$$\frac{d^2v}{dx^2} = 0, \tag{103}$$

$$v|_{x=0} = 0, \quad v|_{x=l} = 0 \tag{103'}$$

can be introduced by means of the following equation:

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*$\widetilde{F}(P) \equiv 0$ implies that $F(P) \equiv 0$ since $\Delta \widetilde{F} = -F$.

$$G(x_0, x) = \begin{cases} x\left(1 - \frac{x_0}{l}\right), & x \leqslant x_0, \\ x_0\left(1 - \frac{x}{l}\right), & x \geqslant x_0. \end{cases} \tag{103''}$$

As a function of x, this function $G(x_0, x)$ can be characterized by the following properties:

(a) in each of the intervals $0<x<x_0$, $x_0<x<l$, it is a twice continuously differentiable solution of the differential equation (103);

(b) at the end points of the interval $0<x<l$, it satisfies the homogeneous boundary conditions (103');

(c) at the point $x=x_0$, it is continuous and its derivative has a jump equal to -1.

The Green's function $G(x_0, x)$ is a symmetric function of the variables x and x_0.*

It is easy to show that for a function $v(x)$ to be a twice continuously differentiable solution of the boundary-value problem

$$\frac{d^2v}{dx^2} = -f^*(x), \quad v|_{x=0} = 0, \quad v|_{x=l} = 0, \tag{104}$$

it is necessary and sufficient that

$$v(x_0) = \int_0^l G(x_0, x) f^*(x)\, dx. \tag{104'}$$

To see this, note that, if $v(x_0)$ is the function defined by Eq. (104'), it satisfies the homogeneous boundary conditions of the problem (104) since the Green's function, as a function of x_0, satisfies these conditions. Furthermore, simple calculation yields

$$\frac{dv(x_0)}{dx_0} = \int_0^{x_0} \frac{dG(x_0, x)}{dx_0} f^*(x)\, dx + \int_{x_0}^{l} \frac{dG(x_0, x)}{dx_0} f^*(x)\, dx,$$

$$\frac{d^2v(x_0)}{dx_0^2} = f^*(x)\left[\frac{dG(x_0, x)}{dx_0}\bigg|_{x=x_0-0} - \frac{dG(x_0, x)}{dx_0}\bigg|_{x=x_0+0}\right] = -f^*(x_0).$$

Consequently, the function defined by Eq. (104') is a solution of the boundary-value problem (104). Conversely, if we integrate the identity

$$v\frac{d^2G(x_0, x)}{dx^2} - G(x_0, x)\frac{d^2v}{dx^2} = \frac{d}{dx}\left[v\frac{dG(x_0, x)}{dx} - G(x_0, x)\frac{dv}{dx}\right]$$

* In the case of one-dimensional boundary-value problems, a study of the Green's functions in the general case will be made in Chapter VI, which deals with the method of separation of variables.

from 0 to l and take account of the homogeneous boundary conditions (104), we obtain

$$\int_0^l G(x_0, x) f^*(x)\, dx = v(x) \frac{dG(x_0, x)}{dx}\Bigg|_{x=x_0-0} -$$

$$- v(x) \frac{dG(x_0, x)}{dx}\Bigg|_{x=x_0+0} = v(x_0),$$

that is, the solution of the boundary-value problem (104) must be represented by Eq. (104′). Therefore, we may assert that solving the boundary-value problem (102) is equivalent to solving the integral equation

$$v(x_0) = \tilde{F}(x_0) + k^2 \int_0^l G(x_0, x) v(x)\, dx, \tag{105}$$

where

$$\tilde{F}(x_0) = \frac{1}{a^2} \int_0^l G(x_0, x) F(x)\, dx. \tag{105′}$$

This integral equation is completely analogous to the integral equation (101) and the conclusions derived from it are applicable to the problem (102) and to the corresponding one-dimensional steady-state oscillations just as in the three- and two-dimensional cases.

In the case of the second and third one-dimensional boundary-value problems

$$\frac{d^2v}{dx^2} + k^2 v = -\frac{F(x)}{a^2}, \quad \varkappa v|_{x=0} = 0, \quad \varkappa v|_{x=l} = 0, \tag{106}$$

by using the general solution of the equation $d^2v/dx^2 - \gamma^2 v = 0$, we can construct in explicit form the Green's function $G(x_0, x)$ corresponding to the boundary-value problem

$$\frac{d^2v}{dx^2} - \gamma^2 u = 0, \quad \varkappa v|_{x=0} = 0, \quad \varkappa v|_{x=l} = 0. \tag{106′}$$

This Green's function must, by definition, possess the same characteristic properties as does the Green's function (103),

(103′). It is always possible to construct such a Green's function of the problem (106′) provided the parameter γ is chosen in such a way that the problem (106′) does not have nonzero solution. This is easily done by starting with the explicit expression for the solutions of the differential equation (106′). After we have constructed this Green's function, the second and third boundary-value problems (106) reduce to solving the integral equation

$$v(x_0) = \widetilde{F}(x_0) + (k^2 + \gamma^2) \int_0^l G(x_0, x)\, v(x)\, dx, \tag{107}$$

where

$$\widetilde{F}(x_0) = \frac{1}{a^2} \int_0^l G(x_0, x)\, F(x)\, dx. \tag{107′}$$

The conclusions drawn from this integral equation are just as in the three- and two-dimensional cases.

These results have a simple physical interpretation. Suppose that the given frequency of ω of the desired steady-state oscillations is such that $k^2 = \omega^2/a^2$ is an eigenvalue of the boundary-value problem (99) or, what amounts to the same thing, of the integral equation (101). Let us consider separately the cases of $F \equiv 0$ and $F \not\equiv 0$. In the case of $F \equiv 0$, the boundary-value problem (99) has a nonzero solution of $v = v_1 + iv_2$, which is an eigenfunction or, as we shall call it, a *"complex eigenamplitude."* Corresponding to this eigenamplitude are the steady-state oscillations $u = Re(ve^{i\omega t})$. A characteristic feature of these steady-state oscillations is the fact that they come into existence in the absence of an exciting force $(F = f_1 + if_2 = 0)$ and with a zero boundary regime $(\varkappa u/_S = 0)$. They are as if "sourceless"; after they have come about for whatever reasons, they do not disappear. Therefore, it is customary to call these oscillations *natural (or free) oscillations*, and the frequencies corresponding to them the *natural frequencies*. In the case in which $F \not\equiv 0$, for the problem (99) to have a solution, it is necessary and sufficient that the function $\widetilde{F}(P)$ be orthogonal to all the eigenfunctions of the problem (99) corresponding to the given eigenvalue. Let us show that this orthogonality condition is equivalent to the condition that the complex disturbing force $F = f_1 + if_2$ be orthogonal to the same eigenfunction of the problem (99), i.e., to the eigenamplitudes corresponding to the given natural frequency. If $\varphi(P)$ is one of these eigenamplitudes, then, by taking into account the symmetry of the Green's function $G(P_0, P)$, we have

$$\iiint_G \varphi(P_0)\,\widetilde{F}(P)\,d\tau_0 = \iiint_G \varphi(P_0)\,d\tau_0 \frac{1}{a^2}\iiint_G G(P_0, P)\,F(P)\,d\tau =$$

$$= \frac{1}{a^2}\iiint_G F(P)\,d\tau \iiint_G G(P_0, P)\,\varphi(P_0)\,d\tau_0 =$$

$$= \frac{1}{a^2}\iiint_G F(P)\,d\tau \iiint_G G(P, P_0)\,\varphi(P_0)\,d\tau_0 =$$

$$= \frac{1}{a^2}\iiint_G F(P)\,k^2\varphi(P)\,d\tau.$$

From this, it is clear that orthogonality of $\widetilde{F}(P)$ and $\varphi(P)$ implies orthogonality of $F(P)$ and $\varphi(P)$ and conversely.

Thus, we arrive at the following conclusion: For the boundary-value problem (99) with eigenvalue k^2 to have a solution and, consequently, for there to exist steady-state oscillations corresponding to it with natural frequency $\omega = ka$, it is necessary and sufficient that the complex exciting force $F = f_1 + if_2$ be orthogonal to the complex eigenamplitudes corresponding to the given eigenvalue k^2. If these orthogonality conditions are satisfied, the solution of problem (99) is determined up to an additive term in the form of a linear combination of eigenamplitudes, and the corresponding steady-state oscillations are determined up to an additive term representing the real part of that linear combination multiplied by $e^{i\omega t}$.

This last assertion can be explained as follows: If the exciting force $f = f_1 \cos \omega t - f_2 \sin \omega t$ acting on the system with a natural frequency of oscillation ω is such that the complex force $F = f_1 + if_2$ does not satisfy these conditions of orthogonality with the complex eigenamplitudes corresponding to the given natural frequency, there can be no steady-state oscillations with the given natural frequency. The situation is such that the exciting force "builds up" the system indefinitely and the complex amplitude $v = v_1 + iv_2$ increases in absolute value without bound, so that the possibility of existence of steady-state oscillations disappears. We may say there is *resonance* in this case. On the other hand, if the frequency of the desired steady-state oscillations ω is not a natural frequency, i.e., if $k^2 = \omega^2/a^2$ is not an eigenvalue of the problem (99), there will be unique steady-state vibrations for arbitrary values of the exciting force $f = f_1 \cos \omega t - f_2 \sin \omega t$. Of course, this does not mean that the natural oscillations of the system corresponding to its different natural frequencies cannot be superimposed on these steady-state oscillations with other-than-natural frequency ω.

Let us illustrate all this with a specific example of the first boundary-value problem on steady-state oscillations for an interval $[0, l]$

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{1}{a^2}(f_1 \cos \omega t - f_2 \sin \omega t), \qquad (108)$$
$$u|_{x=0} = 0, \; u|_{x=l} = 0.$$

Corresponding to this problem on the steady-state oscillations is the boundary-value problem for the amplitude equation (102). The general solution of the amplitude equation for $F=0$ is of the form $v=A \cos kx+B \sin kx$, where A and B are arbitrary (real) constants. Taking the boundary conditions of the problem (102) into account, we obtain from this solution the eigenvalues of this problem $k_n^2 = (n\pi/l)^2$, where n is an arbitrary integer, and the corresponding eigenfunctions $\varphi_n(x) = \sin(n\pi x/l)$ and natural frequencies $\omega_n = n\pi a/l$. Consequently, the general set of natural oscillations corresponding to the problem (108) is given by the equation

$$u(x, t) = \operatorname{Re}\left\{\sum_{n=1}^{\infty} (A_n + iB_n) \sin \frac{n\pi}{l} x e^{i\frac{n\pi a}{l}t}\right\}, \qquad (108')$$

where the A_n and B_n are arbitrary real constants. The condition that the complex exciting force $F = f_1 + if_2$ be orthogonal to the natural amplitude $\varphi_n = (A_n + iB_n) \sin(n\pi x/l)$ is written in the form

$$\int_0^l F(x) \sin \frac{n\pi}{l} x \, dx = 0. \qquad (108'')$$

If this condition is satisfied, there will be steady-state oscillations with natural frequency $\omega_n = n\pi a/l$, and they will be of the form

$$u(x, t) = \operatorname{Re}\left\{\left[v_0 + (A + iB) \sin \frac{n\pi}{l} x\right] e^{i\frac{n\pi a}{l}t}\right\}, \qquad (108''')$$

where $v_0 = v_0(x)$ is some particular solution of the boundary-value problem (102) for $k^2 = (n\pi/l)^2$ and A and B are arbitrary constants. If condition (108″) is not satisfied, there will be no steady-state oscillations with given natural frequency $\omega_n = n\pi a/l$, and we shall have resonance.

For example, if F = const and n is odd, the orthogonality condition (108″) is not satisfied and there will be no steady-state oscillations with natural frequency $\omega_n = n\pi a/l$ and we shall have resonance. On the other hand, if $F = (C+iD) \cos(n\pi x/l)$, where C and D are real constants, the orthogonality conditions (108″) will be satisfied and, consequently, there will be steady-state oscillations with frequency $\omega_n = n\pi a/l$. It is easy to find these steady-state oscillations. Specifically, a particular solution of the problem

$$\frac{d^2 v}{dx^2} + \left(\frac{n\pi}{l}\right)^2 v = -\frac{C+iD}{a^2} \cos \frac{n\pi}{l} x, \; v|_{x=0} = 0, \; v|_{x=l} = 0,$$

is

$$v_0 = -\frac{l}{2n\pi a^2}(C + iD)\, x \sin\frac{n\pi}{l}x.$$

as is easily verified. Consequently, on the basis of (108), the desired steady-state oscillations with natural frequency $\omega_n = n\pi a/l$ can be written in the form

$$u(x, t) = \operatorname{Re}\left\{\left[-\frac{l}{2n\pi a^2}(C + iD)\, x \sin\frac{n\pi}{l}x + \right.\right.$$
$$\left.\left. + (A + iB)\sin\frac{n\pi}{l}x\right] e^{i\frac{n\pi a}{l}t}\right\},$$

where A and B are arbitrary constants.

2. Boundary-value problems for steady-state oscillations for the telegraph equation and problems without initial conditions. Just as with the wave equation, it is possible to pose interior boundary-value problems for steady-state oscillations for the telegraph equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \frac{b}{a^2}\frac{\partial u}{\partial t} + \frac{c}{a^2}u = -\frac{f}{a^2}. \tag{109}$$

In a bounded domain G of x-space, let us seek twice continuously differentiable solutions of Eq. (109) that are of the form

$$u = \operatorname{Re}[ve^{i\omega t}], \quad v = v_1 + iv_2, \tag{94}$$

where v_1 and v_2 are time-independent, with the condition that the exciting force f defined in the domain G and the boundary conditions on the surface S are of the forms

$$f = \operatorname{Re}[Fe^{i\omega t}], \quad F = f_1 + if_2, \tag{94′}$$
$$\varkappa u|_S = \operatorname{Re}[\Phi e^{i\omega t}], \quad \Phi = \Phi_1 + i\Phi_2, \tag{94″}$$

where f_1, f_2, Φ_1 and Φ_2 are independent of t and where ω is a given positive number representing the frequency.

If we set $u = v_1\cos\omega t - v_2\sin\omega t$ in (109), we obtain

$$\cos\omega t\left[\Delta v_1 + \frac{\omega^2}{a^2}v_1 + \frac{c}{a^2}v_1 - \frac{b\omega}{a^2}v_2\right] -$$
$$-\sin\omega t\left[\Delta v_2 + \frac{\omega^2}{a^2}v_2 + \frac{c}{a^2}v_2 + \frac{b\omega}{a^2}v_1\right] =$$
$$= -\frac{f_1}{a^2}\cos\omega t + \frac{f_2}{a^2}\sin\omega t.$$

From this, we obtain

$$\Delta v_1 + \frac{\omega^2 + c}{a^2} v_1 - \frac{b\omega}{a^2} v_2 = -\frac{f_1}{a^2}, \quad \Delta v_2 + \frac{\omega^2 + c}{a^2} v_2 + \frac{b\omega}{a^2} v_1 = -\frac{f_2}{a^2}$$

or, in complex form,

$$\Delta v + \left(\frac{\omega^2 + c}{a^2} + i\frac{b\omega}{a^2}\right) v = -\frac{f}{a^2}.$$

Conversely, if the function $v = v_1 + iv_2$ satisfies this equation, then the function u, defined by Eq. (94) satisfies Eq. (109). To see this, one need only write Eq. (109) in the form

$$\operatorname{Re}\left\{\Delta(ve^{i\omega t}) - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}(ve^{i\omega t}) + \frac{b}{a^2}\frac{\partial}{\partial t}(ve^{i\omega t}) + \frac{c}{a^2} ve^{i\omega t}\right\} = \operatorname{Re}[Fe^{i\omega t}]$$

or

$$\operatorname{Re}\left\{\left[\Delta v + \frac{\omega^2}{a^2} v + \frac{c}{a^2} v + i\frac{b\omega}{a^2} v\right] e^{i\omega t}\right\} = \operatorname{Re}[Fe^{i\omega t}].$$

Thus, the solution of the interior boundary-value problem for steady-state oscillations (109), (94″) is equivalent to the solution of the following boundary-value problem for the amplitude equation:

$$\Delta v + \lambda v = -\frac{F}{a^2}, \quad \varkappa v|_S = \Phi \quad \left(\lambda = \frac{\omega^2}{a^2}\left(1 + \frac{c}{\omega^2} + i\frac{b}{\omega}\right)\right).$$

In this boundary-value problem, we may assume without loss of generality that the boundary function $\Phi = \Phi_1 + i\Phi_2$ is identically equal to 0, so that it is a matter of solving the following boundary-value problem:

$$\Delta v + \lambda v = -\frac{F}{a^2}, \quad \varkappa v|_S = 0 \quad \left(\lambda = \frac{\omega^2}{a^2}\left(1 + \frac{c}{\omega^2} + i\frac{b}{\omega}\right)\right). \tag{110}$$

Suppose that $G(P_0, P)$ is the Green's function of the corresponding interior boundary-value problem for Laplace's equation with three independent variables

$$\Delta v = 0, \quad \varkappa v|_S = 0.$$

Then, every solution of the three-dimensional boundary-value problem (110) satisfies the integral equation

$$v(P_0) = \widetilde{F}(P_0) + \lambda \iiint_G G(P_0, P)\, v(P)\, d\tau, \tag{111}$$

where

$$\widetilde{F}(P_0) = \frac{1}{a^2} \iiint_G G(P_0, P)\, F(P)\, d\tau. \tag{111'}$$

The converse is also true: Every solution of the integral equation (111) is a solution of the boundary-value problem (110). Thus, solution of the three-dimensional problem on steady-state oscillations for the telegraph equation (109) amounts to solving the integral equation (111). The problem on the steady-state oscillations for the telegraph equation (109) in the two- and one-dimensional cases reduces to an analogous integral equation. In the first case this is done by means of the Green's function for Laplace's equation with two independent variables and, in the second case, by means of the Green's function for the operator $L(v) = d^2v/dx^2$ or the operator $L(v) = d^2v/dx^2 - \gamma^2 v$. The integral equation (111) and its analogs in the two- and one-dimensional cases permit us to draw definite conclusions with regard to the boundary-value problem (110) and the steady-state oscillations corresponding to it. These conclusions are completely analogous to those that we drew earlier from the integral equation (101) with regard to the solutions of the boundary-value problem (99) and the steady-state oscillations corresponding to it. However, in the case in which the coefficient b is not zero in the telegraph equation (109), there can be no steady-state natural oscillations. The reason is that in that case, the parameter of the integral equation (111)

$$\lambda = \frac{\omega^2}{a^2}\left(1 + \frac{c}{\omega^2} + i\,\frac{b}{\omega}\right)$$

will be a complex number and, hence, cannot be an eigenvalue of the given integral equation since the eigenvalues of integral equations with real symmetric kernel are always real. This means that, if $b \neq 0$, the solution of the boundary-value problem (110) and the steady-state oscillations corresponding to it as described by the telegraph equation (109) exist and are unique for arbitrary values of $F = f_1 + if_2$, i.e., for an arbitrary exciting force $f = f_1 \cos\omega t - f_2 \sin\omega t$. Here, we see a qualitative difference between the steady-state oscillations corresponding to the wave equation and the steady-state oscillations corresponding to the telegraph equation with $b \neq 0$. We can give a physical interpretation to this difference. The point is that if $b < 0$, the expression $b\,\partial u/\partial t$ can be regarded as the force of friction that comes about when the oscillations take place. When there is friction, it is natural to expect no natural oscillations. As a result of a protracted action of an

exciting force with arbitrary given frequency $f = f_1 \cos\omega t - f_2 \sin\omega t$, the influence of any initial conditions disappears and oscillations will be set up that will "follow" exactly the force causing them. This represents steady-state oscillations.

As a result of this last phenomenon, we find it desirable to examine the following *"boundary-value problems without initial conditions"* for the telegraph equation.

In a bounded region G of x-space, find a twice continuously differentiable solution of the telegraph equation

$$\Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \frac{b}{a^2}\frac{\partial u}{\partial t} + \frac{c}{a^2}u = -\frac{f}{a^2} \quad (b \neq 0), \tag{109'}$$

that satisfies the boundary conditions

$$\varkappa u|_S = \Phi(P, t), \tag{109''}$$

on the surface S bounding the region G, where $\Phi(P, t)$ is a given real function of the point P on the surface S and the time t and where $\varkappa u$ is the linear operator of the boundary conditions. Here, if $\varkappa u|_S = u|_S$, we shall speak of the first interior boundary-value problem; if $\varkappa u\Big|_S = \frac{\partial u}{\partial n}\Big|_S - 0$, we shall speak of the second boundary-value problem; if $\varkappa u\Big|_S = \left(\frac{\partial u}{\partial n} - \alpha u\right)_S$, where α is a positive time-independent function on the surface S, we shall speak of the third interior boundary-value problem.

Let us note first of all that we may assume that the boundary function $\Phi(P, t)$ in the problem (109'), (109'') is equal to zero since, in the opposite case, we would only need to subtract from the desired solution a twice continuously differentiable function satisfying the nonzero boundary conditions. Thus, let us consider the boundary-value problem

$$L(u) = \Delta u - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \frac{b}{a^2}\frac{\partial u}{\partial t} + cu = -\frac{f}{a^2} \quad (b \neq 0), \tag{109'}$$

$$\varkappa u|_S = 0. \tag{109'''}$$

We shall consider two cases separately, depending on the form of the function $f = f(P, t)$.

(a) Suppose that $f(P, t)$ is a twice continuously differentiable function that is periodic with respect to t with period $2l$. Let us show that, in this case, the solution of the problem without initial conditions (109'), (109''') is unique in the class of functions $u(P, t)$ possessing continuous derivatives of the first four orders and periodic with respect to t with period $2l$. Every solution $u(P, t)$ of the problem (109'), (109''') possessing these properties can be

represented in the form of a Fourier series that can be differentiated termwise twice

$$u(P, t) = \frac{v_0}{2} + \sum_{n=1}^{\infty} \left(v_n \cos \frac{n\pi}{l} t - v_n^* \sin \frac{n\pi}{l} t \right), \tag{112}$$

where v_n and v_n^* are the Fourier coefficients of the function $u(P, t)$:

$$v_n = \frac{1}{l} \int_{-l}^{l} u(P, \xi) \cos \frac{n\pi}{l} \xi \, d\xi, \quad -v_n^* = \frac{1}{l} \int_{-l}^{l} u(P, \xi) \sin \frac{n\pi}{l} \xi \, d\xi.$$

If we substitute (112) into (109′), we obtain

$$L(u) = \frac{1}{2} L(v_0) + \sum_{n=1}^{\infty} L \left(v_n \cos \frac{n\pi}{l} t - v_n^* \sin \frac{n\pi}{l} t \right) =$$

$$= -\frac{1}{a^2} \left\{ \frac{f_0}{2} + \sum_{n=1}^{\infty} \left(f_n \cos \frac{n\pi}{l} t - f_n^* \sin \frac{n\pi}{l} t \right) \right\}, \tag{112′}$$

where f_n and $-f_n^*$ are the Fourier coefficients of the function $f(P, t)$:

$$f_n = \frac{1}{l} \int_{-l}^{l} f(P, \xi) \cos \frac{n\pi}{l} \xi \, d\xi, \quad -f_n^* = \frac{1}{l} \int_{-l}^{l} f(P, \xi) \sin \frac{n\pi}{l} \xi \, d\xi.$$

But, as is clear from the expression for the Fourier coefficients of the function $u(P, t)$, each of the functions

$$u_n(P, t) = v_n \cos \frac{n\pi}{l} t - v_n^* \sin \frac{n\pi}{l} t, \quad n = 0, 1, \ldots, \tag{113}$$

satisfies the boundary condition $\varkappa u_n|_S = 0$ and

$$L(u_n) = A_n \cos \frac{n\pi}{l} t - A_n^* \sin \frac{n\pi}{l} t,$$

where the A_n and A_n^* do not depend on t. Therefore, in view of the linear independence of the functions $\cos(n\pi t/l)$ and $\sin(n\pi t/l)$, we conclude that each of the functions $u_n(P, t)$ must be a solution of the corresponding problem on steady-state oscillations:

$$L(u_n) = -\frac{1}{a^2}\left(f_n \cos\frac{n\pi}{l}t - f_n^* \sin\frac{n\pi}{l}t\right),$$
$$\varkappa u_n|_S = 0, \qquad n = 0, 1, \ldots \tag{114}$$

But we have seen that the solution of each of the problems for $b \neq 0$ exists and is unique. Consequently, the solution of the problem without initial conditions (109′), (109″) is unique. We note that this same line of reasoning can be used to prove the existence of a solution of the problem (109′), (109‴) and to obtain it in explicit form. To do this, we need only construct the series (112) of solutions of the boundary-value problems for the steady-state oscillations (114) and then show that this series converges uniformly and admits two termwise differentiations. This last will be possible if we require that the Fourier coefficients of the function $f(P, t)$, namely, f_n and $-f_n^*$, decrease sufficiently rapidly as n increases.

(b) Suppose that $f(P, t)$, as a function of t, is continuous and of bounded variation in every finite interval and absolutely integrable in the interval $(-\infty, \infty)$. Then, as we know, it can be represented in the form of a Fourier integral

$$f(P, t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty f(P, \xi)\cos\omega(t-\xi)\,d\xi \tag{115}$$

or

$$f(P, t) = \int_0^\infty \tilde{f}(P, t, \omega)\,d\omega, \tag{115′}$$

where

$$\tilde{f}(P, t, \omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(P, \xi)\cos\omega(t-\xi)\,d\xi. \tag{115″}$$

Let us show that, in this case, the solution of the problem without initial conditions (109′), (109‴) is unique in the class of functions $u(P, t)$ that have continuous partial derivatives of second order and that satisfy the following conditions as $t \to \pm\infty$:

$$|u|, \left|\frac{\partial u}{\partial x_i}\right|, \left|\frac{\partial^2 u}{\partial x_i^2}\right|, \left|\frac{\partial u}{\partial t}\right|, \left|\frac{\partial^2 u}{\partial t^2}\right| < \frac{A}{|t|^{1+\varepsilon}},\ A, \varepsilon = \text{const}, \varepsilon > 0. \tag{116}$$

Under the conditions listed, the function $u(P, t)$ can be represented in the form of a Fourier integral

$$u(P, t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty u(P, \xi)\cos\omega(t-\xi)\,d\xi] \tag{117}$$

or

$$u(P, t) = \int_0^\infty \tilde{u}(P, t, \omega)\, d\omega, \tag{117'}$$

where

$$\tilde{u}(P, t, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(P, \xi) \cos \omega (t - \xi)\, d\xi. \tag{117''}$$

It is particularly important that the function $u(P, t, \omega)$ satisfy the boundary conditions $\tilde{\varkappa u}|_S = 0$ for arbitrary values of the parameter ω and, by virtue of conditions (116), admit two differentiations with respect to the x_i and t under the integral sign. In view of this, we obtain

$$L(\tilde{u}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \left[\Delta u(P, \xi) + \frac{c}{a^2} u(P, \xi) + \frac{\omega^2}{a^2} u(P, \xi) \right] \cos \omega (t - \xi) - \right.$$

$$\left. - \frac{b\omega}{a^2} u(P, \xi) \sin \omega (t - \xi) \right\} d\xi.$$

If we integrate the last two terms by parts, we obtain, by virtue of conditions (116),

$$L[\tilde{u}(P, t, \omega)] = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos \omega (t - \xi)\, L_\xi (u(P, \xi))\, d\xi,$$

where $L_\xi (u(P, \xi))$ is the operator on the left-hand side of Eq. (109') in which the variable t is replaced with the variable ξ. But, if we replace t in Eq. (109') with ξ and then multiply both sides of the equation by $(1/\pi) \cos \omega (t - \xi)$ and integrate with respect to ξ from $-\infty$ to ∞, we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \cos \omega (t - \xi)\, L_\xi (u(P, \xi))\, d\xi =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(P, \xi) \cos \omega (t - \xi)\, d\xi = \tilde{f}(P, t, \omega).$$

Thus, we conclude that each of the functions $\tilde{u}(P, t, \omega)$ must be a solution of the boundary-value problem on steady-state oscillations

$$L(\tilde{u}(P, t, \omega)) = \tilde{f}(P, t, \omega),$$
$$\varkappa\tilde{u}(P, t, \omega)|_S = 0, \qquad 0 \leqslant \omega < \infty. \tag{118}$$

Since $b \neq 0$, this boundary-value problem has a unique solution. Consequently, the solution of the boundary-value problem without initial conditions (109′), (109‴) is unique. These considerations also show how we might prove the existence of a solution of the boundary-value problem (109′), (109‴) and to find it in explicit form. The point is that if the integral (117′), in which $\tilde{u}(P, t, \omega)$ is a solution of the steady-state oscillation boundary-value problem (118), converges uniformly and can be differentiated twice under the integral sign, it is a solution of the boundary-value problem without initial conditions (109′), (109‴).

We note that if we assume in the statement of the boundary-value problem without initial conditions (109′), (109‴) that the coefficient b vanishes, the results obtained lose their validity. In case (a), one of the frequencies $n\pi/l$ (for $n = 0, 1, \ldots$) may coincide with the natural frequency, and, in this case, the problem (109′), (109‴) may not have any solution or it may have more than one solution. If none of these frequencies coincide with the natural frequency, the problem (109′), (109‴) has a solution but on it are superimposed the natural oscillations, to determine which we need to take some initial conditions or other into consideration. In case (b), the frequencies ω (for $0 \leqslant \omega < \infty$) necessarily include some that will coincide with the natural frequencies. Therefore, there may not be a solution. If there is a solution, again the natural oscillations (in the form of a linear combination of them with constant coefficients) are superimposed on it. A unique determination of the solution again requires that we consider some initial conditions or other; without the initial conditions, the solution will not be unique.

3. Solution of the steady-state-oscillation problem for space. Radiation conditions. Let us now look at the exterior boundary-value problems for steady-state oscillations for the wave equation (1).

Let us consider the problem on steady-state oscillations for Eq. (1) for all space. As was pointed out above (subsection 1), this problem is equivalent to finding solutions $v = v_1 + iv_2$ of the amplitude equation

$$\Delta v + k^2 v = -\frac{F}{a^2}\left(k = \frac{\omega}{a},\ F = f_1 + if_2\right), \tag{96}$$

that are twice continuously differentiable throughout all x-space and that satisfy the radiation conditions at infinity. From a physical point of view, the radiation conditions must single out from all the solutions of Eq. (96) those solutions that represent

steady-state oscillations $u = \mathrm{Re}\,[ve^{i\omega t}]$ that are propagated in the direction toward infinity but not in the direction away from infinity. From a mathematical point of view, these conditions must ensure uniqueness of the desired solution of Eq. (96).

First, let us assume for definiteness that the number of independent variables in Eq. (96) is three. Let us find solutions of Eq. (96) for $F=0$ that depend only on $r=\sqrt{x^2+y^2+z^2}$. We recall that the Laplacian operator in spherical coordinates is

$$\Delta v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2}$$

(see Chapter III, Sect. 3, subsection 1). Therefore, under the conditions stipulated, we have the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) + k^2 v = 0$$

or

$$\frac{1}{r}\frac{d^2}{dr^2}(rv) + k^2 v = 0$$

for finding the solutions mentioned. As can easily be seen, the functions

$$v^* = \frac{e^{-ikr}}{4\pi r}, \quad v^{**} = \frac{e^{ikr}}{4\pi r}, \tag{119}$$

are linearly independent solutions of this equation, and the steady-state oscillations corresponding to them may be written in the form

$$u^* = \mathrm{Re}\,\frac{e^{i(\omega t-kr)}}{4\pi r}, \quad u^{**} = \mathrm{Re}\,\frac{e^{i(\omega t+kr)}}{4\pi r}. \tag{119'}$$

From a physical point of view u^* differs from u^{**} only in that u^* corresponds to a divergent, i.e., outgoing, spherical wave, and the function u^{**} corresponds to a convergent, i.e., incoming, spherical wave. The latter wave comes from infinity to the coordinate origin since, for a fixed value of the variable $\omega t+kr$, the distance r from the coordinate origin decreases with increasing t. It is easy to see that the function $u^* = \cos(\omega t - kr)/4\pi r$ is a wave source at the coordinate origin with intensity equal to $\cos\omega t$ (see Sect. 7, subsection 1). From a mathematical point

of view, the solution $v=v^*$ differs from the solution v^{**} in that for the former the so-called *Sommerfeld radiation conditions**

$$v = O\left(\frac{1}{r}\right), \quad \Omega(v) = \frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \quad (120)$$

are satisfied as $r\to\infty$. This means that the quantity rv remains bounded and the quantity $r\Omega(r)$ approaches 0 as $r\to\infty$. Only the first of conditions (120) are satisfied for the solution $v=v^{**}$.

It is important to note that if we put a wave source of intensity $\cos\omega t$ not at the coordinate origin but at some point P_0 with coordinates x_0, y_0, and z_0, the solution of Eq. (96) for $F=0$ corresponding to it

$$\delta_k(P, P_0) = \frac{e^{-ikr}}{4\pi r} \qquad (r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}) \quad (121)$$

would also satisfy conditions (120). To see this, we set

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = R,$$

$$\sqrt{x^2+y^2+z^2} = r, \ \sqrt{x_0^2+y_0^2+z_0^2} = r_0$$

(see Fig. 34), and get

$$\Omega(\delta_k(P_0, P)) = \frac{\partial\delta_k}{\partial r} + ik\delta_k = \frac{\partial\delta_k}{\partial R}\frac{\partial R}{\partial r} + ik\delta_k.$$

Since

$$R = \sqrt{r^2+r_0^2-2rr_0\cos\varphi}, \ \frac{\partial R}{\partial r} = \frac{r-r_0\cos\varphi}{R} = 1 + O\left(\frac{1}{r}\right),$$

we obtain

$$\Omega(\delta_k) = \frac{\partial\delta_k}{\partial R} + ik\delta_k + \frac{\partial\delta_k}{\partial R}O\left(\frac{1}{r}\right).$$

Since

$$\frac{\partial\delta_k}{\partial R} + ik\delta_k = O\left(\frac{1}{R}\right) = O\left(\frac{1}{r}\right), \ \frac{\partial\delta_k}{\partial R} = O\left(\frac{1}{R}\right) = O\left(\frac{1}{r}\right),$$

we see that the function $\delta_k(P, P_0)$ does indeed satisfy conditions (120). Therefore, it is natural to expect that the Sommerfeld

*It may be shown that the first of these conditions is a consequence of the second. See I. N. Vekua, *Trudy Tbilisskogo matematicheskogo instituta*, 12, 1943, and F. Rellich, *Jahresber. Deut. Math. Vereinigung*, 53, 1943.

conditions (120) will be radiation conditions, i.e., that, when the corresponding steady-state-oscillation problem is solved, conditions (120) will ensure uniqueness of the solution of these problems. From a physical point of view, out of all possible steady-state oscillations in space with the given frequency ω, those steady-state oscillations that correspond to outgoing waves, i.e., to waves that go out to infinity rather than come in from infinity, will be singled out. Now, our problem is to show that this is indeed the case if we assume that the right-hand member of Eq. (96) is not zero only in some bounded region and is Hölder continuous at points in the closure of that region.

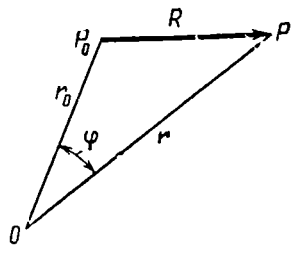

Fig. 34.

Suppose that $v(P)$ is a solution of Eq. (96) that is twice continuously differentiable in the closed region $G \cup S$. Then, at points of the domain G,

$$v(P_0) = \frac{1}{4\pi} \iint_S \left[v \frac{\partial}{\partial n} \left(\frac{e^{-ikr}}{r} \right) - \frac{e^{-ikr}}{r} \frac{\partial v}{\partial n} \right] dS + \\ + \frac{1}{4\pi a^2} \iiint_G F(P) \frac{e^{-ikr}}{r} d\tau \qquad (r = |P - P_0|). \tag{121'}$$

This can be shown by using Ostrogradskiy's formula and reasoning as in the derivation of the analogous formula for solutions of Poisson's equation (see Chapter III, Sect. 2, subsection 2).

Let us suppose now that the function $F(P)$ is identically equal to 0 outside the domain G and let us apply formula (121′) to the interior of a sphere, Σ with sufficiently large radius R, and with center at the coordinate origin. Then, we obtain

$$v(P_0) = -\frac{1}{4\pi} \iint_\Sigma \left[v \frac{\partial}{\partial r} \left(\frac{e^{-ikr}}{r} \right) - \frac{e^{-ikr}}{r} \frac{\partial v}{\partial r} \right] dS + \\ + \frac{1}{4\pi a^2} \iiint_G F(P) \frac{e^{-ikr}}{r} d\tau. \tag{121''}$$

The last term in the right-hand member of this equation obviously satisfies conditions (120) since the function $\delta_k(PP_0) = e^{-ikr}/4\pi r$ satisfies these conditions. In view of conditions (120), we obtain the following expression for the integrand in the first integral in the right-hand member of Eq. (121″):

$$v\frac{\partial\delta_k}{\partial r}-\delta_k\frac{\partial v}{\partial r}=v\left[-ik\delta_k+o\left(\frac{1}{r}\right)\right]-\delta_k\left[-ikv+o\left(\frac{1}{r}\right)\right]=$$
$$=vo\left(\frac{1}{r}\right)-\delta_k o\left(\frac{1}{r}\right)=o\left(\frac{1}{r^2}\right).$$

If we now take the limit in Eq. (121″) as $R\to\infty$, we conclude that every solution of Eq. (96) that satisfies conditions (120) must be of the form

$$v(P_0)=\frac{1}{4\pi a^2}\iiint_G F(P)\frac{e^{-ikr}}{r}d\tau. \tag{122}$$

Therefore, if such a solution exists it is unique and, consequently, we may treat conditions (120) as radiation conditions. It is easy to show that, if the function $F(P)$ is Hölder continuous in the closed region $G \cup S$ and identically equal to 0 outside this region and at points of the surface S, then the right-hand member of Eq. (122) is a solution of the boundary-value problem for Eq. (96) for all space. For this, it will be sufficient to show that the integral in the right-hand member of Eq. (122) has continuous first and second derivatives and satisfies Eq. (96). This is easily done by comparing the integral (122) with the potential

$$I(P_0)=\frac{1}{4\pi}\iiint_G\frac{F(P)}{a^2}\frac{1}{r}d\tau$$

as we did more than once in connection with integrals analogous to the integral (122) [see, for example, Sect. 7, subsection 1].

Let us now look at the question of the radiation conditions and the question of the solution of the boundary-value problem for Eq. (96) for all space in the case of a single independent variable. In this case, the solutions of the homogeneous equation (96) that depend only on $r=|x-x_0|$ are determined from the ordinary differential equation $d^2v/dr^2+k^2v=0$ and they represent a linear combination of the two functions

$$v^*=\delta_k(x,x_0)=\frac{e^{-ikr}}{2ki},\quad v^{**}=\frac{e^{ikr}}{2ki}\qquad(r=|x-x_0|). \tag{123}$$

It is easy to see that the first of these functions corresponds to outgoing waves since the steady-state oscillations corresponding to them are of the form

$$u^*=\mathrm{Re}\frac{e^{i(\omega t-kr)}}{2ki}.$$

From a mathematical point of view, a function $v^* = \delta_k(x, x_0)$ is characterized by the fact that the conditions

$$v = O(1), \quad \Omega(v) = \frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \tag{124}$$

are satisfied as $r \to \infty$. The second of the functions (123) satisfies only the first of these conditions.

Suppose that the right-hand member of the one-dimensional equation (96) $F(x)$ is not zero only on a finite interval $\alpha \leqslant x \leqslant \beta$. If we integrate the identity

$$\delta_k L(v) - v L(\delta_k) = \frac{d}{dx}\left[\delta_k \frac{\partial v}{\partial x} - v \frac{\partial \delta_k}{\partial x}\right], \quad L(v) = \frac{d^2v}{dx^2} + k^2 v$$

from $-l$ to l and remember that the derivative of the function $\delta_k(x, x_0)$ undergoes a jump equal to -1 at the point $x = x_0$, we obtain

$$\delta_k L(v)\Big|_{-l}^{l} = \left(\delta_k \frac{\partial v}{\partial x} - v \frac{\partial \delta_k}{\partial x}\right)\Big|_{-l}^{l} - v(x_0).$$

From this we conclude that for every twice continuously differentiable solution of the one-dimensional equation (96),

$$v(x_0) = \left(\delta_k \frac{\partial v}{\partial x} - v \frac{\partial \delta_k}{\partial x}\right)\Big|_{-l}^{l} + \frac{1}{2a^2 ki} \int_{-l}^{l} e^{-ikr} F(x)\, dx. \tag{125}$$

Assuming that the function $F(x)$ is nonzero only on the interval $[\alpha, \beta]$ and that $v(x)$ satisfies conditions (124) as $l \to \infty$, we obtain

$$v(x_0) = \frac{1}{2a^2 ki} \int_{\alpha}^{\beta} e^{-ikr} F(x)\, dx. \tag{126}$$

It is easy to show that the right-hand member of this equation satisfies conditions (124) and that it satisfies Eq. (96) for every continuous function $F(x)$ which is different from zero only on the interval $[\alpha, \beta]$. This last assertion follows from the equations

$$v(x_0) = \frac{1}{a^2} \int_{\alpha}^{x_0} \delta_k(x_0, x) F(x)\, dx + \frac{1}{a^2} \int_{x_0}^{\beta} \delta_k(x_0, x) F(x)\, dx,$$

$$\frac{dv(x_0)}{dx_0} = \frac{1}{a^2} \int_{\alpha}^{x_0} \frac{\partial \delta_k(x_0, x)}{\partial x_0} F(x)\, dx + \frac{1}{a^2} \int_{x_0}^{\beta} \frac{\partial \delta_k(x_0, x)}{\partial x_0} F(x)\, dx,$$

$$\frac{d^2v(x_0)}{dx_0^2} = \frac{1}{a^2}\int_\alpha^{x_0} \frac{\partial^2\delta_k(x_0, x)}{\partial x_0^2} F(x)\,dx + \frac{1}{a^2}\int_{x_0}^{\beta} \frac{\partial^2\delta_k(x_0, x)}{\partial x_0^2} F(x)\,dx +$$

$$+ \frac{F(x_0)}{a^2}\frac{\partial\delta_k(x_0, x)}{\partial x_0}\Big|_{x_0+0}^{x_0-0} = \frac{1}{a^2}\int_\alpha^\beta -F(x)\,k^2\delta_k(x_0, x)\,dx - \frac{F(x)}{a^2} =$$

$$= -k^2 v(x_0) - \frac{F(x)}{a^2}.$$

Steady-state oscillations in one-dimensional space corresponds to the complex amplitude (126), i.e., solutions of the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = -\frac{f_1\cos\omega t - f_2\sin\omega t}{a^2},$$

of the form $u = v_1(x)\cos\omega t - v_2(x)\sin\omega t$ can be written as follows:

$$u(x_0, t) = \mathrm{Re}\,[v(x_0)\,e^{i\omega t}] = \mathrm{Re}\,\frac{1}{2a^2ki}\int_\alpha^\beta e^{i(\omega t - kr)}F(x)\,dx =$$

$$= \frac{1}{2a^2k}\int_\alpha^\beta [f_2(x)\cos(\omega t - kr) + f_1(x)\sin(\omega t - kr)]\,dx \quad (r = |x - x_0|).$$

As a result, we conclude that in the case of the one-dimensional equation (96), Eqs. (124) are radiation conditions. A solution of the problem for the one-dimensional equation (96) for all space that satisfies the radiation conditions (124) for every continuous right-hand member of the equation $F(x)$ that is different from zero only on the interval $[\alpha, \beta]$ exists, is unique, and is given by Eq. (126).

Let us turn now to the case of two-dimensional space. Let us exhibit the radiation conditions and find the solution of the boundary-value problem for all space for Eq. (96).

The Laplacian operator in polar coordinates takes the form

$$\Delta v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2}.$$

Therefore, solutions of Eq. (96) for $F = 0$ that depend only on $r = \sqrt{x^2 + y^2}$ are determined from the equation

$$\frac{d^2v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} + v = 0, \tag{127}$$

where $\rho = kr$. This last equation is Bessel's equation of order zero (see Sect. 9, subsection 4). A particular solution of Bessel's equation of order zero and imaginary argument

$$\frac{d^2v}{d\rho^2} + \frac{1}{\rho}\frac{dv}{d\rho} - v = 0$$

is already known to us. It takes the form (see Chapter III, Sect. 5, subsection 3)

$$K_0(\rho) = \int_1^\infty \frac{e^{-\rho\xi}}{\sqrt{\xi^2 - 1}}\, d\xi.$$

Therefore, the functions

$$v^* = K_0(i\rho) = \int_1^\infty \frac{e^{-i\rho\xi}}{\sqrt{\xi^2-1}}\, d\xi, \quad v^{**} = K_0(-i\rho) = \int_1^\infty \frac{e^{i\rho\xi}}{\sqrt{\xi^2-1}}\, d\xi \qquad (128)$$

will be solutions of Eq. (127). These functions, like the functions $K_0(\rho)$, have singularities of the type $\ln(1/\rho)$ as $\rho \to 0$.*

Let us look at the asymptotic behavior of these functions as $\rho \to \infty$. If we set $\rho(\xi - 1) = \eta^2$, we have

$$K_0(i\rho) = \int_0^\infty \frac{e^{-i\eta^2 - i\rho}}{\sqrt{\left(1 + \frac{\eta^2}{\rho}\right)^2 - 1}} \frac{2\eta}{\rho} d\eta =$$

$$= \sqrt{\frac{2}{\rho}}\, e^{-i\rho} \int_0^\infty \frac{e^{-i\eta^2}}{\sqrt{1 + \frac{\eta^2}{2\rho}}}\, d\eta, \qquad (129)$$

*If we set $\rho\xi = \eta$, we have

$$K_0(i\rho) = \int_\rho^\infty \frac{e^{-i\eta}}{\sqrt{\eta^2 - \rho^2}}\, d\rho = \int_\rho^A \frac{d\eta}{\sqrt{\eta^2-\rho^2}} + \int_\rho^A \frac{e^{-i\eta} - 1}{\sqrt{\eta^2-\rho^2}}\, d\eta +$$

$$+ \int_A^\infty \frac{e^{-i\eta}}{\sqrt{\eta^2-\rho^2}}\, d\eta.$$

The last two integrals in the right-hand member of this equation remain bounded as $\rho \to 0$. The first of these is equal to the expression

$$\ln(\eta + \sqrt{\eta^2 - \rho^2})\Big|_\rho^A = \ln \frac{A + \sqrt{A^2 - \rho^2}}{\rho}$$

which has a singularity of the type $\ln(1/\rho)$.

Therefore, as $\rho \to \infty$

$$K_0(i\rho) = \sqrt{\frac{2}{\rho}}\, e^{-i\rho}(A+\varepsilon),\quad A = \int_0^\infty e^{-i\eta^2}\,d\eta = \frac{\sqrt{\pi}}{2}\, e^{-i\frac{\pi}{4}}, \tag{130}$$

where ε tends to 0 as $\rho \to \infty$.** Analogously, we obtain

$$K_0(-i\rho) = \sqrt{\frac{2}{\rho}}\, e^{i\rho}(\bar{A}+\varepsilon'),\quad \bar{A} = \int_0^\infty e^{i\eta^2}\,d\eta = \frac{\sqrt{\pi}}{2}\, e^{i\frac{\pi}{4}}, \tag{130'}$$

where ε' tends to 0 as $\rho \to \infty$. Two things follow from the asymptotic formulas (130) and (130'). In the first place, the functions v^* and v^{**} defined by Eqs. (128) are linearly independent solutions of Eq. (127). In the second place, it is clear from these formulas that only the function v^* corresponds to outgoing waves. The order of decrease of both functions v^* and v^{**} as $\rho = kr \to \infty$ is the same. Therefore, to show a difference in the asymptotic behavior of these two functions, we need to look at their derivatives. If we differentiate Eq. (129), we obtain

$$\frac{\partial K_0(i\rho)}{\partial \rho} = -i\sqrt{\frac{2}{\rho}}\, e^{-i\rho}\int_0^\infty \frac{e^{-\eta^2}\,d\eta}{\sqrt{1+\frac{\eta^2}{2\rho}}} +$$

$$+ e^{-i\rho}\frac{\partial}{\partial \rho}\left\{\sqrt{\frac{2}{\rho}}\int_0^\infty \frac{e^{-i\eta^2}}{\sqrt{1+\frac{\eta^2}{2\rho}}}\,d\eta\right\} =$$

$$= -iK_0(i\rho) + e^{-i\rho}\int_0^\infty \frac{e^{-i\eta^2}}{\sqrt{1+\frac{\eta^2}{2\rho}}}\,d\eta\,\frac{\partial}{\partial \rho}\sqrt{\frac{2}{\rho}} +$$

$$+ e^{-i\rho}\sqrt{\frac{2}{\rho}}\,\frac{\partial}{\partial \rho}\int_0^\infty \frac{e^{-i\eta^2}\,d\eta}{\sqrt{1+\frac{\eta^2}{2\rho}}}.$$

Thus, we have

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* The integrals

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

are well known under the name Fresnel integrals (see, for example, V. I. Smirnov, *a Course of Higher Mathematics*, Vol. III, part 2, Reading, Massachusetts, 1964.

$$\frac{\partial K_0(i\rho)}{\partial\rho} + iK_0(i\rho) = e^{-i\rho}\int_0^\infty \frac{e^{-i\eta^2}d\eta}{\sqrt{1+\frac{\eta^2}{2\rho}}}\frac{\partial}{\partial\rho}\sqrt{\frac{2}{\rho}} + $$

$$+ e^{-i\rho}\sqrt{\frac{2}{\rho}}\int_0^\infty \frac{\eta^2 e^{-i\eta^2}d\eta}{4\rho^2\left(1+\frac{\eta^2}{2\rho}\right)^{3/2}}.$$

If we set $\eta^2/2\rho = \xi$ in the last integral, we obtain

$$\int_0^\infty \frac{\eta^2 e^{-i\eta^2}d\eta}{4\rho^2\left(1+\frac{\eta^2}{2\rho}\right)^{3/2}} = \int_0^\infty \frac{\xi e^{-i2\rho\xi}}{2(1+\xi)^{3/2}}d\xi =$$

$$= \int_0^\infty \frac{e^{-i2\rho\xi}d\xi}{2\sqrt{1+\xi}} - \int_0^\infty \frac{e^{-i2\rho\xi}d\xi}{2(1+\xi)^{3/2}}.$$

But the integrals in the right-hand member of the last equation tend to 0 as $\rho\to\infty$ since they can be represented as the sums of alternating series with terms whose absolute values decrease monotonically to zero. Thus, we obtain

$$\frac{\partial K_0(i\rho)}{\partial\rho} + iK_0(i\rho) = o\left(\frac{1}{\sqrt{\rho}}\right).$$

In an analogous manner we obtain the asymptotic equation

$$\frac{\partial K_0(-i\rho)}{\partial\rho} - iK_0(-i\rho) = o\left(\frac{1}{\sqrt{\rho}}\right).$$

As a result, we conclude that the first of the functions (128), $v=v^*$, differs from the second of these functions in that the following asymptotic estimates for it:

$$v = O\left(\frac{1}{\sqrt{r}}\right),\ \Omega(v) = \frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{\sqrt{r}}\right) \qquad (131)$$

are satisfied simultaneously as $r\to\infty$.

The function v^{**} satisfies the first of these conditions but not the second. If we set

$$\delta_k(z, z_0) = \frac{1}{2\pi}K_0(ir) \quad (r=|z-z_0|,\ z=x+iy,\ z_0=x_0+iy_0), \qquad (132)$$

it is easy to show that the function $\delta_k(z, z_0)$ also satisfies conditions (131). Indeed, by setting $R=|z-z_0|$, $r=|z|$, $r_0=|z_0|$ (see Fig. 34), we obtain

$$\Omega(\delta_k(z, z_0)) = \frac{\partial \delta_k}{\partial R}\frac{\partial R}{\partial r} + ik\delta_k = \frac{\partial \delta_k}{\partial R} + ik\delta_k + \frac{\partial \delta_k}{\partial R} O\left(\frac{1}{r}\right).$$

Since

$$\frac{\partial \delta_k}{\partial R} + ik\delta_k = o\left(\frac{1}{\sqrt{R}}\right) = o\left(\frac{1}{\sqrt{r}}\right), \quad \frac{\partial \delta_k}{\partial R} = O\left(\frac{1}{\sqrt{r}}\right) = O\left(\frac{1}{\sqrt{r}}\right),$$

we have

$$\Omega(\delta_k(z, z_0)) = o\left(\frac{1}{\sqrt{r}}\right) + O\left(\frac{1}{\sqrt{r}}\right) O\left(\frac{1}{\sqrt{r}}\right) = o\left(\frac{1}{\sqrt{r}}\right),$$

that is, conditions (131) are satisfied.

Let us show that conditions (131) are radiation conditions in the case of the boundary-value problem for the two-dimensional wave equation (96) for all space, i.e., that these conditions ensure uniqueness of the solution for that problem. In analogy with what we did in the case of the two-dimensional form of Laplace's equation for solutions of Eq. (96) that are twice continuously differentiable in the closed region of $D \cup C$, we obtain the following integral representation:

$$v(z_0) = \frac{1}{2\pi}\int_C \left[v\frac{\partial K_0(ikr)}{\partial n} - K_0(ikr)\frac{\partial v}{\partial n}\right] ds +$$

$$+ \frac{1}{2\pi a^2}\iint_D F(z) K_0(ikr)\, dx\, dy \qquad (z_0 \in D,\ r = |z - z_0|). \tag{133}$$

Let us suppose that the right-hand member of Eq. (96) is identically equal to zero outside the domain D and let us apply formula (133) to the interior of a circle C_1, of sufficiently large radius R, with center at the coordinate origin. We obtain

$$v(z_0) = -\frac{1}{2\pi}\int_{C_1} \left[v\frac{\partial K_0(ikr)}{\partial r} - K_0(ikr)\frac{\partial v}{\partial r}\right] ds +$$

$$+ \frac{1}{2\pi a^2}\iint_D F(z) K_0(ikr)\, dx\, dy. \tag{133'}$$

In view of conditions (131), we obtain for the integrand of the first two integrals in the right-hand member of this equation

$$v\frac{\partial K_0}{\partial r} - K_0\frac{\partial v}{\partial r} = v\left[-ikK_0 + o\left(\frac{1}{\sqrt{r}}\right)\right] - K_0\left[-ikv + o\left(\frac{1}{\sqrt{r}}\right)\right] =$$

$$= vo\left(\frac{1}{\sqrt{r}}\right) - K_0 o\left(\frac{1}{\sqrt{r}}\right) = O\left(\frac{1}{\sqrt{r}}\right)o\left(\frac{1}{\sqrt{r}}\right) -$$

$$-O\left(\frac{1}{\sqrt{r}}\right)o\left(\frac{1}{\sqrt{r}}\right) = o\left(\frac{1}{r}\right).$$

If we now take the limit in Eq. (133′) as $R \to \infty$, we conclude that every twice continuously differentiable solution of Eq. (96) that satisfies conditions (131) must be representable in the form

$$v(z_0) = \frac{1}{2\pi a^2}\iint_D K_0(ikr)\,F(z)\,dx\,dy \qquad (r = |z - z_0|) \tag{134}$$

and hence must be unique. Conversely, the right-hand member of Eq. (134) satisfies conditions (131) since the function $K_0(ikr)$ satisfies these conditions. Thus, we may regard conditions (131) as radiation conditions in two dimensional space.

If the function $F(z) = f_1 + if_2$ is identically equal to 0 outside the domain D and at points of its boundary C and if it is Hölder continuous in the closed region $D \cup C$, then, by comparing the integral (134) with the logarithmic potential of an area

$$I(z_0) = \frac{1}{2\pi}\iint_D \frac{F(z)}{a^2}\ln\frac{1}{r}\,dx\,dy \qquad (r = |z - z_0|)$$

we can show that the integral (134) is a solution of Eq. (96) that is twice continuously differentiable throughout all space. This means that, in the given case, the solution of the boundary-value problem for the two-dimensional equation (96) for space exists and is given by Eq. (134).

This settles the question for us concerning radiation conditions of the boundary-value problem for space for the amplitude equation (96) in three, two, and one independent variable. We note that the form of these radiation conditions remains the same when we consider a number of other noninterior boundary-value problems associated with Eq. (96). For example, when we study problems in the mathematical theory of diffraction, where one is dealing with the three-dimensional equation (96) with piecewise-constant coefficients k^2, to ensure uniqueness of the solution we require that the solution satisfy the radiation conditions (120) as we approach infinity.

In certain cases, in solving boundary-value problems for space and also other noninterior boundary-value problems for Eq.

(96), we impose on the desired solution not the radiation conditions but various other kinds of restrictions ensuring uniqueness of the desired solution. This may be done, for example, by introducing into Eq. (96) a complex coefficient k^2 or, what amounts to the same thing, by introducing into the wave equation an extra term corresponding to friction. In this case, we speak not about radiation conditions but about the principle of complex absorption or vanishing friction. Sometimes, we treat the solution of Eq. (96) as the limit of $ue^{i\omega t}$ as $t\to\infty$, where u is the solution of the Cauchy problem for the wave equation with right-hand member $-Fe^{i\omega t}/a^2$ and with homogeneous initial conditions. In this case, instead of radiation conditions, we speak of the so-called principle of limiting amplitude.*

11. Questions Associated With the Characteristic Equation. The propagation of a Wavefront. The Propagation of Discontinuities. Wave Dispersion

1. The equation of the propagation of discontinuities of solutions and generalized solutions. Many physical applications of the theory of the characteristics of hyperbolic and partial differential equations are based on the fact that the behavior of these equations on the characteristics is distinguished by a number of special peculiarities.

For definiteness, let us look at a linear hyperbolic equation of the following form:

$$L(u)=\sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i\,\partial x_k}-\frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+b\frac{\partial u}{\partial t}+cu=-f, \tag{135}$$

where $a_{ik}=a_{ki}$ are continuously differentiable functions of the variables $x_1, \ldots, x_n$ and where f, the b_i, b, and c are continuously differentiable functions of the variables $x_1, x_2, \ldots, x_n, t$.

THEOREM (on discontinuities of second order). *Let $u(x_i, t)$ denote solutions of equation (135) which are continuous and have continuous first derivatives on the surface $S: \xi=t-\omega(x_i)=0$. Then, among these solutions there exists one with the property that its second derivative along the normal to S has a discontinuity of the first kind on S, while its other second-order derivatives are continuous on S, the surface S will be a characteristic of equation (135).*

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* For greater detail on this question, see A. G. Sveshnikov, "Printsip izlucheniya," *Doklady Akad. nauk*, USSR, *73*, No. 5, 1950 and A. N. Tikhonov and A. A. Samarskiy, *Equations of Mathematical Physics*, New York, MacMillan, 1963, pp. 571-575, 580-591.

Proof: If we introduce new independent variables

$$\xi = t - \omega(x_i), \quad \xi_i = x_i \qquad (i = 1, 2, \ldots, n),$$

we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}, \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2}, \quad \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial \xi_i} - \frac{\partial u}{\partial \xi} p_i, \quad p_i = \frac{\partial \omega}{\partial x_i},$$

$$\frac{\partial^2 u}{\partial x_i \partial x_k} = \frac{\partial^2 u}{\partial \xi_k \partial \xi_i} - \frac{\partial^2 u}{\partial \xi \partial \xi_i} p_k - \frac{\partial^2 u}{\partial \xi \partial \xi_k} p_i + \frac{\partial^2 u}{\partial \xi^2} p_i p_k - \frac{\partial u}{\partial \xi} \frac{\partial^2 \omega}{\partial x_i \partial x_k}.$$

By substituting these equations into Eq. (135), we obtain

$$L(u) = \frac{\partial^2 u}{\partial \xi^2}\left(\sum_{i,k=1}^{n} a_{ik} p_i p_k - 1\right) - \sum_{i,k=1}^{n} a_{ik}\left(p_k \frac{\partial}{\partial \xi_i} + p_i \frac{\partial}{\partial \xi_k}\right)\frac{\partial u}{\partial \xi} -$$

$$- \frac{\partial u}{\partial \xi}\left(\sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 \omega}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial \omega}{\partial x_i} - b\right) + \ldots = -f,$$

where the dots indicate terms not containing derivatives with respect to the variable ξ. Remembering that $a_{ik} = a_{ki}$, we may write the last equation in the form

$$L(u) = \frac{\partial^2 u}{\partial \xi^2}\left(\sum_{i,k=1}^{n} a_{ik} p_i p_k - 1\right) - 2\sum_{i,k=1}^{n} a_{ik} p_k \frac{\partial^2 u}{\partial \xi \partial \xi_i} - \qquad (136)$$

$$- A \frac{\partial u}{\partial \xi} + \ldots = -f,$$

where

$$A = \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 \omega}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial \omega}{\partial x_i} - b \qquad (136')$$

and the dots have the same meaning as before.

It is obvious from Eq. (136) that, for the function u to satisfy the conditions of the theorem on the surface $S: \xi = 0$, it is necessary that

$$F = \sum_{i,k=1}^{n} a_{ik} p_i p_k - 1 = 0, \quad p_i = \frac{\partial \omega}{\partial x_i} \qquad (137)$$

or

$$\sum_{i,k=1}^{n} a_{ik} \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_k} - \left(\frac{\partial \xi}{\partial t} \right)^2 = 0.$$

But, from the definition of characteristics (see Chapter II, Sect. 2), this means that the surface $S: \xi = t - \omega(x_i) = 0$ is a characteristic of Eq. (135).

We shall call any surface S: $t = \omega(x_i)$ in the space $x_1, \ldots, x_n, t$ that satisfies the conditions of the theorem on discontinuities of second order a surface of discontinuities of second order. Thus, the assertion of that theorem is that every surface S of discontinuities of second order must be a characteristic. We shall call the projection of the intersection of the surface S with the plane t = const onto the space $x_1, \ldots, x_n$ a *wavefront of discontinuities of second order* and we shall denote this wavefront by S_t. The equation of the wavefront corresponding to the instant t is written in the form $\omega(x_i) = t = \text{const}$. With the passage of time, the wavefront S_t in the space $x_1, \ldots, x_n$ will be displaced in the direction of the vector $\nabla \omega$. It is easy to evaluate the magnitude of the velocity of this wavefront. Indeed, let $\vec{n}_0$ denote the direction of the vector $\nabla \omega$ at some point P lying on S_t. The wavefront $S_{t+\Delta t}$ at the instant $t+\Delta t$ intersects the direction of $\vec{n}_0$ at some point P_1 at a distance Δn_0 from the point P. We have

$$\lim_{\Delta t \to 0} \frac{\Delta n_0}{\Delta t} = \lim_{\Delta n_0 \to 0} \frac{1}{\frac{\Delta t}{\Delta n_0}} = \lim_{\Delta n_0 \to 0} \frac{1}{\frac{\Delta \omega}{\Delta n_0}} = \frac{1}{\frac{\partial \omega}{\partial n_0}} = \frac{1}{|\nabla \omega|}.$$

Thus, if we denote by $\vec{v}_{\text{fr}}$ the velocity vector of the wavefront S_t, we obtain

$$\vec{v}_{\text{fr}} = \frac{1}{|\nabla \omega|^2} \nabla \omega. \tag{138}$$

In particular, in the case in which $a_{11} = a_{22} = \ldots = a_{nn} = a^2$ and $a_{ik} = 0$ for $i \neq k$, we have, from the characteristic equation (137), $|\nabla \omega| = 1/a$, and Eq. (138) yields $|\vec{v}_{\text{fr}}| = a$.

The surface of discontinuities of second order S: $t = \omega(x_i)$ is a solution of the characteristic equation (137). Therefore, being an integral surface of a first-order partial differentiable equation, it is composed of characteristic curves. Characteristic curves for the first-order partial differential equation

$$F(x_i, u, p_i) = 0, \quad p_i = \frac{\partial u}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

are curves in the space $x_1, \ldots, x_n, u$ corresponding to the solution of the characteristic system of equations

$$\frac{dx_1}{F_{p_1}} = \ldots\ldots = \frac{dx_n}{F_{p_n}} = \frac{du}{\sum_{i=1}^{n} p_i F_{p_i}} = \frac{-dp_1}{p_1 F_u + F_{x_1}} = \ldots = \frac{-dp_n}{p_n F_u + F_{x_n}}$$

with the supplementary condition $F(x_i, u, p_i) = 0$ (see Chapter II, Sect. 3). Consequently, the characteristic curves corresponding to the characteristic equation (137) must be determined as the curves $x_i = x_i(s)$, $t = \omega(s)$ in the space $x_1, \ldots, x_n, t$ which satisfy the equations

$$\frac{dx_1}{2\sum_{k=1}^{n} a_{1k}p_k} = \ldots\ldots = \frac{dx_n}{2\sum_{k=1}^{n} a_{nk}p_k} = \frac{d\omega}{2} = \frac{-dp_1}{\sum_{i,k=1}^{n} p_i p_k \frac{\partial a_{ik}}{\partial x_1}} = \ldots =$$

$$= \frac{-dp_n}{\sum_{i,k=1}^{n} p_i p_k \frac{\partial a_{ik}}{\partial x_n}}, \quad \sum_{i,k=1}^{n} a_{ik}p_i p_k - 1 = 0. \tag{139}$$

These characteristic curves are called the *bicharacteristics* of Eq. (135), and their projections onto the space $x_1, \ldots, x_n$ are called rays.

Noting that $d\omega = dt$ in the system of equations (139), we see that along the rays

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}p_k \qquad (i = 1, 2, \ldots, n). \tag{140}$$

Therefore, taking the characteristic equation into account, we have

$$\sum_{i=1}^{n} \frac{dx_i}{dt} p_i = \sum_{i,k=1}^{n} a_{ik}p_i p_k = 1.$$

This equation means that the rays are not tangent to the front of discontinuities of the second kind S_t anywhere. In particular, if $a_{11} = a_{22} = \ldots = a_{nn} = a^2$, and $a_{ik} = 0$ for $i \neq k$, then $dx_i/dt = a^2 p_i$. Therefore, the rays are orthogonal to S_t.

The vector $\vec{v}_r$, whose components are

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik}p_k,$$

(for $i = 1, 2, \ldots, n$), is naturally called the ray velocity vector for the propagation of discontinuities of second order since it

is the velocity of propagation of these discontinuities in the direction of the rays. For the magnitude of the ray velocity vector, we have

$$|\vec{v}_r| = \sqrt{\sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} p_k\right)^2} \tag{141}$$

and, in particular, in the case in which $a_{ii}=a^2$ and $a_{ik}=0$ for $i \neq k$,

$$|\vec{v}_r| = \sqrt{\sum_{k=1}^{n} a^2 p_k^2} = \sqrt{a^2} = a. \tag{141'}$$

To characterize the behavior of the discontinuities as they are propagated along the rays, let us turn to Eq. (136). We can write the second of the terms in the left-hand side of this equation in the form

$$-2\sum_{i,k=1}^{n} a_{ik} p_k \frac{\partial^2 u}{\partial \xi\, \partial \xi_i} = -2\sum_{i=1}^{n} \frac{\partial^2 u}{\partial \xi\, \partial \xi_i} \sum_{k=1}^{n} a_{ik} p_k =$$

$$= -2\sum_{i=1}^{n} \frac{\partial^2 u}{\partial \xi\, \partial \xi_i} \frac{\partial x_i}{dt} = -2\frac{\partial}{\partial s} \frac{\partial u}{\partial \xi} \frac{ds}{dt} = -2\frac{\partial}{\partial t} \frac{\partial u}{\partial \xi},$$

where $\partial/\partial s$ is the derivative in the direction of the ray. In view of this, we can write Eq. (136) in the form

$$L(u) = -2\frac{\partial}{\partial t} \frac{\partial u}{\partial \xi} - A\frac{\partial u}{\partial \xi} + \ldots = -f, \tag{136''}$$

where the dots mean the same thing as in Eq. (136). Here, we have seen that the coefficient $\left(\sum_{i,k=1}^{n} a_{ik} p_i p_k - 1\right)$ of $\partial^2 u/\partial \xi^2$ vanishes both on the surface of $\xi = t - \omega(x_i) = 0$ and on the surfaces $\xi = t - \omega(x_i) = \text{const}$ since these last surfaces are characteristic surfaces. Let us now differentiate the expression (136'') with respect to ξ at some point P^+ lying on the ray on that side of the surface S_t: $\xi = 0$, in the direction of which the discontinuities are being propagated. Then, let us subtract from this expression the result obtained by differentiating the expression (136'') with respect to ξ at the point P^- lying on the same ray but on the opposite side of the surface S_t: $\xi = 0$. Now, let us take the limit of this difference as we let P^+ and P^- approach the point P lying on the same ray and on the surface S_t: $\xi = 0$. As a result, we obtain the so-called *equation for the propagation of discontinuities of second order.*

$$2\frac{\partial \mu}{\partial t} + A\mu = 0, \tag{142}$$

where $\mu = [\partial^2 u / \partial \xi^2]$ is the same jump in the second derivative $\partial^2 u / \partial \xi^2$ on the given ray as we cross the surface of discontinuities of second order $S: t = \omega(x_i)$. Equation (142) can also be written in the following form:

$$2 \frac{\partial \mu}{\partial s} \frac{ds}{dt} + A\mu = 0, \tag{142'}$$

where d/ds is the derivative in the direction of the ray and $ds/dt = |\vec{v}_r|$ is the magnitude of the vector denoting the ray velocity. In Eq. (142) or (142'). It is important to note that the magnitude A defined by Eq. (136') is known on the surface of discontinuities of second order S and, consequently, on the characteristic curves which form S, and, hence, on the rays. Thus, Eq. (142) can be regarded as an ordinary differential equation for the jump μ in the second derivative $\partial^2 u / \partial \xi^2$. If μ_0 denotes the value of μ on the given ray at the initial instant of time $t=0$, then at subsequent instants, the value of μ on the given ray is given by the equation

$$\mu = \mu_0 e^{-\frac{1}{2}\int_0^t A dt} \tag{142''}$$

Thus, for example, if at $t=0$ the value of μ is different from zero only on a certain portion of the surface constituting the wavefront S_t: $t=\omega(x_i)$, then at all subsequent instants, it will be different from zero only along rays issuing from that portion of the surface S_t, and only along those rays will the discontinuities in the second derivative $\partial^2 u / \partial \xi^2$ be propagated. We can say that here we shall have a sharp edge of a shadow.

All the principles established above with regard to the behavior of the discontinuities of the derivative $\partial^2 u / \partial \xi^2$ remain valid for higher-order discontinuities, i.e., for discontinuities in the derivatives $\partial^n u / \partial \xi^n$ (for $n \geqslant 2$) if we assume that, on the surface $S: t = \omega(x_i)$, the function u and all the quantities appearing in Eq. (135) together with their first $n-1$ derivatives are continuous and that in addition all the nth order derivatives of the function u except for $\partial^n u / \partial \xi^n$ are continuous on that surface. That the surface $S: \xi = t - \omega(x_i) = 0$ in this case must be a characteristic follows, just as for the case of $n = 2$, from Eq. (136). The fact that the jump μ in the derivative $\partial^n u / \partial \xi^n$ along the given ray satisfies the ordinary differential equation (142) follows from Eq. (136'') when differentiated $n-2$ times with respect to ξ.

With regard to first-order discontinuities, in $\partial u / \partial \xi$, and discontinuities in the function u itself, the matter becomes somewhat more complicated. First of all, we note that we can construct a solution of Eq. (135) in a purely formal manner for which the discontinuities in u or $\partial u / \partial \xi$ will exist along the surface

S: $\xi = t - \omega(x_i) = 0$, which is not a characteristic. For example, this may be done in the following manner: Let u_1 be a given solution of Eq. (135) in the strip $0 \leqslant t \leqslant t_0$. Then, to extend this solution, we take the function u which is a solution of the Cauchy problem for the portion of space $t > t_0$ with initial values of u and $\partial u / \partial t$ at $t = t_0$ that do not coincide with the corresponding values of u_1 and $\partial u_1 / \partial t$ at $t = t_0$. Thus, we obtain a solution of Eq. (135) which, together with its derivative $\partial u / \partial t$, has discontinuities on the surface $t - t_0 = \text{const}$, which is not a characteristic. However, if these purely formal ways of joining solutions are excluded from consideration, and if we consider only those solutions of Eq. (135) for which the discontinuities in the function and in the first derivatives originate in a well-defined manner, then, it turns out that for such discontinuities, all the principles that we have established for the discontinuities in $\partial^2 u / \partial \xi^2$ remain in effect. To single out solutions of Eq. (135) which have the property in which we are interested, let us introduce the concept of so-called *generalized solutions*. For definiteness, we introduce this concept as applied to the linear equation

$$L(u) = \sum_{i,k=1}^{n+1} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n+1} b_i \frac{\partial u}{\partial x_i} + cu = -f \quad (a_{ik} = a_{ki}). \tag{74}$$

Let u be an arbitrary solution of Eq. (74) that is twice continuously differentiable in the closed $(n+1)$-dimensional region $G \cup S$, and let v be an arbitrary function that is twice continuously differentiable in $G \cup S$ and vanishes, together with its first derivatives, on the boundary S of the deomain G. Then, if we apply Ostrogradskiy's formula to the equation

$$vL(u) - uM(v) = \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} \left(v \sum_{k=1}^{n+1} a_{ik} \frac{\partial u}{\partial x_i} - u \sum_{k=1}^{n+1} \frac{\partial}{\partial x_k} (a_{ik} v) + b_i u v \right), \tag{76}$$

where $M(v)$ is the operator adjoint to the operator $L(u)$, i.e.,

$$M(v) = \sum_{i,k=1}^{n+1} \frac{\partial^2}{\partial x_i \partial x_k} (a_{ik} v) - \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} (b_i v) + cv, \tag{75'}$$

we obtain

$$\overbrace{\int \cdots \int}^{n+1}_{G} uM(v)\, dx_1 \ldots dx_{n+1} = \overbrace{\int \cdots \int}^{n+1}_{G} vf\, dx_1 \ldots dx_{n+1} \tag{143}$$

(cf. Sect. 9, subsection 1). But if it is easy to see that, under the assumptions made regarding the function v, the relation (143) is

valid not only for twice continuously differentiable solutions of Eq. (74) but also for certain discontinuous solutions of that equation. Therefore, we can say that the function u is a generalized solution of Eq. (74) in the domain G if it satisfies the integral relation (143) for all functions v which have continuous second derivatives in the closed region $G \cup S$ and which vanish, along with their derivatives, on the surface S bounding the domain G. *

For example, suppose that a domain G in the xy-plane is composed of a region G_1 in the upper half-plane and a region G_2 in the lower half-plane. Let $[a, b]$ denote an interval of the x-axis along which G_1 and G_2 are adjacent. Let u denote a solution of the equation

$$L(u) = \frac{\partial^2 u}{\partial x \partial y} = 0,$$

that is twice continuously differentiable in G_1 and G_2 and at points of their boundaries which do not lie on $[a, b]$. In this case,

$$M(v) = \frac{\partial^2 v}{\partial x \partial y}$$

and the identity (76) takes the form

$$vL(u) - uM(v) = \frac{1}{2}\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial x}\right) + \\ + \frac{1}{2}\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right).$$

If we apply Green's formula (cf. footnote to page 315) to this identity for the regions G_1 and G_2, we obtain

$$\iint_{G_1} uM(v)\,dx\,dy = \frac{1}{2}\int_a^b \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dx = \int_a^b v\frac{\partial u}{\partial x}\,dx,$$

$$\iint_{G_2} uM(v)\,dx\,dy = -\frac{1}{2}\int_a^b \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dx = \\ = -\int_a^b v\frac{\partial u}{\partial x}\,dx.$$

* The concept of generalized solutions can also be introduced by means of a limiting process in the sense of convergence in mean by starting with twice continuously differentiable solutions, and also Chapter VI, Sect. 2, subsections 7 and 8 of the present book.

If we add these equations, we obtain

$$\iint_G uM(v)\,dx\,dy = \int_a^b v\left[\frac{\partial u}{\partial x}\right]dx,$$

where the difference

$$\left[\frac{\partial u}{\partial x}\right] = \frac{\partial u}{\partial x}\Big|_{y=+0} - \frac{\partial u}{\partial x}\Big|_{y=-0}$$

is the jump in the derivative $\partial u/\partial x$ on the interval $[a, b]$ of the x-axis. From this, we conclude that we may regard this function u as a generalized solution if $[\partial u/\partial x] = 0$. In particular, an expression of the form $u=\varphi(x)+\psi(y)$, where $\varphi(x)$ is a twice continuously differentiable function of x for $a<x<b$ and $\psi(y)$ is a twice continuously differentiable function of y for $-\infty<y<0$, $0<y<\infty$, will be a generalized solution with a discontinuity on the interval $[a, b]$, if, for $y=0$, the function $\psi(y)$ is discontinuous, and it will be a generalized solution with a discontinuity of first order on the interval $[a, b]$ if the function $\psi(y)$ is itself continuous but its derivative $d\psi(y)/dy$ is discontinuous at $y=0$. It is also easy to see that an expression of the form $u=\varphi(x)+\psi(y)$, where $\varphi(x)$ and $\psi(y)$ are arbitrary functions of their arguments except that $\varphi(x)$ is twice continuously differentiable everywhere away from the points $x=x_1, x_2, \ldots, x_k$ and where $\psi(y)$ is twice continuously differentiable everywhere away from the points $y=y_1, y_2, \ldots, y_m$, is a generalized solution of the equation

$$L(u) = \frac{\partial^2 u}{\partial x \partial y} = 0.$$

This generalized solution has discontinuities on straight lines parallel to the coordinate axes (i.e., on the characteristics) passing through the points (x_i, y_i) (for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$).

THEOREM (on discontinuities of first order). *Suppose that functions $u(x_i, t)$ are generalized solutions of Eq. (135) that are continuous on the surface $S: \xi = t - \omega(x_i) = 0$ and that are twice continuously differentiable at points not on that surface. Then, if one of these solutions of Eq. (135) is such that its derivative along the normal to S has a discontinuity of the first kind on S, while the remaining derivatives of first order are continuous, the surface S is a characteristic of Eq. (135).*

Proof: If we write Eq. (135) in the form (136) and apply Ostrogradskiy's formula (77) to the two subdomains into which the domain G in question in the space $\xi_1, \xi_2, \ldots, \xi_n, \xi$ is partitioned by the plane $L: \xi = 0$, we see that, by virtue of (143), the left-hand member of the equations of Ostrogradskiy's formula vanish and

the right-hand members of these formulas contain only integrals over L since, on the boundary of the domain G, the function v and its derivatives of first order vanish. If we add the results of the transformations performed, we obtain

$$\overbrace{\int\cdots\int}^{n}_{L} v\left[\frac{\partial u}{\partial \xi}\right]\left(\sum_{i,k=1}^{n} a_{ik}\, p_i\, p_k - 1\right) dS = 0. \qquad (143')$$

Remembering that the jumps in $[\partial u/\partial \xi] \neq 0$ and that v is an arbitrary twice continuously differentiable function on L, we conclude from this that

$$\sum_{i,k}^{n} a_{ik} p_i p_k = 1$$

on L, i.e., on the surface $S: \xi = t - \omega(x_i) = 0$. But this, as noted above, means that the surface $S: \xi = t - \omega(x_i) = 0$ is a characteristic of Eq. (135). This completes the proof of the theorem.

If we consider Eq. (136″) on the two sides of the surface S, we see that the jump in the derivative $\mu = [\partial u/\partial \xi]$ satisfies the ordinary differential equation (142) on the surface S.

In the case in which the solution of Eq. (135) is itself discontinuous on some surface, we shall speak of *strong discontinuities* as opposed to discontinuities of first and second order.

THEOREM (on strong discontinuities). *Suppose that $u(x_i, t)$ are generalized solutions of Eq. (135) which, on the surface S: $\xi = t - \omega(x_i) = 0$ have continuous derivatives in directions tangent to S, and which are twice continuously differentiable at points not on the surface S. Then, if among these solutions there is one that has a discontinuity of the first kind on S, the surface S is a characteristic of Eq. (135).*

Proof: Proceeding as in the proof of the preceding theorem, we obtain from Ostrogradskiy's formula (77) and Eq. (143′),

$$\overbrace{\int\cdots\int}^{n}_{L} \left\{ -[u]\left(\sum_{i,k=1}^{n} a_{ik}\, p_i\, p_k - 1\right)\frac{dv}{d\xi} + \ldots + \ldots \right\} dS, \qquad (143'')$$

where $[u]$ is the jump in the function u on the surface S and the dots denote terms which have either v or a derivative of v in a direction tangent to S as a multiplying factor. If we set $v|_L = 0$ and choose v so that the function $\partial v/\partial \xi|_L$ will assume on L the given values and if we recall that $[u] \neq 0$, we conclude that

$$\sum_{i,\,k}^{n} a_{ik} p_i p_k - 1 = 0,$$

i.e., that S is a characteristic of Eq. (135). This completes the proof of the theorem.

Let $u_n(x_i, t)$ (for $n = 1, 2, \ldots$) be a sequence of solutions of Eq. (135) which are twice continuously differentiable in some domain containing the surface S. Suppose that outside every neighborhood of the surface S, the solutions $u_n(x_i, t)$ converge uniformly as $n \to \infty$ to the generalized solution $u(x_i, t)$ mentioned in the conditions of the last theorem, and that the first and second derivatives of $u_n(x_i, t)$ converge uniformly to the corresponding derivatives of $u(x_i, t)$. Then, if we substitute $u_n(x_i, t)$ in (136″), integrate the resulting expression from $-\varepsilon$ to ε (where $\varepsilon>0$), and take the limit as $n \to \infty$, we obtain

$$-2\frac{\partial}{\partial t}\left\{u\Big|_{\xi=\varepsilon} - u\Big|_{\xi=-\varepsilon}\right\} - Au\Big|_{\xi=-\varepsilon}^{\xi=\varepsilon} + \int_{-\varepsilon}^{\varepsilon} u\frac{\partial A}{\partial \xi}\,d\xi = -\int_{-\varepsilon}^{\varepsilon} f\,d\xi.$$

If we take the limit of this as $\varepsilon \to 0$, we see that the jump of the solution $\mu=[u]$ satisfies the ordinary differential equation (142).

2. Solution of the problem of the propagation of a wavefront. The law of refraction of light. For the homogeneous equation corresponding to Eq. (135)

$$Lu = \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i\,\partial x_k} - \frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i} + b\frac{\partial u}{\partial t} + cu = 0 \qquad (135')$$

$$(a_{ik} = a_{ki})$$

at every instant of time t, a closed surface in x-space exists such that $u \not\equiv 0$ in the domain interior to the surface and $u \equiv 0$ in the exterior domain. We have been calling any such surface a wavefront.

The question arises as to how we can find the position of a wavefront at an arbitrary instant t if its position at the initial instant $t=0$ is known. Let us agree to call this problem *the problem of the propagation of a wavefront.*

Let us look at this problem under the assumption that the wavefront is a wavefront of discontinuities either of the solution itself or of its derivatives of the types studied by us in subsection 1. This means that the wavefront coincides with the surface in x-space $S_t : t=\omega(x_i)$. The surface $S_t : t=\omega(x_i)$, which we shall examine in $x_1, \ldots, x_n, t$ space, must be a characteristic of Eq. (135). Therefore, determination of the wavefront $S_t : t=\omega(x_i)$ at an instant t when we are given its position $S_0 : 0=\omega(x_i)$ at the initial instant $t=0$ reduces to solving the Cauchy problem for the characteristic equation

$$F = \sum_{i,k=1}^{n} a_{ik} p_i p_k - 1 = 0, \quad p_i = \frac{\partial \omega}{\partial x_i}, \tag{137}$$

$$x_i = x_i^0(t_1, t_2, \ldots, t_{n-1}), \quad \omega\Big|_{x_i = x_i^0} = 0 \quad (i = 1, 2, \ldots, n). \tag{137'}$$

Here, $t = \omega(x_i)$ is an unknown function, $x_1, \ldots, x_n$ are the independent variables, $x_i = x_i^0(t_1, \ldots, t_{n-1})$ is the equation of the surface S_0, i.e., of the initial wavefront in parametric form, where $t_1, \ldots, t_{n-1}$ are parameters.

The Cauchy problem (137), (137') can be interpreted as the problem of constructing in the $x_1, \ldots, x_n, t$-phase-space an n-dimensional integral surface $S_t : t = \omega(x_i)$ of Eq. (137) that passes through the initial $(n-1)$-dimensional surface $S_0 : 0 = \omega(x_i)$. To solve this equation, it is necessary to use the characteristic system of equations (139) that determines the characteristic curves from which it is possible to construct every integral surface of Eq. (137). To determine the initial values for the p_i (for $i = 1, 2, \ldots, n$) that appear in the system of equations (139), in accordance with the general theory (see Chapter II, Sect. 3), we must use the equations

$$\frac{\partial \omega}{\partial t_\nu} = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_\nu}, \quad F = 0 \quad (\nu = 1, 2, \ldots, n-1). \tag{137''}$$

Thus, construction of the desired integral surface reduces to solving the characteristic system of equations

$$\frac{dx_1}{2\sum_{k=1}^{n} a_{1k} p_k} = \ldots = \frac{dx_n}{2\sum_{k=1}^{n} a_{nk} p_k} = \frac{d\omega}{2} = \frac{-dp_1}{\sum_{i,k=1}^{n} p_i p_k \frac{\partial a_{ik}}{\partial x_1}} = \ldots = \frac{-dp_n}{\sum_{i,k=1}^{n} p_i p_k \frac{\partial a_{ik}}{\partial x_n}} = \frac{ds}{2} \tag{144}$$

with the initial conditions

$$\begin{aligned} x_i(t_1, \ldots, t_{n-1}, s)\Big|_{s=0} &= x_i^0(t_1, \ldots, t_{n-1}), \\ \omega(x_i(t_1, \ldots, t_{n-1}, s))\Big|_{s=0} &= 0 \qquad (i = 1, \ldots, n), \end{aligned} \tag{144'}$$

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t_\nu}\Big|_{s=0} = 0, \quad \sum_{i,k=1}^{n} a_{ik} p_i p_k\Big|_{s=0} - 1 = 0 \quad (\nu = 1, \ldots, n-1). \tag{144''}$$

The solution of this Cauchy problem for the characteristic equation (137), i.e., the desired integral surface of the equation, takes the form

$$x_i = x_i(t_1, \ldots, t_{n-1}, s),\ t = \omega(t_1, \ldots, t_{n-1}, s)\ (i = 1, 2, \ldots, n), \quad (145)$$

where $x_i(t_1, \ldots, t_{n-1}, s)$, $\omega(t_1, \ldots, t_{n-1}, s)$ is a solution of the problem for the system of ordinary differential equations (144), (144′), (144″). The wavefront $S_t: t=\omega(x_i)$ at the instant t is defined as the projection onto x-space of the intersection of the integral surface (145) with the plane t = const.

From the general theory of first-order partial differential equations (see Chapter II, Sect. 3), we note that, for the Cauchy problem for a first-order equation $F(x_i, \omega, p_i) = 0$ to have a unique solution, it is necessary that the surface on which the initial conditions are given not be a characteristic of the given first-order equation $F(x_i, \omega, p_i) = 0$, i.e., that the condition

$$\delta = \begin{vmatrix} F_{p_1} & \ldots\ldots & F_{p_n} \\ \dfrac{\partial x_1}{\partial t_1} & \ldots\ldots & \dfrac{\partial x_n}{\partial t_1} \\ \ldots & \ldots & \ldots \\ \dfrac{\partial x_1}{\partial t_{n-1}} & \ldots\ldots & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} \neq 0$$

be satisfied on the surface in question. Here, the values of the p_i are assumed to be uniquely determined from conditions (137″). This means that for the Cauchy problem (137), (137′) determining the wavefront $S_t: t=\omega(x_i)$ to have a unique solution, it is sufficient that the condition

$$\delta = \begin{vmatrix} 2\sum_{k=1}^{n} a_{1k} p_k^0 & \ldots & 2\sum_{k=1}^{n} a_{nk} p_k^0 \\ \dfrac{\partial x_1}{\partial t_1} & \ldots & \dfrac{\partial x_n}{\partial t_1} \\ \ldots & \ldots & \ldots \\ \dfrac{\partial x_1}{\partial t_{n-1}} & \ldots & \dfrac{\partial x_n}{\partial t_{n-1}} \end{vmatrix} \neq 0, \quad (146)$$

where the p_i^0 are the values of p_i determined by Eqs. (144″), be satisfied on the surface initial wavefront ($S_0: x_i = x_i^0\ (t_1, \ldots, t_{n-1})$, $t = \omega(x_i)\big|_{x_i = x_i^0} = 0$).

Let us illustrate these results for the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{147}$$

Here, if the initial wave front is given for $t=0$ in the form of a curve $S_0: x=x_0(t_1)$, $y=y_0(t_1)$, then the Cauchy problem (137), (137′) can be written in the form

$$a^2(p_1^2 + p_2^2) - 1 = 0, \quad p_1 = \frac{\partial \omega}{\partial x}, \quad p_2 = \frac{\partial \omega}{\partial y}, \tag{148}$$

$$x = x_0(t_1), \quad y = y_0(t_1), \quad \omega\Big|_{\substack{x=x_0\\ y=y_0}} = 0. \tag{148′}$$

The boundary-value problem for the characteristic system of differential equations (144), (144′), (144″) corresponding to this problem takes the form

$$\frac{dx}{2a^2p_1} = \frac{dy}{2a^2p_2} = \frac{d\omega}{2} = \frac{-dp_1}{2a\dfrac{\partial a}{\partial x}(p_1^2+p_2^2)} = \frac{-dp_2}{2a\dfrac{\partial a}{\partial y}(p_1^2+p_2^2)} = \frac{ds}{2}, \tag{149}$$

$$x\Big|_{s=0} = x_0(t_1),\ y\Big|_{s=0} = y_0(t_1),\ \omega\Big|_{s=0} = 0, \tag{149′}$$

$$p_1^0 x_0' + p_2^0 y_0' = 0, \quad a^2(p_1^{0\,2} + p_2^{0\,2}) - 1 = 0, \tag{149″}$$

where the primes denote differentiation with respect to t_1. Condition (146) may be written in the form

$$\delta = \begin{vmatrix} 2a^2 p_1^0 & 2a^2 p_2^0 \\ x_0' & y_0' \end{vmatrix} \neq 0. \tag{150}$$

On the basis of the first of conditions (149″), we have

$$\delta = 2a^2 p_1^0 \begin{vmatrix} 1 & -\dfrac{x_0'}{y_0'} \\ x_0' & y_0' \end{vmatrix} = 2a^2 \frac{p_1^0}{y_0'}(x_0'^2 + y_0'^2).$$

It follows from Eqs. (149″) that one of the values of p_1^0 and p_2^0 is necessarily different from zero. Therefore, condition (150) is equivalent to the condition $x_0'^2 + y_0'^2 \neq 0$. Consequently, this condition will be satisfied for every initial wavefront which is a smooth curve with singularities. To find the wavefront $S_t: t=\omega(x_i)$, we solve the Cauchy problem (148), (148′) by using the method of characteristic curves (see Chapter II, Sect. 3). It follows from

from the last two equations (149) and from the facts that $\partial a/\partial x = 0$ and $\partial a/\partial y = 0$, that

$$p_1 = p_1^0 = \text{const}, \quad p_2 = p_2^0 = \text{const}.$$

The first three of equations (149) yield, by virtue of the initial conditions (149′),

$$x = a^2 p_1^0 s + x_0(t_1), \quad y = a^2 p_2^0 s + y_0(t_1). \tag{151}$$

From Eqs. (149″), we obtain

$$p_1^0 = \frac{y_0'}{\pm a\sqrt{x_0'^2 + y_0'^2}}, \quad p_2^0 = -\frac{x_0'}{\pm a\sqrt{x_0'^2 + y_0'^2}}.$$

Thus, the solution $\omega = \omega(x, y)$ of the Cauchy problem (148), (148′) may be written in the following parametric form:

$$x = \frac{ay_0'(t_1)}{\pm\sqrt{x_0'^2(t_1) + y_0'^2(t_1)}}\omega + x_0(t_1),$$

$$y = -\frac{ax_0'(t_1)}{\pm\sqrt{x_0'^2(t_1) + y_0'^2(t_1)}}\omega + y_0(t_1). \tag{151′}$$

For example, if the initial wavefront is the circle $x_0 = r_0\cos t_1$, $y_0 = r_0\sin t_1$, Eqs. (151′) take the form

$$x = \pm a\omega\cos t_1 + r_0\cos t_1, \quad y = \pm a\omega\sin t_1 + r_0\sin t_1. \tag{151″}$$

If we set $\omega = t$ in these equations and choose the positive or negative sign in accordance with the physical nature of the problem, we shall obtain the desired wavefront at the instant t. If we are interested in the wavefront that is being propagated in the exterior of the initial circle $x_0 = r_0\cos t_1$, $y_0 = r_0\sin t_1$, then, in Eqs. (151″), we choose the positive sign and thus obtain the wavefront at the instant t in the form $x = (at + r_0)\cos t_1$, $y = (at + r_0)\sin t_1$ or $t = \left(\sqrt{x^2 + y^2} - r_0\right)/a$.

We note that these considerations, which reduce the problem of finding the wavefront from its initial position to the Cauchy problem, can be used to justify the method of construction of a wavefront which is familiar from physics, as an envelope of circles with centers at points of the initial wavefront and radius equal to the velocity of propagation of the waves multiplied by the time.

We shall point out yet another property of rays that correspond to the two-dimentional wave equation (147) with piecewise-constant

values of the coefficient a. From a physical point of view, the shift from a single constant value $a=a_1$ to a different constant value $a=a_2$ indicates a shift of oscillatory process described by Eq. (147) from one homogeneous isotropic medium with velocity of propagation a_1 into another homogeneous isotropic medium with velocity of propagation of waves a_2. For definiteness, let us suppose that the boundary separating the two media passes along the y-axis, that $a=a_1$ for $x>0$ and $a=a_2>a_1$ for $x<0$ (see Fig. 35). The rays in the half-plane $x>0$ are both straight lines but, with the bicharacteristics corresponding to them, they form different angles. In the first half-plane, this angle is given by

$$\gamma_1=\arcsin\frac{1}{\sqrt{1+a_1^2}},$$

in the second half-plane, it is given by the equation

$$\gamma_2=\arcsin\frac{1}{\sqrt{1+a_2^2}}.$$

Suppose that a ray passing through the coordinate origin forms an angle φ_1 with the x-axis in the first half-plane and an angle φ_2 in the second half-plane (see Fig. 35). Since the rays must be perpendicular to the wavefront, we have

$$\sin\varphi_1=\left.\frac{p_2}{\sqrt{p_1^2+p_2^2}}\right|_{x>0}=p_2a_1,\quad \sin\varphi_2=\left.\frac{p_2}{\sqrt{p_1^2+p_2^2}}\right|_{x<0}=p_2a_2,$$

$$\left(p_1=\frac{\partial\omega}{\partial x},\ p_2=\frac{\partial\omega}{\partial y}\right). \tag{152}$$

Since $\partial a/\partial y = 0$, the last of Eqs. (149) yields $p_2=\text{const}=p_2^0$ in both these half-planes. Therefore, if we divide the first of Eqs. (152) by the second, we obtain

$$\frac{\sin\varphi_1}{\sin\varphi_2}=\frac{a_1}{a_2}.$$

This is the familiar *law of refraction of light.*

3. **Waves with damping. Waves with dispersion. Distortion of waves.** Every solution of Eq. (135′) that is of the form

$$u(x_i,t)=gF(\omega(x_i)-\beta t),\quad g=T(t)X(\omega), \tag{153}$$

where $\omega=\omega(x_i)$ is a given function of the variables x_1, ..., x_n and T, X, and F are functions of their arguments, $\beta=\text{const}>0$, is called a *wave corresponding to equation (135′).* For $\beta=0$, the wave (153) takes the form

$$u(x_i, t) = T(t)\Phi(\omega(x_i)), \quad \Phi = X(\omega)F(\omega) \tag{153'}$$

and is called a *standing wave.* For $\beta>0$, the wave (153) is called a *traveling wave.*

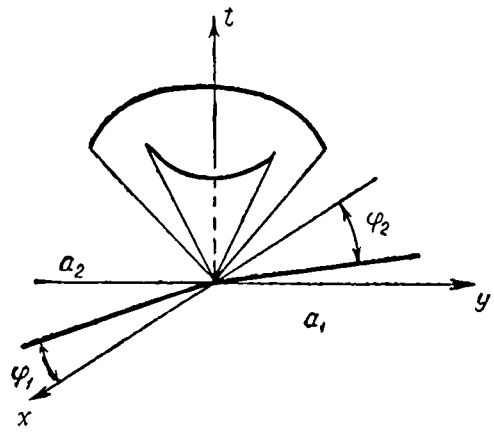

Fig. 35.

Let us consider questions associated with traveling waves. In this case, the function $g=T(t)X(\omega)$ is called the *damping coefficient,* the function F is called the waveform, and the function $\xi=\omega(x_i) - \beta t$ is called *phase of the wave.* The surfaces $\omega(x_i)$ = const in x-space are called the *phase surfaces* of the wave. These phase surfaces are characterized by the fact that the phase of the wave is constant on them at every fixed instant of time. The phase surfaces determine the form of the wave. For example, if $\omega(x_i)$ = const are spheres, we speak of spherical waves. If the surfaces $\omega(x_i)$ = const are constants, we speak of plane waves, etc. If the damping coefficient $g \equiv 1$, we speak of traveling waves without damping. If $g \not\equiv 1$, we speak of traveling waves with damping.

The equation of a surface in x-space on which the phase remains constant with the passage of time may be written in the form $\omega(x_i)=\beta t+\text{const}$. From this it is clear that this surface of constant phase will, with the passage of time, be displaced in x-space. At every point on the surface, this displacement will obviously take place in the direction of the vector $\nabla\omega$. The velocity of this displacement is called the *phase velocity.* It is easy to give an explicit expression for the vector denoting the phase velocity $\vec{v}_{ph}$ and for its magnitude v_{ph}. If $\vec{n}_0$ is the direction of the vector $\nabla\omega$ at some point of the surface $\omega(x_i)=\beta t+\text{const}$, then, at the instant $t+\Delta t$, the surface $\omega(x_i)=\beta(t+\Delta t)+\text{const}$ intersects the direction $\vec{n}_0$ at some point P^* at a distance Δn_0 from the point P. A simple calculation yields

$$v_{ph}=\lim_{\Delta t\to 0}\frac{\Delta n_0}{\Delta t}=\lim_{\Delta n_0\to 0}\frac{1}{\frac{\Delta t}{\Delta n_0}}=\lim_{\Delta n_0\to 0}\frac{1}{\frac{\Delta\omega}{\beta\Delta n_0}}=\beta\frac{1}{\frac{\partial\omega}{\partial n}}=\frac{\beta}{|\nabla\omega|},$$

i.e.,

$$v_{ph}=\frac{\beta}{|\nabla\omega|},\quad \vec{v}_{ph}=\beta\frac{\nabla\omega}{|\nabla\omega|^2}. \tag{154}$$

In particular, it is obvious from this that for $\beta=0$ the phase velocity becomes equal to 0 and the traveling waves (153) degenerate into standing waves (153′) for which surfaces of constant phase are motionless.

Suppose that the function $\omega(x_i)$ which determines the form of the traveling wave (153) is given, i.e., that the set of phase surfaces characterizing the traveling wave (153) is given. Two cases may then present themselves:

(a) The function F is arbitrary and the magnitude of the phase velocity is independent of F.*

(b) The function F is not arbitrary and the magnitude of the phase velocity depends on F.**

In the first case, the waves (153) are called *waves without dispersion;* in the second case, they are called *waves with dispersion.*

To make clearer the sense of these concepts, let us, for simplicity, consider waves without damping. In this case, the wave formed by the superposition of two waves without dispersion will not be distorted as it is propagated. At every point of space, it will be an exact reproduction, as far as shape is concerned, of the original waves though possible with some lag. This is explained by the fact that, because of the independence of the phase velocity on the waveform, the phases of the waves will not be displaced with respect to each other. Quite a different picture results when two waves with dispersion are superimposed on each other. In this case, because of the different phase velocities, i.e., because of the different values of the constant β, at every point of space there will be a phase displacement and, because of this phase displacement, an oscillatory process will be distorted as it is propagated. Wave dispersion involves those difficulties which are encountered, for example, in transmitting signals along conductors. Therefore, the question of singling out those particular cases of Eq. (135′) for which waves without dispersion are possible and the question of the specific forms of waves without dispersion are of great theoretical value.

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*This means that the property β remains the same independently of the choice of the waveform F.

**This dependence by virtue of (154) is felt directly only through β, which has different values for the different possible forms of the wave F.

Let us find out what necessary and sufficient conditions should be imposed on the phase of wave $\xi=\omega(x_i)-\beta t$ and on the coefficient of damping g for Eq. (135′) to allow a wave without dispersion. If we substitute the wave (153) into Eq. (135′) and make some simple calculations

$$\frac{\partial u}{\partial t}=-\beta g F'+\frac{\partial g}{\partial t}F,\quad \frac{\partial^2 u}{\partial t^2}=gF''\beta^2-2\beta\frac{\partial g}{\partial t}F'+\frac{\partial^2 g}{\partial t^2}F,$$

$$\frac{\partial u}{\partial x_i}=gF'\omega_{x_i}+\frac{\partial g}{\partial x_i}F,\quad \frac{\partial^2 u}{\partial x_l\,\partial x_k}=gF''\omega_{x_i}\omega_{x_k}+gF'\omega_{x_l x_k}+$$
$$+F'\left(\frac{\partial g}{\partial x_k}\omega_{x_l}+\frac{\partial g}{\partial x_l}\omega_{x_k}\right)+F\frac{\partial^2 g}{\partial x_l\,\partial x_k}$$

we obtain

$$L(u)=F''g\Big(\sum_{i,k=1}^{n}a_{ik}\omega_{x_i}\omega_{x_k}-\beta^2\Big)+F'\Big[2\Big(\sum_{i,k=1}^{n}a_{ik}\omega_{x_l}g_{x_k}+\beta g_t\Big)+$$
$$+g\Big(\sum_{i,k=1}^{n}a_{ik}\omega_{x_l x_k}+\sum_{i=1}^{n}b_i\omega_{x_i}-b\beta\Big)\Big]+'FL(g)=0.$$

If we set the coefficients of F'', F', and F equal to zero in this equation, we conclude that for waves without dispersion corresponding to Eq. (135′) to exist, it is necessary and sufficient that the following conditions be satisfied simultaneously:

(a) the phase of the wave $\xi=\omega(x_i)-\beta t$ is a characteristic variable; that is, the surfaces $\xi=\omega(x_i)-\beta t=$ const must be characteristics;*

(b) the coefficient of damping g satisfies simultaneously the two equations

$$L(g)=\sum_{i,k=1}^{n}a_{ik}\frac{\partial^2 g}{\partial x_l\,\partial x_k}-\frac{\partial^2 g}{\partial t^2}+\sum_{i=1}^{n}b_i\frac{\partial g}{\partial x_l}+b\frac{\partial g}{\partial t}+cg=0,$$

$$A(g)=2\Big(\sum_{i,k=1}^{n}a_{ik}\omega_{x_i}g_{x_k}+\beta g_t\Big)+$$
$$+g\Big(\sum_{i,k=1}^{n}a_{ik}\omega_{x_l x_k}+\sum_{i=1}^{n}b_i\omega_{x_l}-b\beta\Big)=0. \qquad (155)$$

---

* The significant point is that the equation

$$\sum_{i,k=1}^{n}a_{ik}\xi_{x_l}\xi_{x_k}-\xi_t^2=0$$

is the equation of the characteristic variables for Eq. (135′).

Condition (a) obviously can always be satisfied for arbitrary coefficients in Eq. (135′). This condition makes a restriction only on the form of waves without dispersion and indicates the possible structure of their phase surfaces $\omega(x_i) = \text{const}$. Condition (b) imposes restrictions on the coefficients in Eq. (135′) and brings out the fact that the existence of waves without dispersion is by no means possible for every equation of the form (135′).

Let us give some important cases of equations with constant coefficients for which there exist waves without dispersion of certain well-defined types.

1. In the case of the telegraph equation

$$L(u) \equiv a^2 u_{xx} - u_{tt} + cu = 0 \tag{156}$$

the characteristics are of the form

$$x - at = \text{const}, \quad -x - at = \text{const}.$$

Consequently, we need to set $\omega = \pm x$. Here, the second of conditions (155) yields

$$\pm a^2 g_x + a g_t = 0, \quad g = Ae^{\mu(\pm x - at)}, \quad A = \text{const}, \quad \mu = \text{const},$$

and the first of conditions (155) takes the form $a^2 g_{xx} - g_{tt} + cg = 0$. For $c \neq 0$, it is incompatible with the preceding condition. Therefore, Eq. (156) does not have waves without dispersion when $c \neq 0$. On the other hand, when $c = 0$, each of the functions $\omega = x$ and $\omega = -x$ satisfies both of conditions (155) if we set $g \equiv 1$. Consequently, for $c = 0$, Eq. (156) has waves without dispersion and without damping

$$u = F(x - at), \quad u = F(-x - at),$$

where F is an arbitrary function of its arguments. The term cu in Eq. (156) is sometimes called the *dispersion term*.

Let us show that, in the case of Eq. (156), the presence of wave dispersion is equivalent to the dependence of the phase velocity of harmonic solutions of this equation

$$u(x, t) = e^{-i(kx - \beta t)} \tag{156′}$$

on the frequency β.* The phase velocity of harmonic oscillations (156′) is equal to $|\beta/k|$. Here, the constant k is, by virtue of Eq. (156), determined from the equation $-a^2k^2 + \beta^2 + c = 0$.

*In physics, this dependence of the phase velocity of harmonic oscillations (156′) on the frequency is often used as the definition of wave dispersion.

Consequently, if $c=0$, this phase velocity will be equal to a and the wave (156′) will take the form

$$u(x,t) = e^{-ik(\pm x-at)}.$$

But, the Fourier integral theorem,

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\, dk \int_{-\infty}^{\infty} F(\xi)\, e^{ik\xi}\, d\xi$$

shows that every function of the form

$$F(\pm x - at) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(\pm x-at)}\, dk \int_{-\infty}^{\infty} F(\xi)\, e^{ik\xi}\, d\xi$$

is a solution of Eq. (156). Thus, there exist wave solutions of Eq. (156) of arbitrary form with velocity of propagation equal to a; that is, there is no wave dispersion. If $c \neq 0$, then $k = \pm\sqrt{\beta^2+c}/a$ and harmonic oscillations (156′) will have a phase velocity dependent on the frequency β that is equal to $a(1 + c/\beta^2)^{-\frac{1}{2}}$. This means that harmonic waves (156′) are waves with dispersion in the sense of the general definition given above.

2. For the wave equation with three space variables

$$L(u) = a^2(u_{xx} + u_{yy} + u_{zz}) - u_{tt} = 0 \tag{157}$$

the characteristics are the planes $\alpha_1 x+\alpha_2 y+\alpha_3 z - at$ = const, where α_1, α_2, and α_3 are the direction cosines: $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. If we set $\omega=\alpha_1 x+\alpha_2 y+\alpha_3 z$ and $g \equiv 1$, we see that the first and second conditions (155) become identities. Consequently, Eq. (157) has waves without dispersion and without damping that are of the form

$$u = F(\alpha_1 x + \alpha_2 y + \alpha_3 z - at). \tag{157′}$$

These waves are called plane waves since the phase surfaces corresponding to them $\omega(x_i) = \alpha_1 x+\alpha_2 y+\alpha_3 z$ = const are planes.

For equation (157), the surfaces

$$r - at = \text{const}, \quad -r - at = \text{const},$$

where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ and (x_0, y_0, z_0) is a fixed point in x-space, are also characteristics. If we set $\omega = \pm r$ and $g = 1/4\pi r$, it is easy to see that the first and second of conditions (155) become identities. Consequently, for Eq. (157), waves without dispersion but with damping that are of the form

$$u = \frac{F(r-at)}{4\pi r}, \quad u = \frac{F(-r-at)}{4\pi r} \tag{157''}$$

exist. These are spherical waves: the first of them is an outgoing wave, the second one—an incoming wave.

3. For the wave equation with two space variables

$$L(u) = a^2(u_{xx} + u_{yy}) - u_{tt} = 0 \tag{158}$$

the planes $\alpha_1 x + \alpha_2 y - at = \text{const}$, where $\alpha_1^2 + \alpha_2^2 = 1$, are characteristics. If we set $\omega = \alpha_1 x + \alpha_2 y$ and $g \equiv 1$, we see that conditions (155) are satisfied and, consequently, for Eq. (158), there exist rectilinear waves without dispersion and without damping:

$$u = F(\alpha_1 x + \alpha_2 y - at). \tag{158'}$$

For the wave equation (158), the surfaces $r - at = \text{const}$ and $-r - at = \text{const}$, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, and (x_0, y_0) is a fixed point in the plane, are also characteristics. But, as we shall immediately show, there are no waves without dispersion corresponding to these characteristics, i.e., there are no cylindrical waves without dispersion. To show this, note that if we set $\omega = \pm r$, conditions (155) take the form

$$L(g) = a^2(g_{xx} + g_{yy}) - g_{tt} = 0,$$

$$\pm A(g) = 2\left(a^2 g_x \frac{\partial r}{\partial x} + a^2 g_y \frac{\partial r}{\partial y} + a g_t\right) + g\frac{a^2}{r} = 0.$$

By using the two-dimensional Laplace operator in polar coordinates

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$$

and remembering that g depends only on ω and t, we can write these last two conditions in the form

$$L(g) = a^2\left(g_{rr} + \frac{1}{r}g_r\right) - g_{tt} = 0,$$

$$\pm A(g) = 2a^2\left(g_r + \frac{1}{a}g_t\right) + g\frac{a^2}{r} = 0$$

or, since $g = T(t)X(\omega)$,

$$L(g) = a^2 T(t)\left(X'' + \frac{1}{r}X'\right) - XT'' = 0,$$

$$\pm A(g) = 2a^2\left(TX' + \frac{1}{a}XT'\right) + XT\frac{a^2}{r} = 0.$$

Here, the primes denote differentiation of T with respect to t and differentiation of X with respect to r. After we divide these last equations by XT, we obtain

$$\frac{T''}{T} = a^2 \frac{X'' + \frac{1}{r} X'}{X} = \lambda = \text{const},$$
$$-\frac{1}{a}\frac{T'}{T} = \frac{X'}{X} + \frac{1}{2r} = \mu = \text{const}. \qquad (155')$$

The second of these equations yields

$$X = C \frac{1}{\sqrt{r}} e^{\mu r} \text{ where } C = \text{const}.$$

From this we obtain

$$\frac{X''}{X} + \frac{1}{r}\frac{X'}{X} = \frac{3}{4} r^{-2} - \mu r^{-1} + \mu^2 + \frac{1}{r}\left(\mu - \frac{1}{2r}\right) \neq \text{const}.$$

But this contradicts the first of Eqs. (155′). Consequently, Eq. (158) has no cylindrical waves without dispersion.

4. For the telegraph equation which describes electrical oscillations in conductors

$$L^*(u) = \frac{\partial^2 u}{\partial x^2} - LC\frac{\partial^2 u}{\partial t^2} - (RC + LG)\frac{\partial u}{\partial t} - RGu = 0 \qquad (159)$$

(cf. Chapter I, Sect. 2, subsection 8), the surfaces

$$x - at = \text{const}, \quad -x - at = \text{const} \left(a = \frac{1}{\sqrt{LC}}\right)$$

will be characteristics. If we set $\omega = \pm x$ and $L(u) = a^2 L^*(u)$, then Eqs. (155) take the form

$$L(g) = a^2 g_{xx} - g_{tt} - a^2 (RC + LG) g_t - a^2 RGg = 0,$$
$$A(g) = 2(\pm a^2 g_x + a g_t) + g a^2 (RC + LG) a = 0. \qquad (155'')$$

If we assume that g does not depend on $\omega = \pm x$, the second of Eqs. (155″) yields

$$g = Ae^{-\frac{a^2}{2}(RC+LG)t}, \quad A = \text{const}$$

where A is a constant. If we substitute this into the first of Eqs. (155″), we obtain

$$-\frac{a^2}{4}(RC+LG)^2+\frac{a^4}{4}(RC+LG)^2-a^2RG=0$$

or, after some transformations,

$$(RC+LG)^2-4LCRG=(RC-LG)^2=0.$$

Thus, when the condition

$$RC=LG \tag{159'}$$

is satisfied, Eq. (159) will have waves without dispersion but with damping of the following form:

$$u=e^{-kt}F(x-at),\quad u=e^{-kt}F(-x-at),$$

where

$$a=\frac{1}{\sqrt{LC}},\quad k=\frac{a^2}{2}[RC+LG]=\frac{1}{2}\left[\frac{R}{L}+\frac{G}{C}\right].$$

When the coefficients in Eq. (159) that characterize the conducting cable, the resistance R, the capacitance C, the inductance L, and the leakage G satisfy condition (159'), then waves of arbitrary form will be propagated without distortion in the cable; they can only be damped. But this damping in the place of a wave-signal device can always be compensated by strengthening the signal, and thus we are enabled to reproduce exactly the signals that are transmitted along the cable. This phenomenon is very useful in telephony and in the the transmission of signals along conductors and cables over great distances.

4. **Numerical integration along characteristics.** One of the more important applications of characteristics is a method of approximate solution of boundary-value problems for hyperbolic equations. This method, known as the method of integrating along characteristics, has had very extensive application in gas dynamics in the solution of boundary-value problems associated with the nonlinear system of equations of supersonic gas flow.*

To make the matter easier to grasp and to make the calculations simpler, we shall confine ourselves to an illustration of this method by using it to solve the Cauchy problem for a linear hyperbolic system of equations of the following form:

*See, for example, R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, New York, Interscience, 1948, and L. Howarth (ed), Modern *Developments in Fluid Dynamics (High Speed Flow)*, Oxford, Oxford University Press, 1953.

$$\left.\begin{aligned} \alpha\frac{\partial u}{\partial x}+\beta\frac{\partial v}{\partial x}&=0\\ \gamma\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial y}&=0 \end{aligned}\right\}\quad \alpha\delta-\gamma\beta\neq 0, \tag{160}$$

$$u|_C=\varphi(s),\quad v|_C=\psi(s), \tag{160'}$$

where α, β, γ and δ are given functions of x and y, where C is a curve varying monotonically with respect to x and to y, and where $\varphi(s)$ and $\psi(s)$ are given functions of the arclength s along the curve C (see Fig. 33a).

The characteristic determinant of the system (160) will be of the form

$$\Delta(\alpha)=\begin{vmatrix}\alpha\alpha_1 & \beta\alpha_1\\ \gamma\alpha_2 & \delta\alpha_2\end{vmatrix}=(\alpha\delta-\gamma\beta)\,\alpha_1\alpha_2,$$

and the equation of the characteristics can be written in the form

$$\begin{vmatrix}\alpha\varphi_x & \beta\varphi_x\\ \gamma\varphi_y & \delta\varphi_y\end{vmatrix}=0,\quad \varphi(x,y)=\text{const},\quad |\varphi_x|^2+|\varphi_y|^2\neq 0.$$

From this it is clear that the straight lines parallel to the x- and y-axes, i.e., the lines

$$y=\text{const},\quad x=\text{const}$$

are characteristics.

Let us choose on C a sufficiently dense sequence of points P_1, q_1, ..., q_n, P_2 (see Fig. 33a). We note that, by virtue of the initial conditions (160'), these points are mapped into well-defined points P_1^*, q_1^*, ..., q_n^*, P_2^* in the uv-plane. On the basis of the first of Eqs. (160), the equation

$$\alpha\,du+\beta\,dv=0 \tag{161}$$

will hold along the characteristic y = const issuing from the point P_1. On the basis of the second of Eqs. (160), the equation

$$\gamma\,du+\delta\,dv=0 \tag{161'}$$

will hold along the characteristic x = const issuing from the point q_1 (see Fig. 33a).

We note that Eqs. (161), (161') should, from the point of view of the general theory of the method of integrating along characteristics, be regarded as the simplest example of what are called differential relationships along the characteristics.*

*See, for example, the books cited in the previous footnote.

If we draw the straight line (161) from the point P_1 and the straight line (161′) from the point q_1, at the intersection of these two lines, we obtain a point, which will be the image of the point q_1' (see Fig. 33*a*). Similarly, we can obtain the images of the points q_2', ..., q_n', P_2' (see Fig. 33*a*). If we now proceed the same way with the points q_1', q_2', ..., q_n', P_2' as with the points P_1, q_1, ..., q_n, P_2, and continue this process, we obtain, after a finite number of operations in the uv-plane, the image of the given point $P(x, y)$ and thus an approximate value of the solution of the Cauchy problem at the given point $P(x, y)$.

This method shows convincingly how simple it is in some cases to solve boundary-value problems by shifting to calculations involving discrete quantities. However, in the present book we cannot go into detail regarding this, or regarding the connection with the method of characteristics or with other numerical methods of solving boundary-value problems for partial differential equations of different types. This is due in part to the space limitations of the book and in part to the fact that numerical methods of solving the problems of mathematical physics in university curricula in connection with the development of computational mathematics has at the present time become the subject of a special discipline, namely, approximate and numerical methods.

The reader who is interested in these questions is referred to the university textbooks on this discipline: I. S. Berezin and N. P. Zhidkov, Computating Methods, Vols. 1 and 2, Oxford and New York, Pergamon, 1965, G. N. Polozhiy, N. A. Pakhareva, I. Z. Stepanenko, P. S. Bondarenko, I. M. Velikoivanenko, *Matematicheskiy praktikum* (Mathematical exercise book), edited by G. N. Polozhiy, Fizmatgiz, 1960 (this has been translated into German as *Mathematisches Praktikum*, Leipzig, Teubner).*

*See, also, L. Collatz, *The Numerical Treatment of Differential Equations*, Berlin, Springer, 1960; V. E. Milne, *Numerical Solution of Differential Equations*, New York, Wiley, 1953; V. K. Saul'yev, *Integration of Equations of Parabolic Type by the Method of Nets*, Oxford and New York, Pergamon, 1964; G. N. Polozhiy, *The Method of Summary Representation for Numerical Solution of Problems of Mathematical Physics*, Oxford, Pergamon, 1965; G. E. Forsythe and W. R. Wassow, *Finite Difference Methods for Partial Differential Equations*, New York, Wiley, 1960.

CHAPTER 6

The Method of Separation of Variables As Applied To Hyperbolic, Parabolic, and Elliptic Equations. The Eigenvalue Problem. Elements of the Theory of Special Functions

In this chapter, we shall study the method of separation of variables as applied to all three types of equations: elliptic, parabolic, and hyperbolic. In connection with this, we introduce the general theory of the Sturm–Liouville problem and the so-called generalized Sturm–Liouville problem. We shall give the asymptotics of the eigenvalues and eigenfunctions. We shall prove certain theorems regarding the Fourier coefficients in the expansions in the eigenfunctions of the Sturm–Liouville problem and we shall apply them to the justification of the method of separation of variables for obtaining solutions in the classical sense. We shall also examine the method of separation of variables from the point of view of obtaining generalized solutions. We shall give certain variational properties of the eigenvalues and we shall present the idea of the Ritz approximation method of finding the eigenvalues and eigenfunctions. At the end of the chapter, we shall give some of the simpler features of special functions and some examples of their application.

1. Application of the Method of Separation of Variables to the Solution of Boundary-Value Problems Associated with Trigonometric Functions.

One of the more generally applicable methods of solving boundary-value problems for partial differential equations is the so-called method of separation of variables. This method is sometimes called the *Fourier method* or the *method of eigenfunctions*. The basic idea behind the method consists in the fact that, in each

specific case, the solution of the given boundary-value problem for a partial differential equation can be reduced to solving auxiliary boundary-value problems for ordinary differential equations, or for partial differentiation equations with fewer independent variables. Solutions of the original partial differential equation are found in the form of products of solutions of auxiliary boundary-value problems and then a solution of the original boundary-value problem is found in the form of a linear combination of these solutions with constant coefficients.

The essential requirements for this method to be applicable is that the partial differential equations and the given boundary conditions be linear. Otherwise, this method is equally applicable to all the basic equations of mathematical physics and even to equations of a more general form whether they be hyperbolic, elliptic, or parabolic.

Let us consider a few very simple examples of boundary-value problems that can be solved by means of trigonometric functions of various integral multiples of the independent variables. In the process of solving these boundary-value problems, which are of independent interest, we shall clarify as much as possible the basic idea of the method of separation of variables.

1. Solution of the first boundary-value problem for the one-dimensional wave equation. First, let us note that the solution of the most general first boundary-value problem for the one-dimensional wave equation can be reduced to the case in which the boundary conditions are homogeneous. To do this, we need only subtract from the desired solution some twice continuously differentiable function which satisfies the given nonhomogeneous boundary conditions. When we do this we change only the right-hand member of the equation, and the solution of the new problem will satisfy the homogeneous boundary conditions. Furthermore, the solution of the problem with homogeneous boundary conditions which has the general form

$$u_{xx} - \frac{1}{a^2} u_{tt} = -\frac{f(x, t)}{a^2}, \tag{1}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x), \tag{1'}$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0 \tag{1''}$$

can be represented as the sum of the solution of the boundary-value problem for the homogeneous equation

$$u_{xx} - \frac{1}{a^2} u_{tt} = 0, \tag{2}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x), \tag{2'}$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0 \tag{2''}$$

and the solution of the boundary-value problem for the nonhomogeneous equation with homogeneous initial conditions

$$u_{xx} - \frac{1}{a^2} u_{tt} = -\frac{f(x, t)}{a^2}, \tag{3}$$

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{3'}$$

$$u|_{x=0} = 0, \; u|_{x=l} = 0. \tag{3''}$$

Let us apply the method of separation of variables to the solution of the problem for the homogeneous equation (2)–(2′)–(2″). We seek solutions of Eq. (2) in the form

$$u = X(x)\,T(t), \tag{4}$$

where $T(t)$ is a function of t only and $X(x)$ is a function of x only, satisfying the zero boundary conditions (2″). If we substitute (4) into Eq. (2), we obtain

$$X''(x)\,T(t) - \frac{1}{a^2} X(x)\,T''(t) = 0,$$

where the primes denote differentiation of X with respect to x and differentiation of T with respect to t. If we divide this equation by $X(x)T(t)$, we obtain the so-called condition of separability of the variables

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = -\lambda, \tag{5}$$

where λ must be a constant since X''/X does not depend on t and T''/T does not depend on x.

Using the zero boundary conditions (2″), we obtain from Eq. (5) the simplest case of a so-called *eigenvalue and eigenfunction problem.*

We need to find nontrivial solutions of the boundary-value problem

$$X'' + \lambda X = 0, \tag{6}$$

$$X|_{x=0} = 0, \quad X|_{x=l} = 0, \tag{6'}$$

and also the values of the parameter λ for which these nontrivial solutions exist.

This problem is often called the *Sturm–Liouville problem.* The values of λ with the property indicated are called the *eigenvalues* and the nontrivial solutions corresponding to them are called *eigenfunctions.*

For $\lambda=0$ and $\lambda<0$, the general solution of Eq. (6) takes, respectively, the forms

$$X = Cx + C', \quad X = Ce^{\sqrt{-\lambda}x} + C'e^{-\sqrt{-\lambda}x},$$

where C and C' are arbitrary constants. From this it follows that the boundary-value problem (6)–(6′) has no eigenvalues when $\lambda \leqslant 0$. For $\lambda>0$, the general solution of Eq. (6) can be written in the form

$$X = C\cos\sqrt{\lambda}x + C'\sin\sqrt{\lambda}x,$$

where C and C' are arbitrary constants. If we substitute this equation into the first of the boundary conditions (6′), we obtain

$$C=0.$$

The second of the boundary conditions (6′) yields $C'\sin\sqrt{\lambda}\,x|_{x=l}=0$. From this we conclude that $\sqrt{\lambda_n} = n\pi/l$ (where $n = 1, 2, \ldots$). Thus, the eigenvalues of the boundary-value problem (6)–(6′) are the numbers

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad (n = 1, 2, \ldots), \tag{7}$$

and the eigenfunctions corresponding to them may be written in the form

$$X_n = C_n' \sin\frac{n\pi}{l}x \qquad (n = 1, 2, \ldots), \tag{7′}$$

where the C_n' are arbitrary constants. If we replace λ in the condition for separability of the variables (5) with the eigenvalue $\lambda_n = (n\pi/l)^2$ for determining the function $T(t)=T_n(t)$ corresponding to the given eigenvalue, we obtain the equation

$$T_n'' + \lambda_n a^2 T_n = 0. \tag{8}$$

The general solution of this equation is of the form

$$T_n = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t, \tag{8'}$$

where the A_n and B_n are arbitrary constants. If we substitute (7') and (8') into Eq. (4), we obtain the solution of Eq. (2)

$$u_n(x, t) = \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t\right) \sin \frac{n\pi}{l} x, \tag{9}$$

where A_n and B_n are arbitrary constants. Each of these solutions, corresponding to $n = 1, 2, \ldots$, satisfies the homogeneous boundary conditions (2''). If we write (9) in the form

$$u_n(x, t) = \operatorname{Re}\left[(A_n - iB_n) e^{i \frac{n\pi a t}{l}} \sin \frac{n\pi}{l} x\right],$$

we see that $u_n(x, t)$ represents the steady-state natural oscillations of frequency $\omega_n = n\pi a/l$ (see Chapter V, Sect. 10, subsection 2). Let us now seek a solution of the problem (2)–(2')–(2'') in the form of a series of the functions $u_n(x, t)$ (for $n = 1, 2, \ldots,$):

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t\right) \sin \frac{n\pi}{l} x. \tag{10}$$

The function $u(x, t)$ defined by (10) satisfies the homogeneous boundary conditions (2''). It is also a twice continuously differentiable solution of Eq. (2) if the series obtained by twice differentiating the right-hand member of equation (10) termwise converges uniformly. It now remains to choose the coefficients A_n and B_n in the series (10) in such a way that the function defined by this series will satisfy initial conditions (2'). If we assume that the functions $\psi(x)$ and $\psi_1(x)$ can be expanded in Fourier series in the interval $[0, l]$, then the initial conditions (2') yield

$$u|_{t=0} = \psi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} B_n \sin \frac{n\pi}{l} x.$$

From this we obtain

$$A_n = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi}{l} x \, dx, \quad B_n = \frac{2}{n\pi a} \int_0^l \psi_1(x) \sin \frac{n\pi}{l} x \, dx. \tag{11}$$

Now, it only remains to see what restrictions must be imposed on the functions $\psi(x)$ and $\psi_1(x)$ for the series obtained from the series (10) by two termwise differentiations to converge uniformly. For this, we use well-known properties of Fourier series. If the function $F(x)$, which is periodic with period $2l$, is k times continuously differentiable, then its Fourier coefficients

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi}{l}x\,dx,\quad b_n = \frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{n\pi}{l}x\,dx$$

go to zero faster than $1/n^k$ as $n \to \infty$ and*

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)n^{k-1}<\infty. \tag{12}$$

For the series obtained by differentiating the series (10) twice to converge uniformly, it is sufficient that

$$\sum_{1}^{\infty}|A_n|n^2<\infty,\quad \sum_{1}^{\infty}|B_n|n^2<\infty. \tag{13}$$

But, on the basis of (12), for the series (13) to converge it will be sufficient for the function $\psi(x)$, extended as an odd function to the interval $[-l, 0]$ and further extended as a periodic function with period $2l$, to be three times continuously differentiable and for the function $\psi_1(x)$ to be twice continuously differentiable under the same conditions. When these restrictions on the functions $\psi(x)$ and $\psi_1(x)$ are satisfied, the series (10) with coefficients A_n and B_n defined by Eqs. (11) yields a twice continuously differentiable solution of the first boundary-value problem (2)—(2′)—(2″).

Let us turn now to the solution of the boundary-value problem for the nonhomogeneous equation with homogeneous initial conditions (3)–(3′)–(3″). Let us seek a solution of this problem in the form of a series, which we assume to be twice differentiable termwise with respect to x and t:

$$u(x,t)=\sum_{n-1}^{\infty}u_n(t)\sin\frac{n\pi}{l}x. \tag{14}$$

Substitution of this series into Eq. (3) yields

*See, for example, I. I. Privalov, *Rady Fur'e* (Fourier Series), GNTI, Moscow and Leningrad, 1931, pp. 82–86 or V. I. Smirnov, *A Course of Higher Mathematics*, Reading, Massachusetts, Addison Wesley, 1961, Vol. II, pp. 456–459.

$$\sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{l}\right)^2 u_n(t)-\frac{1}{a^2}u_n''(t)\right]\sin\frac{n\pi}{l}x=-\frac{f(x,t)}{a^2}.$$

From this we obtain

$$u_n''(t)+\left(\frac{n\pi}{l}\right)^2 u_n(t)=f_n(t), \tag{15}$$

where the $f_n(t)$ are the Fourier coefficients of the function $f(x, t)$, i.e.,

$$f_n(t)=\frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi}{l}x\,dx.$$

The zero initial conditions (3′) yield

$$u_n|_{t=0}=0,\quad u_n'|_{t=0}=0. \tag{15′}$$

Thus, to determine the functions $u_n(t)$, we have the ordinary differential equation (15) and the initial conditions (15′). We then get the solution of the problem (15)–(15′) in the form

$$u_n(t)=\frac{l}{n\pi a}\int_0^t f_n(t')\sin\frac{n\pi a}{l}(t-t')\,dt', \tag{15″}$$

as is easily verified by direct differentiation. Thus, the desired solution of the problem (3)–(3′)–(3″) is given by the series (14) with coefficients $u_n(t)$ determined by Eqs. (15″). Conversely, if the series obtained by two termwise differentiations with respect to x and t of the series (14) with coefficients determined by Eqs. (15″) converges uniformly, it clearly represents a twice continuously differentiable solution of the problem (3)–(3′)–(3″).

This solution of the problem (3)–(3′)–(3″) can be represented in a somewhat different form. From Eqs. (14) and (15″), we obtain

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{l}x\,\frac{l}{n\pi a}\int_0^t\sin\frac{n\pi a}{l}(t-t')f_n(t')\,dt'=$$

$$=\int_0^t\left[\sum_{n=1}^{\infty}\frac{l}{n\pi a}\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}(t-t')\frac{2}{l}\int_0^l f(x',t')\sin\frac{n\pi}{l}x'\,dx'\right]dt'.$$

Thus, the solution of the problem (3)—(3′)—(3″) can be written in the form

$$u(x, t) = \int_0^t \int_0^l G(x, x', t-t') f(x', t')\, dx'\, dt', \tag{16}$$

where

$$G(x, x', t-t') = \sum_{n=1}^{\infty} \frac{2}{n\pi a} \sin\frac{n\pi}{l} x \sin\frac{n\pi}{l} x' \sin\frac{n\pi a}{l}(t-t'). \tag{16'}$$

The function $G(x, x', t-t')$ defined by Eq. (16') has a simple physical meaning quite analogous to the physical meaning of the Green's function $G(P, P', t-t')$ of the first basic boundary-value problem for the heat equation (see Chapter IV, Sect. 5). Specifically, $G(x, x', t-t')/\rho$, where ρ is the linear density of the string, is the displacement of points of the string from the equilibrium position that results from the action of a unit impulse applied to the string at rest at the point x' at the instant t' under the condition that the ends of the string $x = 0$ and $x = l$ are at all times fastened. In fact, if the function $f(x, t)$ is different from zero only in a sufficiently small neighborhood of the point (x', t'), i.e., for $x'-\varepsilon<x<x'+\varepsilon$, $t'<t<t'+\Delta t'$, the force $F(t)$ applied to the segment of the string $[x'-\varepsilon, x'+\varepsilon]$ is given by

$$F(t) = \rho \int_{x'-\varepsilon}^{x'+\varepsilon} f(x, t)\, dx,$$

and the impulse of this force is given by

$$I = \int_{t'}^{t'+\Delta t'} F(t)\, dt = \rho \int_{t'}^{t'+\Delta t'} \int_{x'-\varepsilon}^{x'+\varepsilon} f(x, t)\, dx\, dt. \tag{16''}$$

The displacement of the string from the equilibrium position as a result of this impulse can, in accordance with (16), be written in the form

$$\begin{aligned} u(x, t) &= \int_0^t \int_0^l G(x, x'', t-t'') f(x'', t'')\, dx''\, dt'' = \\ &= \int_{t'}^{t'+\Delta t'} \int_{x'-\varepsilon}^{x'+\varepsilon} G(x, x'', t-t'') f(x'', t'')\, dx''\, dt'' = \\ &= G(x, x^*, t-t^*) \int_{t'}^{t'+\Delta t'} \int_{x'-\varepsilon}^{x'+\varepsilon} f(x'', t'')\, dx''\, dt'', \end{aligned} \tag{16'''}$$

where x^* and t^* are such that $x'-\varepsilon<x^*<x'+\varepsilon$, $t' \leqslant t^* \leqslant t'+\Delta t'$. If we take the limit as ε and $\Delta t'$ approach zero in Eqs. (16'') and (16'''), we obtain

$$u(x, t) = G(x, x', t - t') \frac{1}{\rho}.$$

This means that $G(x, x', t - t')/\rho$ represents the displacement of the string from the equilibrium position caused by a unit impulse applied at the point x' at the instant t' to the string at rest with fastened ends. In other words, in accordance with the general concept of a Green's function for hyperbolic equations (see Chapter V, Sect. 4), the function $G(x, x', t-t')$ defined by Eq. (16′) represents Green's function of the first boundary-value problem for Eq. (1) of the string for the interval $[0, l]$. By means of this Green's function, the solution of the first boundary-value problem for the equation of a string in its most general form

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = -\frac{f(x, t)}{a^2}, \tag{1}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x), \quad 0 \leqslant x \leqslant l, \tag{17}$$

$$u|_{x=0} = \varphi_1(t), \; u|_{x=l} = \varphi_2(t), \quad t \geqslant 0, \tag{17′}$$

where $\psi(x)$, $\psi_1(x)$, $\varphi_1(t)$, and $\varphi_2(t)$ are given functions of their arguments, may, in accordance with the formula for integral representations (15′) in Chapter V, Sect. 4, be written in the form

$$u(x, t) = a^2 \int_0^{t+0} \left(\left.\frac{\partial G(x, x', t-t')}{\partial x'}\right|_{x'=0} \varphi_1(t') - \left.\frac{\partial G(x, x', t-t')}{\partial x'}\right|_{x'=l} \varphi_2(t') \right) dt' -$$

$$- \int_0^l \left(G(x, x', t-0)\, \psi_1(x') - \left.\frac{\partial G(x, x', t-t')'}{\partial t'}\right|_{t'=0} \psi(x') \right) dx' +$$

$$+ \int_0^{t+0} dt' \int_0^l G(x, x', t-t')\, f(x', t')\, dx'. \tag{17″}$$

2. Solution of the first boundary-value problem for the one-dimensional heat equation. Let us solve the boundary-value problem for the homogeneous heat equation

$$u_{xx} - \frac{1}{a^2} u_t = 0, \tag{18}$$

$$u|_{t=0} = \psi(x), \tag{18′}$$

$$u|_{x=0} = 0, \quad u|_{x=l} = 0, \tag{18″}$$

by the method of separation of variables and let us also find the solution of the boundary-value problem for the nonhomogeneous heat equation with the homogeneous initial conditions

$$u_{xx} - \frac{1}{a^2} u_t = -\frac{f(x, t)}{k}, \tag{19}$$

$$u|_{t=0} = 0, \tag{19'}$$

$$u|_{x=0} = 0, \quad u_{x=l} = 0. \tag{19''}$$

As in the case of the wave equation, we seek a solution of Eq. (18) of the form

$$u = X(x)\,T(t), \tag{4}$$

where $X(x)$ is a function of x and $T(t)$ is a function of t. When we substitute (4) into Eq. (18), we obtain the condition for separability of the variables:

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda. \tag{20}$$

From this, we obtain, by taking into account the zero boundary conditions (18″) the following eigenvalue problem for determining $X(x)$:

$$X'' + \lambda X = 0, \tag{6}$$

$$X|_{x=0} = 0, \quad X|_{x=l} = 0. \tag{6'}$$

As we saw in the preceding section, the eigenvalues of this boundary-value problem are $\lambda_n = (n\pi/l)^2$ (for $n = 1, 2, \ldots$) and the corresponding eigenfunctions are of the form

$$X_n = C_n' \sin\frac{n\pi}{l} x. \tag{7'}$$

When we substitute the eigenvalues into Eq. (20), we obtain the equation

$$T_n' + \left(\frac{n\pi a}{l}\right)^2 T_n = 0$$

for determining $T(t) = T_n(t)$. From this we obtain

$$T_n = C_n e^{-\left(\frac{n\pi a}{l}\right)^2 t},$$

where C_n are constants. If we substitute X_n and T_n into (4), we obtain solutions of Eq. (18) which satisfy the homogeneous boundary conditions (18″):

$$u_n = A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi}{l} x \quad (n = 1, 2, \ldots),$$

where the A_n are arbitrary constants. If we now put the desired solution of the problem (18)–(18′)–(18″) in the form

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} \sin \frac{n\pi}{l} x \tag{21}$$

and assume that the function $\psi(x)$ can be expanded in a uniformly and absolutely convergent Fourier series, we obtain by virtue of the initial conditions (18′)

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x = \psi(x).$$

From this, we obtain

$$A_n = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi}{l} x \, dx. \tag{21′}$$

If we substitute this into Eq. (21), we obtain a twice continuously differentiable solution of the problem (18)–(18′)–(18″).

Let us now turn to the solution of the problem for the nonhomogeneous equation (19)–(19′)–(19″). We seek a solution of this problem in the form of a series which can be differentiated termwise once with respect to t and twice with respect to x:

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi}{l} x. \tag{22}$$

If we substitute this into Eq. (19), we obtain

$$\sum_{n=1}^{\infty} \left[-\left(\frac{n\pi}{l}\right)^2 u_n(t) - \frac{1}{a^2} u_n'(t) \right] \sin \frac{n\pi}{l} x = -\frac{1}{k} f(x, t)$$

or

$$\left(\frac{n\pi}{l}\right)^2 u_n(t) + \frac{1}{a^2} u_n'(t) = \frac{1}{k} f_n(t), \quad f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi}{l} x \, dx.$$

From this, making use of the homogeneous initial conditions to determine $u_n(t)$, we arrive at the Cauchy problem for the ordinary differential equation

$$u_n' + \left(\frac{n\pi a}{l}\right)^2 u_n = \frac{a^2}{k} f_n(t), \quad u_n|_{t=0} = 0.$$

We immediately get a solution of this problem in the form

$$u_n(t) = \frac{a^2}{k} \int_0^t f_n(t') e^{-\left(\frac{n\pi a}{l}\right)^2 (t-t')} \, dt'.$$

Substituting this into Eq. (22), we conclude that the desired solution of the problem (19)–(19′)–(19″) must be of the form

$$u(x, t) = a^2 \int_0^t \int_0^l \left[\frac{2}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi}{l} x \sin \frac{n\pi}{l} x' e^{-\left(\frac{n\pi a}{l}\right)^2 (t-t')} \right] \times \times \frac{f(x', t')}{k} dx' \, dt'. \tag{22′}$$

The solution of the general first boundary-value problem for the one-dimensional heat equation with homogeneous boundary conditions is obtained as a sum of solutions of the boundary-value problem (18)–(18′)–(18″) and the boundary-value problem (19)–(19′)–(19″).

Comparison of Eq. (22′) with formula (15) in Chapter IV, Sect. 5, which gives an integral representation of the solution to the first boundary-value problem for the heat-flow equation in terms of Green's function, gives us some basis for hoping that the function

$$G(x, x', t-t') = \sum_{n=1}^{\infty} \frac{2}{l} \sin \frac{n\pi}{l} x \sin \frac{n\pi}{l} x' e^{-\left(\frac{n\pi a}{l}\right)^2 (t-t')} \tag{22″}$$

will be the Green's function of the first boundary-value problem for the one-dimensional heat-flow equation (19) for the interval $[0, l]$. Indeed, the function (22″) satisfies the homogeneous boundary conditions for the first boundary-value problem. It remains only to show that it represents the temperature at the point x at the instant t that is caused by an instantaneous concentrated heat source at the point x' at the instant t' of magnitude $\gamma\rho$ (where γ

is the specific heat and ρ is the linear density). And this actually is the case for the following reason.

Under the condition that the intensity of the heat sources $f(x, t)$ is different from zero only in an arbitrarily small neighborhood of the point (x', t') and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, t)\,dx\,dt = \gamma\rho,$$

it follows from (22′) that

$$u(x, t) = \frac{a^2\gamma\rho}{k} G(x, x', t-t') = G(x, x', t-t') \quad \left(a = \sqrt{\frac{k}{\gamma\rho}}\right).$$

Thus, the function (22″) is indeed the *Green's function of the first boundary-value problem for the heat equation (19) in the interval* $[0, l]$.

It is easy to see that the method of finding Green's function in explicit form that we have used here and in the preceding subsection can be useful and convenient in many other cases in solving the boundary-value problems for partial differential equations.

3. Solution of the Dirichlet problem for an annulus. Suppose that we wish to solve the first boundary-value problem for Laplace's equation for the annulus $r < \sqrt{x^2+y^2} < R$

$$u_{xx} + u_{yy} = 0, \tag{23}$$

$$u|_{\rho=r} = \Phi(\varphi), \quad u|_{\rho=R} = \Phi^*(\varphi), \tag{23′}$$

where $\rho = \sqrt{x^2+y^2}$ and φ are polar coordinates and $\Phi(\varphi)$ and $\Phi^*(\varphi)$ are given functions of φ. Because of the single-valuedness of the desired solution, the functions $\Phi(\varphi)$ and $\Phi^*(\varphi)$ must be periodic with period 2π.

Let us write Eq. (23) in polar coordinates:

$$\Delta u = \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0.$$

For solutions of this equation of the form

$$u = X(\rho)Y(\varphi),$$

where $X(\rho)$ is a function of ρ only and $Y(\varphi)$ is a function of φ only, we obtain the condition for separability of the variables,

$$\frac{Y''(\varphi)}{Y(\varphi)} = -\frac{\rho^2 X''(\rho) + \rho X'(\rho)}{X(\rho)} = -n^2, \tag{24}$$

where n^2 is a constant. From Eq. (24), we obtain

$$Y(\varphi) = A\cos n\varphi + B\sin n\varphi,$$

where A and B are constants. From this we see that, for the desired solution of the boundary-value problem to be a single-valued function in the annulus $r<\rho<R$, it is necessary and sufficient that n be an integer. From Eq. (24), we obtain the equation

$$\rho^2 X_n'' + \rho X_n' - n^2 X = 0.$$

for determining the functions $X(\rho) = X_n(\rho)$. For $n \neq 0$, this equation is Euler's equation, which has the linearly independent solutions ρ^n and ρ^{-n}. For $n=0$, we can verify directly that $\ln \rho$ and 1 are linearly independent solutions of this equation. Thus, as a solution of Eq. (23) we obtain the series

$$u(\rho, \varphi) = \alpha_0 \ln \rho + \beta_0 + \sum_{n=1}^{\infty} [(\alpha_n \rho^n + \beta_n \rho^{-n})\cos n\varphi + (\gamma_n \rho^n + \delta_n \rho^{-n})\sin n\varphi], \tag{25}$$

where α_0, β_0, the α_n, the β_n, the γ_n, and the δ_n are constants. The boundary conditions (23′) for determining these constants yield the equations

$$\Phi(\varphi) = \alpha_0 \ln r + \beta_0 + \sum_{n=1}^{\infty} [(\alpha_n r^n + \beta_n r^{-n})\cos n\varphi + (\gamma_n r^n + \delta_n r^{-n})\sin n\varphi], \tag{26}$$

$$\Phi^*(\varphi) = \alpha_0 \ln R + \beta_0 + \sum_{n=1}^{\infty} [(\alpha_n R^n + \beta_n R^{-n})\cos n\varphi + (\gamma_n R^n + \delta_n R^{-n})\sin n\varphi]. \tag{26′}$$

Let us suppose that the functions $\Phi(\varphi)$ and $\Phi^*(\varphi)$ can be expanded in uniformly convergent Fourier series:

$$\Phi(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi), \tag{27}$$

$$\Phi^*(\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi). \tag{27′}$$

Then, when we substitute these series into Eqs. (26) and (26′) and equate the coefficients of $\cos n\varphi$ and $\sin n\varphi$ to zero, we obtain

$$\alpha_0 \ln r + \beta_0 = \frac{a_0}{2}, \quad \alpha_0 \ln R + \beta_0 = \frac{A_0}{2},$$

$$\alpha_n r^n + \beta_n r^{-n} = a_n, \quad \alpha_n R^n + \beta_n R^{-n} = A_n,$$

$$\gamma_n r^n + \delta_n r^{-n} = b_n, \quad \gamma_n R^n + \delta_n R^{-n} = B_n.$$

From these equations, we easily obtain

$$\alpha_0 = \frac{A_0 - a_0}{2 \ln \frac{R}{r}}, \quad \beta_0 = -\frac{A_0 \ln r - a_0 \ln R}{2 \ln \frac{R}{r}},$$

$$\alpha_n = \frac{A_n r^{-n} - a_n R^{-n}}{\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n}, \quad \beta_n = -\frac{A_n r^n - a_n R^n}{\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n},$$

$$\gamma_n = \frac{B_n r^{-n} - b_n R^{-n}}{\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n}, \quad \delta_n = -\frac{B_n r^n - b_n R^n}{\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n}.$$

When we substitute this into the series (25), we obtain a solution of the Dirichlet problem (23)–(23′) in the following form:

$$u(\rho, \varphi) = \frac{A_0 - a_0}{2 \ln \frac{R}{r}} \ln \rho - \frac{A_0 \ln r - a_0 \ln R}{2 \ln \frac{R}{r}} +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{\left(\frac{R}{r}\right)^n - \left(\frac{r}{R}\right)^n} \{[(A_n r^{-n} - a_n R^{-n}) \rho^n -$$

$$- (A_n r^n - a_n R^n) \rho^{-n}] \cos n\varphi +$$

$$+ [(B_n r^{-n} - b_n R^{-n}) \rho^n - (B_n r^n - b_n R^n) \rho^{-n}] \sin n\varphi\}. \qquad (25')$$

It is easy to see that the terms of the series (25′) are quantities of higher order than $(\rho/R)^n + (r/\rho)^n$ as $n \to \infty$ since A_n, B_n, a_n, and b_n approach zero as $n \to \infty$. Therefore, outside the annulus (i.e., for $r < \rho < R$), the series (25′) can be differentiated twice termwise. Consequently, this series is a twice continuously differentiable solution of the Dirichlet problem (23)–(23′). In particular, if we set $r = 0$ as in the preceding case, we obtain from the series (25′) a solution of the Dirichlet problem for the circle

$$u(\rho, \varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{R}\right)^n (A_n \cos n\varphi + B_n \sin n\varphi).$$

If we set $R = \infty$ in Eq. (25′), we obtain a solution of the exterior Dirichlet problem for $\rho > r$:

$$u(\rho, \varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{\rho}\right)^n (a_n \cos n\varphi + b_n \sin n\varphi).$$

By just such a procedure, we can obtain a solution of the second boundary-value problem for Laplace's equation for the annulus $r<\rho<R$ if the coefficients of the series (25) are determined from the condition that the values of $\partial u/\partial \rho$ are given for $\rho=r$ and $\rho=R$.

4. Solution of the Dirichlet problem for a rectangle. Let us solve the first boundary-value problem for Laplace's equation for the rectangle $0<x<p$, $0<y<q$

$$u_{xx} + u_{yy} = 0, \tag{28}$$

$$u|_{y=0} = a(x), \quad u|_{y=q} = A(x), \tag{28'}$$

$$u|_{x=0} = b(y), \quad u|_{x=p} = B(y), \tag{28''}$$

where $a(x)$ and $A(x)$ are given functions of x which are continuous on the interval $0 \leqslant x \leqslant p$ and vanish at its end points, where $b(y)$ and $B(y)$ are given functions of y which are continuous on the interval $0 \leqslant y \leqslant q$ and vanish at its end points (see Fig. 36).

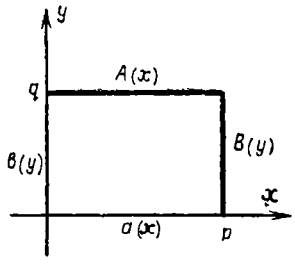

Fig. 36.

Let us note that the vanishing of the desired solution of the problem (28)–(28′)–(28″) at the corners of the rectangle does not in any way restrict the generality of our reasoning since it is always possible to achieve this situation by subtracting from the desired solution the harmonic function $A+Bx+Cy+Dxy$, where A, B, C, and D are suitably chosen constants.

To solve the problem (28)–(28′)–(28″), we first find solutions of Eq. (28) of the form

$$u = X(x)Y(y) \tag{4}$$

which satisfy the homogeneous boundary conditions for $x=0$ and $x=p$, i.e., they satisfy the boundary conditions (28″) for $b(y)=B(y)=0$. By substituting (4) into Eq. (28), we obtain the condition for separability of the variables

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \tag{29}$$

where λ is a constant. From this, we obtain the following eigenvalue problem for determining the functions $X(x)$:

$$X'' + \lambda X = 0,$$

$$X|_{x=0} = 0, \quad X|_{x=p} = 0.$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_n = \left(\frac{n\pi}{p}\right)^2, \quad X_n = \sin\frac{n\pi}{p}x \qquad (n = 1, 2, \ldots).$$

If we substitute λ_n into Eq. (29) in place of λ, we obtain the equation

$$Y_n'' - \left(\frac{n\pi}{p}\right)^2 Y_n = 0$$

for determining $Y(y) = Y_n(y)$. The general solution of this equation is

$$Y_n = \alpha_n \cosh\frac{n\pi}{p}y + \beta_n \sinh\frac{n\pi}{p}y \qquad (\alpha_n, \beta_n = \text{const}).$$

If we substitute the X_n and Y_n into (4) and add all possible solutions of the form (4), we obtain a solution of Eq. (28) satisfying the homogeneous boundary conditions for $x=0$ and $x=p$:

$$u_1(x, y) = \sum_{n=1}^{\infty}\left(\alpha_n \cosh\frac{n\pi}{p}y + \beta_n \sinh\frac{n\pi}{p}y\right)\sin\frac{n\pi}{p}x. \qquad (30)$$

Let us choose the coefficients α_n and β_n in the series (30) in such a way that it will satisfy the boundary conditions (28′). Let us suppose that the functions $a(x)$ and $A(x)$ can be expanded in a uniformly convergent Fourier series

$$a(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{p}x, \quad A(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{p}x.$$

If we first set $y=0$ and then $y=q$ in Eq. (30), we obtain by comparing terms with like values of n in the argument of the sine

$$\alpha_n = a_n, \quad \alpha_n \cosh\frac{n\pi}{p}q + \beta_n \sinh\frac{n\pi}{p}q = A_n.$$

From this we obtain

$$\alpha_n = a_n, \quad \beta_n = \frac{A_n - a_n \cosh\dfrac{n\pi}{p}q}{\sinh\dfrac{n\pi}{p}q}.$$

If we substitute this into Eq. (30), we obtain a solution* of the problem (28)—(28′)—(28″) for $b(y)=B(y)\equiv 0$:

$$u_1(x, y)=\sum_{n=1}^{\infty}\left(a_n \sinh\frac{n\pi}{p}(q-y)+A_n \sinh\frac{n\pi}{p}y\right)\frac{1}{\sinh\frac{n\pi}{p}q}\sin\frac{n\pi}{p}x \quad *.$$

If we sought a solution of the problem (28)—(28′)—(28″) with $a(x)=0$, $A(x)=0$, all the above would remain true without change except that x and y would reverse roles and, consequently, we would obtain a solution in the form

$$u_2(x, y)=\sum_{n=1}^{\infty}\left(b_n \sinh\frac{n\pi}{q}(p-x)+B_n \sinh\frac{n\pi}{q}x\right)\frac{1}{\sinh\frac{n\pi}{q}p}\sin\frac{n\pi}{q}y, \tag{31}$$

where the b_n and the B_n are the Fourier coefficients of the functions $b(x)$ and $B(x)$. If we add the series (30′) and (31), we obtain

$$u(x, t)=\sum_{n=1}^{\infty}\left(\frac{a_n \sinh\frac{n\pi}{p}(q-y)+A_n \sinh\frac{n\pi}{p}y}{\sinh\frac{n\pi}{p}q}\sin\frac{n\pi}{p}x+\right.$$

$$\left.+\frac{b_n \sinh\frac{n\pi}{q}(p-x)+B_n \sinh\frac{n\pi}{q}x}{\sinh\frac{n\pi}{q}p}\sin\frac{n\pi}{q}y\right).$$

Keeping in mind the rate of growth of the hyperbolic sine, we see that the series obtained by differentiating the series (32) twice converges uniformly within the rectangle $0<x<p$, $0<y<q$. Consequently, this series yields a solution of the Dirichlet problem for the rectangle (28)—(28′)—(28″).

In a completely analogous manner, we can find the solution of the second boundary-value problem for Laplace's equation for a rectangle.

5. Solution of the problem of the vibrations of a rectangular membrane. The problem of the free transverse vibrations of a rectangular membrane $0<x<p$, $0<y<q$ with fixed edges reduces to solving the first boundary-value problem of the following form:

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*Here, we use the property that $\sinh(a-b)=\sinh a \cosh b-\cosh a \sinh b$.

$$u_{xx} + u_{yy} - \frac{1}{a^2} u_{tt} = 0, \tag{33}$$

$$u|_{t=0} = \psi(x, y), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x, y), \tag{33'}$$

$$u|_S = 0, \tag{33''}$$

where u is the displacement of the membrane from the equilibrium position, which coincides with the xy-plane, and where S is the boundary of the rectangle $0<x<p$, $0<y<q$. If we set

$$u = V(x, y)\, T(t), \tag{34}$$

where $V(x, y)$ is a function of x and y only and $T(t)$ is a function of t only, we obtain from Eq. (33)

$$T(V_{xx} + V_{yy}) - \frac{1}{a^2} VT'' = 0,$$

where the primes denote differentiation with respect to t. If we divide this last equation by XT, we obtain the condition for separability of the variables:

$$\frac{V_{xx} + V_{yy}}{V} = \frac{T''}{a^2 T} = -\lambda, \tag{35}$$

where λ is a constant. By using the zero boundary conditions (33''), we obtain from (35) the following "two-dimensional" eigenvalue problem for determining $V(x, y)$:

$$V_{xx} + V_{yy} + \lambda V = 0, \tag{36}$$

$$V|_S = 0. \tag{36'}$$

To find the eigenvalues and eigenfunctions of the problem (36)–(36'), we set

$$V(x, y) = X(x)\, Y(y),$$

where $X(x)$ is a function of x only and $Y(y)$ is a function of y only. From Eq. (36), we obtain the condition for separability of the variables,

$$\frac{Y''}{Y} + \lambda = -\frac{X''}{X} = \alpha = \text{const}, \tag{37}$$

where the primes denote differentiation of X with respect to x and differentiation of Y with respect to y. By using the zero

boundary conditions (36′), we obtain the following eigenvalue problems for determining the functions $X(x)$ and $Y(y)$:

$$X'' + \alpha X = 0, \quad X|_{x=0} = 0, \quad X|_{x=p} = 0, \tag{38}$$

$$Y'' + \beta Y = 0, \quad Y|_{y=0} = 0, \quad Y|_{y=q} = 0, \tag{38′}$$

where α and β are constants such that

$$\alpha + \beta = \lambda. \tag{38″}$$

The eigenvalues and eigenfunctions of the problems (38) and (38′) may be written, respectively, in the forms

$$\alpha_n = \left(\frac{n\pi}{p}\right)^2, \quad X_n = \sqrt{\frac{2}{p}} \sin \frac{n\pi}{p} x, \qquad n = 1, 2, \ldots,$$

$$\beta_m = \left(\frac{m\pi}{q}\right)^2, \quad Y_m = \sqrt{\frac{2}{p}} \sin \frac{m\pi}{q} y, \qquad m = 1, 2, \ldots.$$

From this it follows that the eigenvalues and eigenfunctions of the two-dimensional Sturm–Liouville problem (36)–(36′) are *

$$\lambda_{nm} = \pi^2 \left(\frac{n^2}{p^2} + \frac{m^2}{q^2}\right), \quad V_{n,m}(x,y) = \frac{2}{\sqrt{pq}} \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \tag{39}$$

$$(n, m = 1, 2, \ldots).$$

It is interesting to note that these eigenvalues of the problem (36)–(36′) may be multiple eigenvalues, i.e., eigenvalues to which there correspond not one but several linearly independent eigenfunctions. This will be the case when the lengths of the sides of the rectangle p and q are commensurable, so that p/q is a rational number and the numbers m, n, m', and n' are such that $\lambda_{nm} = \lambda_{n'm'}$. For example, if $p = q = 1$, the eigenvalue $\lambda = 5\pi^2$ will be a double eigenvalue since corresponding to it are, in accordance with Eqs. (39), the two linearly independent eigenfunctions

$$V_{12} = 2 \sin \pi x \sin 2\pi y, \quad V_{21} = 2 \sin 2\pi x \sin \pi y.$$

In the case of multiple eigenvalues, the eigenfunctions corresponding to them may have *nodal lines*, i.e., curves along which these functions vanish. These curves may have a quite complicated shape. They are sometimes called Lissajous figures. For example, for the eigenvalue $\lambda = 5\pi^2$ for $p = q = 1$ that we referred to above, we may take as eigenfunctions the functions

*That Eqs. (39) exhaust all the eigenvalues and eigenfunctions of the two-dimensional Sturm-Liouville problems (36)–(36′) follows from the fact that the system of functions (39) is complete in the rectangle $0 < x < p$, $0 < y < q$ (see Sect. 2, subsections 2 and 4).

$$\sin \pi x \sin 2\pi y + \sqrt{\frac{2}{3}} \sin 2\pi x \sin \pi y,$$

and

$$\sin \pi x \sin 2\pi y + \sin 2\pi x \sin \pi y.$$

The nodal curves of these functions are shown, respectively, in Figs. 37 *a* and b.

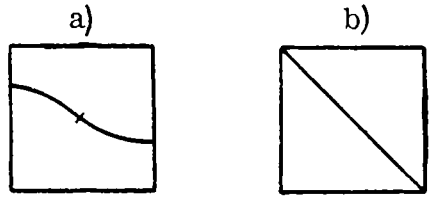

Fig. 37.

The eigenfunctions defined by Eqs. (39) constitute an orthonormal system of functions; that is,

$$\int_0^p \int_0^q V_{nm}(x, y) V_{n'm'}(x, y)\, dx\, dy = \begin{cases} 1, & \text{if } n = n',\ m = m', \\ 0, & \text{if } n \neq n' \text{ or } m \neq m'. \end{cases}$$

If we replace λ in Eq. (25) with λ_{nm}, we have the equation

$$T''_{nm} + a^2 \lambda_{nm} T_{nm} = 0$$

for determining $T = T_{nm}$. The general solution of this equation is

$$T_{nm} = A_{nm} \cos a \sqrt{\lambda_{nm}}\, t + B_{nm} \sin a \sqrt{\lambda_{nm}}\, t,$$

where A_{nm} and B_{nm} are arbitrary constants. If we substitute $V_{nm}(x, y)$ and $T_{nm}(t)$ into Eqs. (34) and add all possible solutions of the form (34), we obtain the solution of Eq. (33) satisfying the homogeneous boundary conditions (33″)

$$u(x, y, t) = \sum_{n,m=1}^{\infty} V_{nm}(x, y) [A_{n,m} \cos \sqrt{\lambda_{nm}}\, t + B_{nm} \sin a \sqrt{\lambda_{nm}}\, t]. \tag{40}$$

It now remains for us to choose the coefficients A_{nm} and B_{nm} in such a way that the series (40) satisfies the initial conditions (33′). To do this, we assume that the functions $\psi(x, y)$ and $\psi_1(x, y)$ can be expanded in a uniformly convergent double Fourier series in the sines of integral multiples of the argument:

$$\psi(x, y) = \sum_{n,m=1}^{\infty} a_{nm} V_{nm}(x, y), \quad \psi_1(x, y) = \sum_{n,m=1}^{\infty} b_{nm} V_{nm}(x, y), \tag{41}$$

where

$$a_{nm} = \int_0^p \int_0^q \psi(x, y) V_{nm}(x, y)\, dx\, dy, \quad b_{nm} = \int_0^p \int_0^q \psi_1(x, y) V_{nm}(x, y)\, dx\, dy. \tag{41'}$$

If we set $t=0$ in Eq. (40), and if we equate the coefficients of sines of like multiples of the argument, we obtain from the first of the initial conditions (33′)

$$A_{nm} = a_{nm}.$$

Similarly, by differentiating the series (40) termwise with respect to t and setting $t=0$, we obtain from the second of the initial conditions (33′)

$$B_{nm} = \frac{b_{nm}}{a\sqrt{\lambda_{nm}}}.$$

If we substitute these values of A_{nm} and B_{nm} into the series (40), we obtain a solution of the boundary-value problem (33)–(33′)–(33″) of the following form:

$$u(x, y, t) = \sum_{n,m=1}^{\infty} V_{n,m}(x, y)\left[a_{nm} \cos a\sqrt{\lambda_{nm}}t + \frac{b_{nm}}{a\sqrt{\lambda_{nm}}} \sin a\sqrt{\lambda_{nm}}t\right], \tag{40'}$$

where the $V_{nm}(x, y)$ are the eigenfunctions and λ_{nm} are the eigenvalues determined by Eqs. (39) and where a_{nm} and the b_{nm} are the Fourier coefficients of the initial functions $\psi(x, y)$ and $\psi_1(x, y)$ determined by Eqs. (41′).

Let us show that for the series obtained from the series (40′) after two termwise differentiations to converge uniformly, it is sufficient to assume that the functions $\psi(x, y)$ and $\psi_1(x, y)$, when continued as odd functions onto the rectangle $|x| \leqslant p$, $|y| \leqslant q$ and then continued onto the entire xy-plane as periodic functions with period $2p$ with respect to x and $2q$ with respect to y, are sufficiently many times continuously differentiable, for example, four times.* It follows from the assumption made that the functions $\psi(x, y)$ and $\psi_1(x, y)$ and their derivatives of even order vanish

*As can be seen from the calculations below, these restrictions can be weakened somewhat.

at $x=y=0$ and at $|x|=p$, $|y|=q$. Therefore, if we integrate by parts in Eq. (41'), we obtain

$$a_{nm}=\left(\frac{n\pi}{p}\right)^{-2}\left(\frac{m\pi}{q}\right)^{-2}\int_0^p\int_0^q\frac{\partial^4\psi(x,y)}{\partial x^2\,\partial y^2}V_{nm}(x,y)\,dx\,dy=$$

$$=\left(\frac{n\pi}{p}\right)^{-2}\left(\frac{m\pi}{q}\right)^{-2}\alpha_{nm}.$$

But the α_{nm} can be regarded as the coefficients in the Fourier series of the function $\partial^4\psi(x,y)/\partial x^2\,\partial y^2$. From the inequality

$$\int_0^p\int_0^q\left(\frac{\partial^4\psi}{\partial x^2\,\partial y^2}-\sum_{n,m=1}^{N}\alpha_{nm}V_{nm}\right)dx\,dy>0$$

or, what is the same thing, from Bessel's inequality

$$\sum_{n,m=1}^{N}\alpha_{nm}^2\leqslant\int_0^p\int_0^q\left[\frac{\partial^4\psi}{\partial x^2\,\partial y^2}\right]^2dx\,dy$$

we see that the series $\sum_{n,m=1}^{\infty}\alpha_{nm}^2$ converges. Convergence of this series implies convergence of the series $\sum_{n,m=1}^{\infty}\frac{|\alpha_{nm}|}{nm}$, since

$$\sum_{n,m=1}^{\infty}\frac{|\alpha_{nm}|}{nm}<\frac{1}{2}\sum_{n,m=1}^{\infty}\left(\alpha_{nm}^2+\frac{1}{n^2m^2}\right).$$

Consequently, the series

$$\sum_{n,m=1}^{\infty}|a_{nm}|\,mn \tag{42}$$

converges since

$$a_{nm}nm=\frac{p^2q^2}{\pi^4}\frac{\alpha_{nm}}{nm}.$$

Analogously, we see that the following series * converge:

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*For this, it will be sufficient to express a_{nm} and b_{nm} in terms of the Fourier coefficients of the corresponding fourth-order derivatives of the functions $\psi(x,y)$ and $\psi_1(x,y)$.

$$\sum_{n,m=1}^{\infty} |a_{nm}| n^2, \quad \sum_{n,m=1}^{\infty} |a_{nm}| m^2, \quad \sum_{n,m=1}^{\infty} |b_{nm}| nm, \quad \sum_{n,m=1}^{\infty} |b_{nm}| n^2, \quad \sum_{n,m=1}^{\infty} |b_{nm}| m^2. \tag{42'}$$

But it is easy to see that the series obtained by differentiating the series (40′) twice with respect to the variables x, y, and t are dominated by the sum of the series (42) and (42′). This means that the series (40′) can be differentiated twice and that this series yields a twice continuously differentiable solution of the problem (33)–(33′)–(33″).

2. The General Theory of the One-Dimensional Eigenvalue Problem and Its Application To the Justification of the Method of Separation of Variables

1. Adjoint and self-adjoint boundary-value problems. As was mentioned above (see Chapter V, Sect. 9, subsection 1), the adjoint of the differential operator

$$L(u) = \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \qquad (a_{ik} = a_{ki}), \tag{43}$$

where the a_{ik}, b_i, and c are given functions of $x_1, x_2, \ldots, x_n$, is a differential operator of the form

$$M(v) = \sum_{i,k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k} (a_{ik} v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (b_i v) + cv. \tag{43'}$$

The property of adjointness of two differential operators is reciprocal, i.e., the adjoint of the differential operator $M(v)$ is the differential operator $L(u)$. If $L(u) \equiv M(u)$, the operator $L(u)$ is said to be *self-adjoint*. A characteristic property of adjoint differential operators $L(u)$ and $M(v)$ is the following: for arbitrary twice continuously differentiable functions u and v, the expression $vL(u) - uM(v)$ is an expression which has the form of a divergence, specifically,

$$vL(u) - uM(v) = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}, \tag{44}$$

where

$$P_i = v\sum_{k=1}^{n} a_{ik}\frac{\partial u}{\partial x_k} - u\sum_{k=1}^{n}\frac{\partial}{\partial x_k}(a_{ik}v) + b_i uv.$$

This property can be thought of as the defining property of adjointness of differential operators $L(u)$ and $M(v)$. Integration of Eq. (44) by parts yields the most general form of Ostrogradskiy's formula

$$\int_G \cdots \int [vL(u) - uM(v)]\,dx_1 \ldots dx_n = -\overbrace{\int_S \cdots \int}^{n-1} \sum_{i=1}^{n} P_i \cos nx_i\, dS, \qquad (45)$$

where G is a region in n-dimensional space $x_1, \ldots, x_n$, S is the surface bounding it, and $\vec{n}$ is the inner normal to the surface S.

Let $\{u\}$ and $\{v\}$ be two sets of functions that satisfy on the surface S certain homogeneous boundary conditions (A) and (B), respectively.* Then, if, for arbitrary functions u and v belonging to these sets, the right-hand member of Eq. (45) vanishes, the boundary conditions (A) and (B) are said to be *adjoint boundary conditions* corresponding to the differential operators $L(u)$ and $M(v)$.

The boundary-value problem for the equation $L(u) = -f$ with boundary conditions (A) and the boundary-value problem for the equation $M(v) = -\varphi$ with adjoint boundary conditions (B) (where f and φ are given functions of $x_1, x_2, \ldots, x_n$) are called *adjoint boundary-value problems*. If the differential operator $L(u)$ is self-adjoint in this case and if the boundary conditions (A) are also self-adjoint, i.e., if they coincide with the boundary conditions (B), then the boundary-value problem for the equation $L(u) = -f$ with boundary conditions (A) is called a *self-adjoint boundary-value problem*.

A characteristic feature of two adjoint boundary-value problems is the fact that

$$\int_G \cdots \int [vL(u) - uM(v)]\,dx_1 \cdots dx_n = 0 \qquad (46)$$

where u is any solution of one and v is any solution of the other.

A self-adjoint boundary-value problem is characterized by the fact that, for arbitrary solutions of u and v of it,

*This means that a given linear combination (with constant or variable coefficients) of the function u and its first derivatives vanish on S. Analogous remarks hold for the case (B).

$$\int_G \cdots \int [vL(u) - uL(v)]\, dx_1 \ldots dx_n = 0. \tag{47}$$

As examples of self-adjoint boundary-value problems, we may cite the first, second, and third boundary-value problems for Poisson's equation $\Delta u = -f$ with homogeneous boundary conditions. Indeed, the Laplacian operator is a self-adjoint differential operator and formula (45) as applied to the Laplacian operator takes the form

$$\int_G \cdots \int [v\Delta u - u\Delta v]\, dx_1 \ldots dx_n = -\int_S \cdots \int \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS.$$

From this, it is clear that the right member of this equation vanishes if u and v satisfy homogeneous boundary conditions of the first, second, or third boundary-value problem. By definition, this means that these boundary-value problems are self-adjoint.

2. The basic properties of the eigenvalues and eigenfunctions of the one-dimensional Sturm-Liouville problem. In solving the boundary-value problems of mathematical physics by the method of separation of variables, we frequently encounter the following *Sturm–Liouville* eigenvalue *problem.*

Find the nontrivial twice continuously differentiable solutions, on an interval $[a, b]$, of the boundary-value problem

$$L(u) + \lambda \rho u = 0 \quad \left(L(u) = \frac{d}{dx}\left(p \frac{du}{dx}\right) - qu\right), \tag{48}$$

$$\begin{aligned} R_0(u) &= u(a)\cos\alpha + u'(a)\sin\alpha = 0, \\ R_1(u) &= u(b)\cos\beta + u'(b)\sin\beta = 0 \end{aligned} \tag{49}$$

and determine the values of the parameter λ corresponding to them. Here, α and β are constants such that $0 \leqslant \alpha$ and $\beta \leqslant 2\pi$ and p, q, and ρ are functions of x such that, on the interval $[a, b]$, the functions p, dp/dx, q, and ρ are continuous and p and ρ are both positive. *

These nontrivial solutions are called eigenfunctions and the values of the parameter λ corresponding to them are called eigenvalues.

We have given the usual formulation of the Sturm-Liouville problem (48)–(49). In connection with questions that we shall encounter later, we introduce the concept of the *"generalized"*

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*Obviously, the boundary condition $R_0(u) = 0$ can be written in the form $Au(a) + Bu'(a) = 0$, where A and B are arbitrary constants which do not vanish simultaneously. The same is true of the boundary condition $R_1(u) = 0$.

Sturm–Liouville problem. The formulation of this problem differs from that of the problem (48)–(49) only with regard to the following four particulars: (1) boundary conditions (49) are replaced with boundary conditions of the form

$$\begin{aligned} \sqrt{p(a)}\, R_0(u) &= \sqrt{p(a)}\,[u(a)\cos\alpha + u'(a)\sin\alpha] = 0, \\ \sqrt{p(b)}\, R_1(u) &= \sqrt{p(b)}\,[u(b)\cos\beta + u'(b)\sin\beta] = 0; \end{aligned} \tag{50}$$

(2) the requirement that p, dp/dx, q, and ρ (with p and ρ positive) be continuous, and that u have continuous second derivatives in the closed interval $[a, b]$ is replaced with the same requirement in the open interval (a, b); (3) the function $\rho\,|u|^2$ is required to be integrable; (4) the limits

$$\lim_{x\to a}\{\sqrt{p(x)}\,[u(x)\cos\alpha + u'(x)\sin\alpha]\}$$

and

$$\lim_{x\to b}\{\sqrt{p(x)}\,[u(x)\cos\beta + u'(x)\sin\beta]\}$$

exist and are equal to zero [in accordance with (50)].

It is easy to see that the differential operator

$$L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) - qu \tag{50'}$$

is self-adjoint and that formulas (44) and (45) as applied to this operator take the form

$$vL(u) - uL(v) = \frac{d}{dx}[p(vu' - uv')], \tag{44'}$$

$$\int_a^b [vL(u) - uL(v)]\,dx = [p(vu' - uv')]_a^b. \tag{45'}$$

Furthermore, if two functions u and v satisfy the boundary conditions

$$\sqrt{p(a)}\,[u(a)\cos\alpha + u'(a)\sin\alpha] = 0,$$

$$\sqrt{p(a)}\,[v(a)\cos\alpha + v'(a)\sin\alpha] = 0,$$

then, treating $\cos\alpha$ and $\sin\alpha$ as our unknowns, we conclude that $p(a)[vu' - uv']_{x=a} = 0$. Similarly, if $\sqrt{p(b)}\,R_1(u) = 0$ and $\sqrt{p(b)}\,R_1(v) = 0$, then $p(b)[vu' - uv']_{x=b} = 0$. From this we conclude that the

Sturm–Liouville problem (48)–(49) and the generalized form of it (48)–(50) are self-adjoint boundary-value problems.* If u and v are two solutions of either of these problems, then

$$\int_a^b [vL(u) - uL(v)]\,dx = 0. \tag{47'}$$

Let us establish certain very simple properties of the eigenvalues and eigenfunctions of the original and generalized Sturm-Liouville problem, (48)–(49) and (48)—(50), respectively.

1. *The eigenfunctions* u_1 *and* u_2 *corresponding to different eigenvalues* λ_1 *and* λ_2 *are orthogonal with weight* ρ; *that is,*

$$\int_a^b \rho(x)\,u_1(x)\,u_2(x)\,dx = 0. \tag{51}$$

Proof: If we multiply the equations

$$L(u_1) + \lambda_1 \rho u_1 = 0,\quad L(u_2) + \lambda_2 \rho u_2 = 0$$

by u_2 and u_1, respectively, and then subtract the second of these from the first and integrate the results from a to b, we obtain

$$\int_a^b [u_2 L(u_1) - u_1 L(u_2)]\,dx + (\lambda_1 - \lambda_2)\int_a^b \rho u_1 u_2\,dx = 0.$$

The first term in the left-hand member of this equation vanishes by virtue of formula (47'). But $\lambda_1 \neq \lambda_2$, so that Eq. (51) is valid.

2. *All the eigenvalues are real.*

Proof: Suppose that there exists a complex eigenvalue λ with eigenfunctions u. Then, its complex conjugate $\overline{\lambda}$ would also be an eigenvalue and the complex conjugate $\overline{u}$ of the function u would be its eigenfunction. From the orthogonality condition

$$\int_a^b \rho u\overline{u}\,dx = \int_a^b \rho |u|^2\,dx = 0$$

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* We note that if, instead of the boundary-value problem (48)–(49), we consider the more general boundary-value problem

$$L(u) = -r(x),\ \alpha_i u(a) + \beta_i u'(a) = \gamma_i u(b) + \delta_i u'(b)$$
$$(\alpha_i, \beta_i, \gamma_i, \delta_i = \text{const},\ i = 1, 2),$$

we can show that a necessary and sufficient condition for this problem to be self-adjoint is that

$$p(b)(\alpha_1\beta_2 - \alpha_2\beta_1) = p(a)(\gamma_1\delta_2 - \gamma_2\delta_1).$$

we conclude that $u \equiv 0$, i.e., that the number λ is not an eigenvalue.

3. *Any two eigenfunctions corresponding to the same eigenvalue are linearly dependent.*

Proof: Let us suppose that u and v are two linearly independent eigenfunctions corresponding to an eigenvalue λ. From identity (44′), we have

$$p(vu' - uv') = \text{const}.$$

Since the left-hand member of this equation vanishes at $x=a$, then, at some point at which $p \neq 0$, we have $vu' - uv' = 0$. Because of a known property of the Wronskian determinant, this last equation means that u and v are linearly dependent, thus proving our assertion.

Obviously, *every eigenfunction u can be normalized:*

$$\int_a^b \rho u^2 dx = 1. \tag{52}$$

The function u which satisfies condition (53) is said to be *normalized* with *weight* ρ.

3. Green's function and the integral equation of an eigenvalue problem. Suppose that $\lambda = 0$ is not an eigenvalue of the Sturm-Liouville problem (48)–(49); that is, suppose that the boundary-value problem

$$L(u) = 0, \quad \left(L(u) = \frac{d}{dx}\left(p \frac{du}{dx}\right) - qu \right), \tag{48′}$$

$$R_0(u) = 0, \quad R_1(u) = 0 \tag{49}$$

does not have nontrivial solutions. Suppose that $u(x)$ and $v(x)$ are twice continuously differentiable solutions of Eq. (48′) satisfying respectively the boundary conditions

$$R_0(u) = 0, \quad R_1(v) = 0.$$

Under the assumptions made regarding the coefficients in the problem (48′)–(49), such solutions u and v exist and can be obtained as solutions of the Cauchy problem with initial conditions $u(a) = -\sin\alpha$, $u'(a) = \cos\alpha$, $v(b) = -\sin\beta$, and $v'(b) = \cos\beta$. The functions u and v will be linearly independent since the supposition that $u = Cv$, where C is a constant, implies that $R_0(u) = 0$ and $R_1(u) = 0$. This means that the problem (48′)–(49) has a nontrivial solution. On the basis of Eq. (44′), the quantity

$$\Delta = p(vu' - uv') \tag{53}$$

is equal to a constant which, because of the linear independence of u and v, will be different from zero.

The function of x, depending on a parameter x_0,

$$G(x_0, x) = \begin{cases} \frac{1}{\Delta} u(x) v(x_0) & x \leqslant x_0, \\ \frac{1}{\Delta} u(x_0) v(x) & x \geqslant x_0 \end{cases} \tag{54}$$

is called the *Green's function* of the boundary-value problem (48′)–(49).

The Green's function $G(x_0, x)$ as a function of x possesses the following properties:

(a) It is continuous on the interval $[a, b]$.

(b) On each of the intervals $[a, x_0]$ and $[x_0, b]$, it is twice continuously differentiable and satisfies the equation $L(u)=0$.

(c) $R_0(G)=0$, $R_1(G)=0$.

(d) $\left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0+0} - \left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0-0} = -\frac{1}{p(x_0)}$

(e) $G(x_0, x)$ is a symmetric function of x_0 and x; that is, $G(x_0, x) = G(x, x_0)$.

Properties (a), (b), and (c) follow immediately from the construction of the Green's function. Property (d) is obtained as follows:

$$\frac{\partial G(x_0, x)}{\partial x} = \frac{1}{\Delta} u(x_0) v'(x), \quad \text{if} \quad x > x_0,$$

$$\frac{\partial G(x_0, x)}{\partial x} = \frac{1}{\Delta} u'(x) v(x_0), \quad \text{if} \quad x < x_0 .$$

Consequently,

$$\left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0+0} - \left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0-0} = \frac{1}{\Delta}(uv' - vu')_{x=x_0} =$$

$$= -\frac{\Delta}{p(x_0)} \frac{1}{\Delta} = -\frac{1}{p(x_0)}.$$

To see that Property (e) is true, we note that, by virtue of (54),

$$G(x_0, x) = \frac{1}{\Delta} u(x_0) v(x) \text{ for } x_0 \leqslant x,$$

$$G(x, x_0) = \frac{1}{\Delta} u(x) v(x_0) \text{ for } x_0 \geqslant x .$$

Consequently, $G(x, x_0) = G(x_0, x)$.

If $\lambda=0$ is an eigenvalue of the problem (48)–(49), i.e., if the problem (48′)–(49) has a nontrivial solution $U(x)$, we can construct a so-called generalized Green's function of the boundary-value

problem (48′)–(49). This generalized function is constructed as follows: Let $u(x)$ and $v(x)$ be twice continuously differentiable solutions of the equation $L(u)=\rho(x)\,U(x)$ which satisfy the conditions $u(a)=-\sin\alpha$, $u'(a)=\cos\alpha$, $v(b)=-\sin\beta$, and $v'(b)=\cos\beta$.

Suppose that $G(x_0, x)$ is a function of x, depending on a parameter x_0, defined by

$$G(x_0, x)=\begin{cases}\dfrac{1}{\Delta_1}(U(x_0)u(x)+U(x)v(x_0))+AU(x_0)U(x), & x\leqslant x_0,\\ \dfrac{1}{\Delta_1}(U(x_0)v(x)+U(x)u(x_0))+AU(x_0)U(x), & x\geqslant x_0,\end{cases}\tag{55}$$

where Δ_1 and A are quantities independent of x. This function $G(x_0, x)$ is a generalized Green's function of the boundary-value problem (48′)–(49). The quantity Δ_1 is defined by the condition that the derivative $\partial G(x_0, x)/\partial x$ has a jump equal to $-1/p(x_0)$ at the point $x=x_0$. This is possible because, if this derivative were continuous at the point $x=x_0$, it would follow from the uniqueness of the solution to the Cauchy problem that the function $G(x_0, x)$ would be a twice continuously differentiable solution of the boundary-value problem

$$L(G)=\frac{1}{\Delta_1}U(x)U_1(x_0)\rho(x),\quad R_0(G)=0,\quad R_1(G)=0.$$

But this is impossible since the opposite assumption, on the basis of formula (47′), means that

$$\frac{1}{\Delta_1}U(x_0)\int_a^b\rho(x)U^2(x)\,dx=\int_a^b UL(G)\,dx=\int_a^b GL(U)\,dx=0,$$

i.e., that $U(x)\equiv 0$. This contradicts the assumption that $U(x)$ is a nontrivial solution of the problem (48′)–(49). The quantity A in Eq. (55) is determined from the requirement that the function of $G(x_0, x)$ be orthogonal to the function $U(x)$ with weight $\rho(x)$, i.e., from the requirement that

$$\int_a^b G(x_0, x)U(x)\rho(x)\,dx=0.$$

As a function of x, the generalized Green's function $G(x_0, x)$ possesses the following properties:

(a′) It is continuous on the interval $[a, b]$;

(b′) On each of the intervals $[a, x_0]$ and $[x_0, b]$, it is twice continuously differentiable and satisfies the equation

$$L(G)=\frac{1}{\Delta_1}U(x_0)U(x)\rho(x);$$

(c′) $R_0(G) = 0,\ R_1(G) = 0;$

(d′) $\left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0+0} - \left.\frac{\partial G(x_0, x)}{\partial x}\right|_{x=x_0-0} = -\frac{1}{p(x_0)};$

(e′) $G(x_0, x)$ is a symmetric function of x and x_0; that is, $G(x_0, x) = G(x, x_0)$.

Properties (a′), (b′), (c′), and (d′) are obvious from the construction of the generalized Green's function. Property (e′) is easily obtained from formula (44′). Specifically, if we set $u = G(\xi, x)$ and $v = G(\eta, x)$ in formula (44′), we obtain

$$G(\eta, x) = \frac{1}{\Delta_1} U(\xi) U(x) \rho(x) - G(\xi, x) \frac{1}{\Delta_1} U(\eta) U(x) \rho(x) =$$

$$= \frac{d}{dx}[p(G(\eta, x) G'(\xi, x) - G(\xi, x) G'(\eta, x)].$$

Let us integrate this equation from a to b. Keeping in mind the orthogonality of the functions $G(x_0, x)$ and $U(x)$ with weight $\rho(x)$ and the discontinuities in the derivatives $G(\xi, x)$ and $G'(\eta, x)$ at the point $x = \xi$, and $x = \eta$, we obtain

$$0 = G(\eta, \xi) - G(\xi, \eta),$$

that is, $G(x_0, x) = G(x, x_0)$. *

THEOREM (on integral representation of solutions by means of the Green's function). *If $\lambda = 0$ is not an eigenvalue of the Sturm–Liouville problem (48)–(49), i.e., if the boundary-value problem (48′)–(49) has no nontrivial solutions, then, for the function $u(x)$ to be a twice continuously differentiable solution of the boundary-value problem*

$$L(u) = -r(x), \tag{48″}$$

*It is easy to show that the quantity Δ_1 in Eq. (55) is independent of x_0. If we set $u(x) = \varphi(x) + V(x)$ and $v(x) = \psi(x) + V(x)$, where $V(x)$ is a solution of the equation $L(u) = U(x)\rho(x)$, we obtain

$$-\frac{1}{p(x_0)} = G_x(x_0, x)\Big|_{x=x_0-0}^{x=x_0+0} = \frac{1}{\Delta_1}[U(x_0)\psi'(x_0) + \varphi(x_0)U'(x_0) -$$

$$- U(x_0)\varphi'(x_0) - U'(x_0)\psi(x_0)] = -\frac{\text{const}}{\Delta_1 p(x_0)},$$

since

$$p(x)(U(x)\psi'(x) - \psi(x)U'(x)) = \text{const},$$

$$p(x)(\varphi(x)U'(x) - \varphi'(x)U(x)) = \text{const}.$$

Consequently, $\Delta_1 = \text{const}$. From this and from the symmetry of the function $G(x_0, x)$, it follows that A in Eq. (55) is independent not only of x but also of x_0.

$$R_0(u) = 0, \quad R_1(u) = 0, \tag{49}$$

on an interval $[a, b]$, *where* $r(x)$ *is a continuous function on* $[a, b]$, *it is necessary and sufficient that*

$$u(x) = \int_a^b G(x, s)\, r(s)\, ds, \tag{56}$$

where $G(x, s)$ *is the Green's function of the boundary-value problem (48′)–(49).*

Proof: The function $u(x)$ defined by Eq. (56) satisfies the boundary conditions $R_0(u)=0$ and $R_1(u)=0$ since $G(x, s)$ as a function of x satisfies these boundary conditions. Furthermore,

$$u(x) = \int_a^x G(x, s)\, r(s)\, ds + \int_x^b G(x, s)\, r(s)\, ds,$$

$$u'(x) = \int_a^x G_x(x, s)\, r(s)\, ds + \int_x^b G_x(x, s)\, r(s)\, ds,$$

$$u''(x) = \int_a^x G_{xx}(x, s)\, r(s)\, ds + \int_x^b G_{xx}(x, s)\, r(s)\, ds - G_x(s, x)\Big|_{s=x-0}^{s=x+0} r(x).$$

Consequently,

$$L(u) = \int_a^x L(G)\, r(s)\, ds + \int_x^b L(G)\, r(s)\, ds + p(x)\,[G_x(x, x+0) - G(x, x-0)]\, r(x) = -r(x).$$

This means that the function defined by Eq. (56) is a twice continuously differentiable solution of the boundary problem (48″)–(49). Conversely, if $u(x)$ is a twice continuously differentiable solution of the boundary-value problem of (48″)–(49), then

$$G(s, x)\, L(u) - uL(G) = -G(s, x)\, r(x)$$

and, on the basis of formula (44′),

$$-G(s, x)\, r(x) = \frac{d}{dx}\,[p(x)\,(G(s, x)\, u' - uG_x(s, x))].$$

If we integrate this equation from a to b, we obtain

$$-\int_a^b G(s, x)\, r(x)\, dx = [p(x)\,(G(s, x)\, u' - uG_x(s, x))]_a^b + [p(x)\,(G(s, x)\, u' - uG_x(s, x))]_{x=s-0}^{x=s+0}.$$

The first term in the right-hand member of this equation is equal to zero since the functions $G(s, x)$ and $u(x)$ satisfy the boundary conditions (49) and, by virtue of the fact that $G_x(s,x)|_{x=s+0} - G_x(s,x)|_{x=s-0} = -\frac{1}{p(s)}$, the second term is equal to $-u(s)$. Thus, every twice continuously differentiable solution of the problem (48″)–(49) must be represented by the equation

$$u(s) = \int_a^b G(s, x)\, r(x)\, dx$$

or, equivalently, by Eq. (56).

Remark: In Eq. (48″), let us set $r(x) = r_\varepsilon(x)$, where $r_\varepsilon(x) = 0$ for $|x-s| > \varepsilon$, $r_\varepsilon(x) > 0$ for $|x-s| \leqslant \varepsilon$ and $\int_a^b r_\varepsilon(x)\, dx = 1$, where ε is an arbitrarily small positive number. Then, the solution of the problem (48″)–(49) can, by virtue of (56), be written in the form $u_\varepsilon(x) = \int_a^b G(x, t)\, r_\varepsilon(t)\, dt$. Taking the limit, we obtain $\lim_{\varepsilon \to 0} u_\varepsilon(x) = G(x, s)$. Therefore, we can assert that the Green's function $G(x, s)$ satisfies the equation $L(G) = -\delta(x-s)$, where $\delta(x-s)$ is Dirac's delta-function generated by the sequence of functions $r_\varepsilon(x)$ as $\varepsilon \to 0$.

Let us show now that the eigenvalue problem (48)–(49) has at least one value of the parameter λ which is not an eigenvalue. If $\lambda = 0$ is not an eigenvalue, we already have an example. If $\lambda = 0$ is an eigenvalue, let $U(x)$ denote the corresponding eigenfunction. Then, every eigenfunction $u(x)$ of the problem (48)–(49) corresponding to the eigenvalue $\lambda \neq 0$ must be orthogonal to the function $U(x)$ with weight $\rho(x)$. If we now substitute the expressions

$$L(u) = -\lambda \rho u, \quad L(G) = \frac{1}{\Delta_1} U(s)\, U(x)\, \rho(x)$$

into the equation

$$G(s, x)\, L(u) - u L(G) = \frac{d}{dx}[p(G(s, x)\, u' - u\, G_x(s, x))], \qquad (57)$$

where $G(s, x)$ is the generalized Green's function of the boundary-value problem (48′)–(49), and if we then integrate this equation from a to b, we obtain, by virtue of the orthogonality of the functions $u(x)$ and $U(x)$ with weight $\rho(x)$,

$$-\lambda \int_a^b G(s, x)\, \rho(x)\, u(x)\, dx = [p(Gu' - uG_x)]_a^b +$$

$$+ [p(x)(G(s, x)\, u' - uG_x(s, x))]_{x=s+0}^{x=s-0}$$

or

$$u(s) = \lambda \int_a^b G(x, s)\, \rho(x)\, u(x)\, dx.$$

This means that every eigenfunction $u(x)$ of the boundary-value problem (48)–(49) must be a solution of the integral equation

$$u(x) = \lambda \int_a^b G(x, s)\, \rho(s)\, u(s)\, ds.$$

But this integral equation, as we know from the theory of integral equations, can have nontrivial solutions only for discrete values of λ, and, consequently, only discrete values of λ can be eigenvalues of the boundary-value problem (48)–(49).

In what follows, we may, without loss of generality, assume that $\lambda=0$ is not an eigenvalue of this problem since, if it is an eigenvalue, we need only to write this problem in the form

$$L(u) + \lambda_0 \rho u + (\lambda - \lambda_0)\, \rho u = 0,\ R_0(u) = 0,\ R_1(u) = 0,$$

where λ_0 is not an eigenvalue of the problem (48)–(49), and then denote $L(u)+\lambda_0\rho u$ by $L(u)$ and $\lambda-\lambda_0$ by λ.

THEOREM. If $\lambda=0$ is not an eigenvalue of the Sturm–Liouville problem

$$L(u) + \lambda \rho u = 0, \tag{48}$$

$$R_0(u) = 0,\ R_1(u) = 0, \tag{49}$$

then, for the function $u(x)$ to be a twice continuously differentiable solution of this problem on the interval $[a, b]$, it is necessary and sufficient that it be a solution of the integral equation

$$u(x) = \lambda \int_a^b G(x, s)\, \rho(s)\, u(s)\, ds, \tag{58}$$

where $G(x, s)$ is the Green's function of the boundary-value problem (48′)–(49) defined by Eq. (54).

The assertions of this theorem are obtained immediately from the preceding theorem if we set the $r(x)=\lambda\rho u$ in Eq. (48″).

We shall call integral equation (58) the *integral equation of the eigenvalue problem (48)–(49).* The kernel of this integral equation $G(x, s)\ \rho(s)$ has the property of being closed; that is, the equation

$$\int_a^b G(x, s)\, \rho(s)\, f(s)\, ds = 0$$

implies that $f(x) \equiv 0$. To see this, set

$$F(x) = \int_a^b G(x, s)\, \rho(s) f(s)\, ds.$$

On the basis of the theorem on the integral representation of solutions of the boundary-value problem (48″)–(49) by means of Green's function, we obtain

$$L(F) = -\rho(x) f(x).$$

But $L(F) \equiv 0$ since $F \equiv 0$. Therefore, since $\rho(x) > 0$ we have $f(x) \equiv 0$.

4. An expansion theorem. Complete orthonormal systems of functions. A completeness theorem.

THEOREM (V. A. Steklov) (an expansion theorem). *An arbitrary twice continuously differentiable function $f(x)$ defined on an interval $[a, b]$, which satisfies the boundary conditions $R_0(f) = 0$ and $R_1(f) = 0$, can be expanded on that interval in the eigenfunctions of the Sturm–Liouville problem*

$$L(u) + \lambda \rho u = 0, \tag{48}$$

$$R_0(u) = 0, \quad R_1(u) = 0 \tag{49}$$

in an absolutely and uniformly convergent Fourier series

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x), \tag{59}$$

where the $u_n(x)$ are the eigenfunctions of the problem (48)–(49) corresponding to the eigenvalues λ_n satisfying the condition of orthonormality with weight $\rho(x)$:

$$\int_a^b u_n(x)\, u_k(x)\, \rho(x)\, dx = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \neq k, \end{cases} \tag{59′}$$

where the c_n are the Fourier coefficients of the function $f(x)$:

$$c_n = \int_a^b f(x)\, u_n(x)\, \rho(x)\, dx. \tag{59″}$$

Proof: Since the function $f(x)$ is twice continuously differentiable and the function $\rho(x)$ is positive, we can write

$$L(f) = -\sqrt{\rho(x)}\, r(x),$$

where $r(x)$ is a continuous function of x. Since the function $f(x)$ satisfies the boundary conditions $R_0(f)=0$ and $R_1(f)=0$, it follows from the theorem of the integral representation of solutions of the problem (48″)–(49) by means of the Green's function (see subsection 3) that

$$f(x) = \int_a^b G(x, s) \sqrt{\rho(s)}\, r(s)\, ds.$$

If we multiply both members of this equation by $\sqrt{\rho(x)}$, we obtain

$$f(x)\sqrt{\rho(x)} = \int_a^b K(x, s)\, r(s)\, ds, \tag{60}$$

where

$$K(x, s) = G(x, s)\sqrt{\rho(x)\,\rho(s)}. \tag{60'}$$

Equation (60) means that the function $f(x)\sqrt{\rho(x)}$ is a source-like function, which can be represented by means of a symmetric kernel (60′). Since the function $r(s)$ is continuous, it follows from the Hilbert–Schmidt theorem in the theory of integral equations with symmetric kernel that the function $f(x)\sqrt{\rho(x)}$ can be represented in the form of an absolutely and uniformly convergent Fourier series of orthonormal eigenfunctions $\varphi_n(x)$ of the integral equation

$$\varphi(x) = \lambda \int_a^b K(x, s)\, \varphi(s)\, ds. \tag{61}$$

This Fourier series is of the form

$$f(x)\sqrt{\rho(x)} = \sum_{n=1}^{\infty} c_n' \varphi_n(x), \tag{62}$$

where

$$c_n' = \int_a^b f(x)\sqrt{\rho(x)}\, \varphi_n(x)\, dx.$$

Thus, if we compare integral equations (61) with the integral equation of the Sturm-Liouville problem (48)–(49)

$$u(x) = \lambda \int_a^b G(x, s)\, \rho(s)\, u(s)\, ds, \tag{58}$$

we conclude that the latter integral equation reduces to Eq. (61) if we multiply both members of it by $\sqrt{\rho(x)}$ and set

$u(x)\sqrt{\rho(x)}=\varphi(x)$. This means that the eigenvalues of the integral equations (61) and (58) coincide and that the corresponding eigenfunctions $\varphi_n(x)$ of the integral equation (61) and $u_n(x)$ of the integral equation (58) are related by $\varphi_n(x)=u_n(x)\sqrt{\rho(x)}$. If we divide both sides of Eq. (62) by $\sqrt{\rho(x)}$, we obtain

$$f(x)=\sum_{n=1}^{\infty} c_n' u_n(x).$$

As before, this series will converge absolutely and uniformly, and, by virtue of the equations

$$c_n'=\int_a^b f(x)\sqrt{\rho(x)}\,\varphi_n(x)\,dx=\int_a^b f(x)\,\rho(x)\,u_n(x)\,dx=c_n$$

this series coincides with the series (59). This completes the proof of the theorem.

Let $u_1(x)$, $u_2(x)$, ... be a system of eigenfunctions of the problem (48)–(49) or an arbitrary system of twice continuously differentiable functions on the interval $[a, b]$ that are normalized with weight $\rho(x)$, i.e., that satisfy the conditions* (59′). Then, this system of functions is said to be *complete* on the interval $[a, b]$, if, for every function $f(x)$ which is piecewise continuous and square-integrable on the interval $[a, b]$, i.e., which satisfies the inequality

$$\int_a^b f^2(x)\,dx<\infty,$$

the equation

$$\sum_{k=1}^{\infty} c_k^2=\int_a^b f^2(x)\,\rho(x)\,dx, \tag{63}$$

where the c_k are the Fourier coefficients of the function $f(x)$

$$c_k=\int_a^b f(x)\,u_n(x)\,\rho(x)\,dx \tag{63′}$$

holds. Equation (63) is called the *completeness* relation or the *Parseval-Steklov relation.***

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*Here, as problem (48)–(49), the function $\rho(x)$ is assumed positive and continuous on the interval $[a, b]$.

**Steklov called Eq. (63) the equation of closure (see V. A. Steklov, *Osnovnyye zadachi matematicheskoy fiziki* (Basic problems of mathematical physics), St. Petersburg, 1922.

By virtue of the orthonormality of the functions $u_1, u_2, \ldots$, we have

$$\int_a^b \rho(x)[f(x) - \sum_{k=1}^n c'_k u_k(x)]^2 dx = \int_a^b \rho f^2 dx - 2\sum_{k=1}^n c'_k c_k + \\ + \sum_{k=1}^n c'^2_k = \int_a^b \rho f^2 dx - \sum_{k=1}^n c_k^2 + \sum_{k=1}^n (c_k - c'_k)^2, \quad (64)$$

where the c'_k are arbitrary constants, and the c_k are the Fourier coefficients of the function $f(x)$. The quantity

$$I = \sqrt{\int_a^b \rho(x)[f(x) - \sum c'_k u_k(x)]^2 dx} \quad (65)$$

is called the *mean-square deviation* of the functions $f(x)$ and $\sum_{k=1}^n c'_k u_k(x)$. We see from Eq. (64) that the mean-square deviation attains its smallest value if we set $c'_k = c_k$. By virtue of (64), this smallest value of the mean-square deviation is given by the equation

$$I^2 = \int_a^b \rho(x) f^2(x) dx - \sum_{k=1}^n c_k^2. \quad (64')$$

It follows from Eq. (64′) that, in particular,

$$\sum_{k=1}^n c_k^2 \leqslant \int_a^b \rho(x) f^2(x) dx. \quad (64'')$$

This inequality is called *Bessel's inequality*. On the basis of Eq. (64′), we may make the following assertion:

For a system of functions $u_1, u_2, \ldots$ to be complete, it is necessary and sufficient that, for every function $f(x)$ which is piecewise continuous and square-integrable on the interval $[a, b]$

$$\lim_{n\to\infty} \int_a^b \rho(x)[f(x) - \sum_{k=1}^n c_k u_k(x)]^2 dx = 0.$$

This equation means that the mean-square deviation of the sum $\sum_{k=1}^n c_k u_k(x)$ from the function $f(x)$ approaches zero as $n \to \infty$. Whenever the mean-square deviation of some particular functions $f_n(x)$ from $f(x)$ approaches zero as $n \to \infty$, we say that the sequence of functions $f_n(x)$ *converges in mean* to the function $f(x)$. Thus, for an orthonormal system of functions $u_1, u_2, \ldots$, with weight $\rho(x)$ to be complete, it is necessary and sufficient that the Fourier series of an arbitrary function $f(x)$ that is piecewise continuous

and square-integrable on an interval $[a, b]$ converge to it in the mean or, what amounts to the same thing, it is necessary and sufficient that it be possible to approximate any of the functions $f(x)$ in the sense of mean-square deviation with an arbitrary prescribed degree of accuracy.

THEOREM (on completeness). *An orthonormal system of eigenfunctions of the Sturm–Liouville problem with weight* $\rho(x)$

$$L(u) + \lambda \rho u = 0, \tag{48}$$

$$R_0(u) = 0, \quad R_1(u) = 0 \tag{49}$$

is complete on the interval $[a, b]$.*

Proof: Let $f(x)$ be an arbitrary piecewise-continuous square-integrable function on the interval $[a, b]$. Let us choose a function $u(x)$ which is twice continuously differentiable on the interval $[a, b]$ and which satisfies the boundary conditions $R_0(u)=0$ and $R_1(u)=0$ and the inequality

$$\int_a^b \rho(x)[f(x) - u(x)]^2 dx < \varepsilon,$$

where ε is an arbitrarily small positive number. It is possible to choose such a function $u(x)$ in the form of a piecewise-linear function with a sufficiently large number of segments, the corners of which are smoothed over by means of sufficiently small arcs with continuous curvature. For the function $u(x)$ there exists a polynomial $\sigma_n(x) = \sum_{k=1}^{n} c_k u_k(x)$ composed of the eigenfunctions of the problem (48)–(49) such that the inequality

$$|u(x) - \sigma_n(x)| < \varepsilon$$

will be satisfied on the interval $[a, b]$. Since $(a+b)^2 \leqslant 2(a^2+b^2)$, we have

$$\int_a^b \rho(f-\sigma_n)^2 dx - \int_a^b \rho[(f-u)+(u-\sigma)]^2 dx \leqslant 2\int_a^b \rho(f-u)^2 dx + 2\int_a^b \rho(u-\sigma_n)^2 dx$$

and, consequently,

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*Here, the assumptions about the coefficients p, q, and ρ are the same as in the general case of the statement of the problem (48)–(49): Everywhere on the interval $[a, b]$, the functions p, q, and ρ are functions of x such that p, dp/dx, q, and ρ are continuous, with p and ρ positive, and α and β are constants such that $0 \leqslant \alpha, \beta < 2\pi$.

$$\int_a^b \rho\,(f-\sigma_n)^2\,dx \leqslant 2\varepsilon + 2\varepsilon \int_a^b \rho\,dx = \varepsilon',$$

where ε' is an arbitrarily small positive number. But if we substitute the Fourier coefficient c_k of the function $f(x)$ for the c_k' in this inequality, the new inequality will a fortiori hold since, for given n, the Fourier coefficients minimize the mean-square deviation of the polynomial $\sigma_n(x) = \sum_{k=1}^{n} c_k' u_k(x)$ from the function $f(x)$. Thus, we have

$$\int_c^b \rho\,[f(x) - \sum_{k=1}^{n} c_k u_k(x)]^2\,dx < \varepsilon'.$$

This completes the proof of the theorem.

Remark: If the function $\rho(x)$ were not positive in the problem (48)–(49), the assertions of the expansion and completeness theorems would not hold. A counterexample would be the problem $u''+\lambda\rho u=0$, $R_0(u)=0$, $R_1(u)=0$ with $\rho\equiv 0$. The general solution of the equation $u''=0$ is a linear function of x by means of which we cannot approximate every piecewise-continuous square integrable function in the sense of mean square with an arbitrary degree of accuracy.

Just as in the case of a single independent variable, the concept of orthonormal systems of functions of two or more independent variables is useful.

Let $u_1(x, y)$, $u_2(x, y)$, ..., $u_n(x, y)$, ... be a system of continuous functions in the closed rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, and let $\rho(x, y)$ be a positive continuous function in that rectangle. Then, the system of functions is said to be *orthogonal* with weights $\rho(x, y)$ if

$$\int_a^b \int_c^d u_n(x, y)\,u_m(x, y)\,\rho(x, y)\,dx\,dy = 0 \quad (m \neq n),$$

and it is said to be *orthonormal* with weight $\rho(x, y)$ if

$$\int_a^b \int_c^d u_n(x, y)\,u_m(x, y)\,\rho(x, y)\,dx\,dy = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

An orthonormal system of functions $u_1(x, y)$, $u_2(x, y)$, ... with weight $\rho(x, y)$ is said to be complete in a closed rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, if, for every function $f(x, y)$ that is continuous and square integrable in the rectangle $a<x<b$, $c<y<d$, i.e., such that

$$\int_a^b \int_c^d f^2(x, y)\,dx\,dy < \infty,$$

the Parseval–Steklov relation

$$\sum_{n=1}^{\infty} c_n^2 = \int_a^b \int_c^d f^2(x, y)\, \rho(x, y)\, dx\, dy,$$

holds, where the c_n are the Fourier coefficients of the function $f(x, y)$:

$$c_n = \int_a^b \int_c^d f(x, y)\, u_n(x, y)\, \rho(x, y)\, dx\, dy.$$

In solving the problems of mathematical physics by the method of separation of variables, we often encounter such orthonormal functions of two variables which are products of a function of x only and a function of y only (cf. the example, Sect. 1, subsection 5). A remarkable property of such orthonormal systems of functions of two variables is the fact that, in a number of cases, their completeness follows from the completeness of systems of the functions of a single variable from which they are formed. Specifically, we have the following.

LEMMA (on the completeness of a system of functions of two variables). *Suppose that* $X_1(x), X_2(x), \ldots, X_n(x), \ldots$ *is a complete system of orthonormal functions on an interval* $[a, b]$ *with weight* $\rho_1(x)$ *and suppose that each of the systems of functions*

$$Y_{n1}(y), Y_{n2}(y), \ldots, Y_{nm}(y), \ldots \quad (n = 1, 2, \ldots)$$

defined on the interval $[c, d]$ *is complete and orthonormal with weight* $\rho_2(y)$. *Then, the system of functions of two variations*

$$u_{nm}(x, y) = X_n(x)\, Y_{nm}(y) \quad (n = 1, 2, \ldots, m = 1, 2, \ldots)$$

will be complete and orthonormal with weight $\rho(x, y) = \rho_1(x)\, \rho_2(y)$ *in the closed rectangle* $a \leqslant x \leqslant b,\ c \leqslant y \leqslant d$.

Proof: The fact that the functions $u_{nm}(x, y)$ are orthonormal with weight $\rho(x, y)$ is obvious since the integral

$$\int_a^b \int_c^d u_{nm} u_{n'm'}\, \rho\, dx\, dy = \int_a^b X_n X_{n'} \rho_1\, dx \int_c^d Y_m Y_{m'} \rho_2\, dy$$

is equal to unity for $n = n'$, $m = m'$ and is equal to zero in all other cases.

Thus, it only remains to show that, for every function $f(x, y)$, which is continuous and square integrable in the rectangle $a < x < b$, $c < y < d$, the Parseval–Steklov relation

$$\int_a^b \int_c^d f^2(x, y)\, \rho(x, y)\, dx\, dy = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}^2,$$

holds, where the c_{nm} are the Fourier coefficients:

$$c_{nm} = \int_a^b \int_c^d f(x, y)\, u_{nm}(x, y)\, \rho(x, y)\, dx\, dy.$$

Reasoning as in the proof of the completeness theorem that we have proved, we see that, without loss of generality, it will be sufficient for us to prove that the Parseval–Steklov relation is satisfied for every function $f(x, y)$ which is continuous in the closed rectangle $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. By virtue of the completeness of the system of functions $X_1(x), X_2(x), \ldots, X_n(x), \ldots$, we have

$$\int_a^b f^2(x, y)\, \rho_1(x)\, dx = \sum_{n=1}^{\infty} g_n^2(y), \tag{65'}$$

where

$$g_n(y) = \int_a^b f(x, y)\, X_n(x)\, \rho_1(x)\, dx.$$

The left-hand member of Eq. (65′) and $g_n(y)$ (for $n = 1, 2, \ldots$) are continuous functions of y. Therefore, by Dini's theorem, the series on the right-hand side of Eq. (65′) converges uniformly* on the interval $[c, d]$. If we multiply both sides of Eq. (65′) by $\rho_2(y)$ and integrate with respect to y and from c to d, we obtain

$$\int_a^b \int_c^d f^2(x, y)\, \rho(x, y)\, dx\, dy = \sum_{n=1}^{\infty} \int_c^d g_n^2(y)\, \rho_2(y)\, dy.$$

But, by virtue of the completeness of each of the systems of functions $Y_{n1}(y), Y_{n2}(y), \ldots, Y_{nm}(y), \ldots$ (for $n = 1, 2, \ldots$), we have

$$\int_c^d g_n^2(y)\, \rho_2(y)\, dy = \sum_{m=1}^{\infty} c_{nm}^2.$$

Therefore,

$$\int_a^b \int_c^d f^2(x, y)\, \rho(x, y)\, dx\, dy = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}^2.$$

This completes the proof of the lemma.

5. Asymptotic formulas for eigenvalues, eigenfunctions, and their derivatives. To see how the eigenvalues λ_n and the corresponding eigenfunctions $u_n(x)$ of the Sturm–Liouville problem

*Let us recall Dini's theorem: *For a series of functions that are continuous and nonnegative on an interval $[a, b]$ to converge uniformly on that interval, it is necessary and sufficient that its sum be a continuous function on the interval $[a, b]$.*

$$L(u) + \lambda\rho u = 0 \quad \left(L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) - qu\right), \tag{48}$$

$$R_0(u) = u(a)\cos\alpha + u'(a)\sin\alpha = 0, \quad R_1(u) = u(b)\cos\beta + \\ + u'(b)\sin\beta = 0 \tag{49}$$

behave as $n \to \infty$, let us reduce the problem (48)–(49) to a simpler, so-called *canonical form*. Here, in addition to the restrictions placed above on the functions p and ρ, let us assume that these functions are twice continuously differentiable on the segment $[a, b]$. Thus, what we are assuming is, essentially, that the functions p and ρ are positive and twice continuously differentiable on $[a, b]$ and that the function q is continuous on $[a, b]$.

Let us introduce a new independent variable t and a new unknown function $v = v(t)$ defined by

$$u = \frac{1}{\sqrt[4]{p\rho}}v, \quad t = \int_a^x \sqrt{\frac{\rho}{p}}\,dx, \quad 0 \leqslant t \leqslant l, \quad l = \int_a^b \sqrt{\frac{\rho}{p}}\,dx. \tag{66}$$

Then, as direct verification shows, Eq. (48) is reduced to the equation

$$\frac{d^2v}{dt^2} - hv + \lambda v = 0, \tag{67}$$

where

$$h(t) = -\frac{\sqrt[4]{p\rho}}{\rho}\frac{d}{dx}\left(p\frac{d}{dx}\frac{1}{\sqrt[4]{p\rho}}\right) + \frac{q}{\rho} = \frac{1}{\sqrt[4]{p\rho}}\frac{d^2}{dt^2}\sqrt[4]{p\rho} + \frac{q}{\rho}, \tag{67'}$$

and λ is the same as in Eq. (48). Indeed, if we set

$$u = \psi v, \quad \frac{dt}{dx} = \varphi,$$

we have

$$p\frac{du}{dx} = p\left(\frac{dv}{dx}\psi + v\frac{d\psi}{dx}\right) = p\left(\frac{dv}{dt}\psi\varphi + v\frac{d\psi}{dx}\right),$$

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) = \frac{d^2v}{dt^2}p\psi\varphi^2 + \frac{dv}{dt}\frac{d}{dx}(p\psi\varphi) + \frac{dv}{dt}p\frac{d\psi}{dx}\varphi + v\frac{d}{dx}\left(p\frac{d\psi}{dx}\right)$$

and, consequently,

$$L(u) + \lambda\rho u = \frac{d^2v}{dt^2}p\psi\varphi^2 + \frac{dv}{dt}\left[\frac{d}{dx}(p\psi\varphi) + p\frac{d\psi}{dx}\varphi\right] + \\ + v\frac{d}{dx}\left(p\frac{d\psi}{dx}\right) - q\psi v + \lambda\rho\psi v = 0.$$

If we divide the last equation by $p\psi\varphi^2$ and set

$$\frac{\rho}{p\varphi^2} = 1, \quad \frac{d}{dx}(p\psi\varphi) + p\frac{d\psi}{dx}\varphi = 0,$$

we obtain

$$\varphi = \sqrt{\frac{\rho}{p}}, \quad \frac{d}{dx}(\psi\sqrt{p\rho}) + \sqrt{p\rho}\frac{d\psi}{dx} = 0, \quad \psi = \frac{c}{\sqrt[4]{p\rho}} \ (c = \text{const}).$$

Consequently, the transformation (66) does reduce Eq. (48) to the form (67) with the function h defined by Eq. (67′).*

Let us see now how the operators in the boundary conditions (49) are transformed when we make the change of variables (66). We have

$$R_0(u) = \cos\alpha \left.\frac{v}{\sqrt[4]{p\rho}}\right|_{t=0} + \sin\alpha\, v \left.\frac{d}{dx}\frac{1}{\sqrt[4]{p\rho}}\right|_{t=0} +$$

$$+ \sin\alpha \left.\frac{1}{\sqrt[4]{p\rho}}\sqrt{\frac{\rho}{p}}\frac{dv}{dt}\right|_{t=0} = 0.$$

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* For completeness, we note three other forms of second-order differential equations to which Eq. (48) can be reduced:

$$\frac{d^2u}{dt^2} - pqu + \lambda p\rho u = 0, \quad \frac{d}{dx}\left(p^*\frac{dv}{dx}\right) - q^*v + \lambda v = 0$$

$$\frac{d^2v}{dx^2} - q^{**}v + \lambda\frac{\rho}{p}v = 0.$$

The first of the equations is obtained from (48) by the substitution $t = \int_a^x \frac{dx}{p(x)}$, the second by the substitutions

$$u = \frac{v}{\sqrt{\rho}}, \quad p^* = \frac{p}{\rho}, \quad q^* = -\frac{1}{\sqrt{\rho}}\frac{d}{dx}\left(p\frac{d}{dx}\frac{1}{\sqrt{\rho}}\right) + \frac{q}{\rho},$$

and the third by the substitutions

$$u = \frac{v}{\sqrt{p}}, \quad q^{**} = \frac{q}{p} + \frac{1}{\sqrt{p}}\frac{d^2}{dx^2}\sqrt{p}.$$

An equation of the general form $u'' + pu' - qu + \lambda\rho u = 0$ is, after multiplication by $e^{\int p\,dx}$, reduced to a self-adjoint equation of the type (48). After we make the change of unknown function $v = ue^{\frac{1}{2}\int p\,dx}$, it takes the form $v'' + I(x)\,v + \lambda\rho v = 0$, where

$$I(x) + \lambda\rho = -q - \frac{p^2}{4} - \frac{p'}{2} + \lambda\rho$$

is a so-called *invariant*.

This means that the boundary conditions (49) for the function $u(x)$ become boundary conditions of the following form for the function $v(t)$:

$$R_0^*(v) = v(0)\cos A + v'(0)\sin A = 0,\ R_1^*(v) = v(l)\cos B + v'(l)\sin B = 0, \tag{68}$$

where A and B are given angles such that $0 \leqslant A, B < 2\pi$. Here, it is essential that $\sin\alpha$ and $\sin A$ both be zero or not zero, simultaneously, and similarly for $\sin\beta$ and $\sin B$.

Thus, the eigenvalue problem (48)–(49) reduces to an eigenvalue problem of the following form:

$$\frac{d^2v}{dt^2} - hv + \lambda v = 0, \tag{67}$$

$$R_0^*(v) = v(0)\cos A + v'(0)\sin A = 0,\ R_1^*(v) = v(l)\cos B + v'(l)\sin B = 0. \tag{68}$$

Here, the eigenvalues of the problem (48)–(49) and the problem (67)–(68) coincide and the eigenfunctions corresponding to them are expressed in terms of each other by means of Eqs. (66). Therefore, let us examine the problem (67)–(68) in detail. Here, we need to distinguish between four cases into which it is natural to break up the general case of the problem. These four cases are as follows:

(a) $\sin A = 0$, $\sin B = 0$;
(b) $\sin A \neq 0$, $\sin B \neq 0$;
(c) $\sin A = 0$, $\sin B \neq 0$;
(d) $\sin A \neq 0$, $\sin B = 0$;

Let us show first that the set of eigenvalues of the problem (67)–(68) is bounded below.

Let $v(t)$ denote an arbitrary twice continuously differentiable nontrivial solution of Eq. (67), and let $z(t)$ denote a twice continuously differentiable nowhere-zero solution of the equation

$$\frac{d^2z}{dt^2} - \mu^2 z = 0, \tag{69}$$

where μ is a sufficiently large positive number. From Eqs. (67) and (69), we obtain

$$\frac{d}{dt}\left[v^2\left(\frac{v'}{v} - \frac{z'}{z}\right)\right] = \frac{d}{dt}\left(vv' - z'\frac{v^2}{z}\right) =$$

$$= vv'' + v'^2 - z''\frac{v^2}{z} - z'\frac{2vv'z - z'v^2}{z^2} = v^2(h-\lambda) + v'^2 - \mu^2 v^2 -$$

$$- z'\frac{2vv'z - z'v^2}{z^2} = v^2(-\mu^2 + h - \lambda) + \frac{(v'z - vz')^2}{z^2}.$$

Suppose now that λ is a negative number of sufficiently great absolute value that $-\mu^2+h-\lambda>0$. If we integrate the above equation from 0 to l, we obtain the strict inequality

$$v^2(l)\left[\frac{v'}{v}-\frac{z'}{z}\right]_{t=l}-v^2(0)\left[\frac{v'}{v}-\frac{z'}{z}\right]_{t=0}>0. \tag{70}$$

It follows from this inequality that in each of the cases (a), (b), (c), and (d), the set of eigenvalues is bounded below. Indeed, we obtain a contradiction in each of the four cases by assuming the opposite. In case (a), we take $v(0)=0$ and $v(l)=0$; this contradicts the strict inequality (70). In case (c), we take $v(0)=0$ and $v(l)\neq 0$. If we set $z=e^{-\mu t}$, inequality (70) takes the form

$$\frac{v'(l)}{v(l)}>\mu.$$

But this is a contradiction since μ can be made arbitrarily large and $v'(l)/v(l)=\cot B\neq\infty$. In case (d), we set $v(0)\neq 0$ and $v(l)=0$. If we set $z=e^{\mu t}$, inequality (70) takes the form

$$-\frac{v'(0)}{v(0)}>\mu.$$

But this is a contradiction since $-v'(0)/v(0)=\cot A\neq\infty$ and we can take μ arbitrarily large if λ is a negative number of sufficiently large absolute value. Finally, in case (b), we take $v(0)\neq 0$ and $v(l)\neq 0$. If we set $z=\cosh\mu t$, inequality (70) takes the form

$$\frac{v'(l)}{v(l)}>\mu\frac{\sinh\mu l}{\cosh\mu l}-\frac{v^2(0)}{v^2(l)}\cot A. \tag{70'}$$

But it is clear from the differential equation (67) that if we take (without loss of generality) $v(0)>0$, then for negative values of λ of sufficiently large absolute value, the graph of the function $v=v(t)$ will be convex downward and $v(l)>v(0)$. Therefore, the second term on the right-hand side of inequality (70') is bounded. Consequently, this inequality leads us to a contradiction since the number μ can be made arbitrarily large and $v'(l)/v(l)=\cot B\neq\infty$. This completes our assertion that the eigenvalues of the problem (67)–(68) are bounded below.

In view of this, when we study the eigenvalues of the problem (67)–(68) from the point of view of their asymptotic behavior, we need consider only the positive eigenvalues. Considering the parameter λ positive in Eq. (67), we introduce the notation $\lambda=\mu^2$, where μ is a positive number.

Suppose that $w(t)=w(t,\lambda)$ is a solution of Eq. (67) that satisfies the initial conditions $w(0)=-\sin A$ and $w'(0)=\cos A$ and hence satisfies the first of the boundary conditions (68).

By differentiating, we easily see that this solution can be written in the form

$$w(t) = -\sin A \cos \mu t + \cos A \frac{\sin \mu t}{\mu} + \int_0^t \frac{\sin \mu (t-\xi)}{\mu} h(\xi) w(\xi) d\xi. \qquad (72)$$

Define

$$M = \max_{0 \leqslant t \leqslant l} |w(t)|, \quad L = \int_0^l |h(t)| dt.$$

Then, Eq. (72) yields

$$M < |\sin A| + \frac{|\cos A|}{\mu} + \frac{ML}{\mu}$$

or

$$M\left(1 - \frac{L}{\mu}\right) < |\sin A| + \frac{1}{\mu} |\cos A|.$$

This means that for large values of $\mu = \sqrt{\lambda}$

$$M = \begin{cases} O\left(\frac{1}{\mu}\right), & \text{if} \quad \sin A = 0, \\ O(1), & \text{if} \quad \sin A \neq 0. \end{cases} \qquad (73)$$

If we substitute (72) into the operator of the boundary conditions $R_1^{\bullet}(w)$, we obtain

$$R_1^{\bullet}(w) = \cos B \left(-\sin A \cos \mu l + \frac{1}{\mu} \cos A \sin \mu l\right) +$$

$$+ \sin B (\mu \sin A \sin \mu l + \cos A \cos \mu l) +$$

$$+ \int_0^l \left[\cos B \frac{\sin \mu (l-\xi)}{\mu} + \sin B \cos \mu (l-\xi)\right] h(\xi) w(\xi) d\xi. \qquad (74)$$

To find those values of μ at which $R_1^{\bullet}(w)$ vanishes, i.e., to find the eigenvalues $\lambda = \mu^2$ of the problem (67)–(68), we examine the principal parts of $R_1^{\bullet}(w)$ and $dR_1^{\bullet}(w)/d\mu$ for sufficiently large μ. In cases (a), (b), (c), and (d), we obtain from Eq. (74) the following values respectively:

(a) $$R_1^{\bullet}(w) = \cos A \cos B \frac{\sin \mu l}{\mu} + O\left(\frac{1}{\mu^2}\right),$$

$$\frac{dR_1^{\bullet}(w)}{d\mu} = \cos A \cos B \frac{l \cos \mu l}{\mu} + O\left(\frac{1}{\mu^2}\right);$$

(b) $R_1^*(w) = \mu \sin A \sin B \sin \mu l + O(1)$,

$$\frac{dR_1^*(w)}{d\mu} = \mu \sin A \sin B \cdot l \cos \mu l + O(1);$$

(c), (d) $R_1^*(w) = \sin(B - A) \cos \mu l + O\left(\frac{1}{\mu}\right)$,

$$\frac{dR_1^*(\omega)}{d\mu} = -\sin(B - A)\, l \sin \mu l + O\left(\frac{1}{\mu}\right).$$

It follows from these expressions for $dR_1^*(w)/d\mu$ that the function $R_1^*(w)$ increases or decreases in a neighborhood of those values of μ in which the main term in $R_1^*(w)$ vanishes. Thus, from these expressions, we conclude that, for every sufficiently large integer, there exists exactly one number μ_n such that $R_1^*(w)=0$. Also,

$$\mu_n = \begin{cases} \dfrac{n\pi}{l} + \delta_n & \text{in cases (a), (b),} \\ \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{l} + \delta_n & \text{in cases (c), (d),} \end{cases}$$

where δ_n is a quantity that approaches zero as $n \to \infty$. If we substitute these values of μ_n into equation $R_1^*(w)=0$ in cases (a), (b), (c), and (d), we obtain

(a) $(-1)^n \cos A \cos B \dfrac{\sin l\delta_n}{\mu_n} + O\left(\dfrac{1}{\mu_n^2}\right) = 0;$

(b) $(-1)^n \mu_n \sin A \sin B \sin l\delta_n + O(1) = 0;$

(c), (d), $(-1)^{n+1} \sin(B - A) \sin l\delta_n + O\left(\dfrac{1}{\mu_n}\right) = 0.$

Consequently, $\delta_n = O\left(\dfrac{1}{n}\right)$ in all cases. If we substitute this into the expression found above for μ_n, we obtain the following *asymptotic formulas for the square root of the eigenvalues of the problem (67)–(68):*

$$\mu_n = \sqrt{\lambda_n} = \begin{cases} \dfrac{n\pi}{l} + O\left(\dfrac{1}{n}\right) & \text{in cases (a), (b),} \\ \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{l} + O\left(\dfrac{1}{n}\right) & \text{in cases (c), (d),} \end{cases} \tag{75}$$

On the basis of (72), the function $w = w_n(t)$, corresponding to the given eigenvalue $\lambda_n = \mu_n^2$, is given by

(a) $w_n(t) = \dfrac{1}{\mu_n} \cos A \sin \dfrac{n\pi}{l} t + O\left(\dfrac{1}{\mu_n^2}\right);$

(b) $$w_n(t) = -\sin A \cos \frac{n\pi}{l} t + O\left(\frac{1}{\mu_n}\right);$$

(c) $$w_n(t) = \frac{1}{\mu_n} \cos A \sin \left(n + \frac{1}{2}\right) \frac{\pi}{l} t + O\left(\frac{1}{\mu_n^2}\right);$$

(d) $$w_n(t) = -\sin A \cos \left(n + \frac{1}{2}\right) \frac{\pi}{l} t + O\left(\frac{1}{\mu_n}\right).$$

In the four cases considered, corresponding to this eigenvalue $\lambda_n = \mu_n^2$ are the eigenfunctions

(a) $$w_n^*(t) = \sin \frac{n\pi}{l} t + O\left(\frac{1}{\mu_n}\right);$$

(b) $$w_n^*(t) = \cos \frac{n\pi}{l} t + O\left(\frac{1}{\mu_n}\right);$$

(c) $$w_n^*(t) = \sin \left(n + \frac{1}{2}\right) \frac{\pi}{l} t + O\left(\frac{1}{\mu_n}\right);$$

(d) $$w_n^*(t) = \cos \left(n + \frac{1}{2}\right) \frac{\pi}{l} t + O\left(\frac{1}{\mu_n}\right).$$

To obtain normalized eigenfunctions from the eigenfunctions, i.e., functions satisfying the condition

$$\int_0^l v_n^2(t)\, dt = 1,$$

we only need to multiply them by the normalizing factor

$$\left[\int_0^l w_n^{*2}\, dt\right]^{-\frac{1}{2}} = \left[\frac{l}{2} + O\left(\frac{1}{n}\right)\right]^{-\frac{1}{2}}.$$

Thus, in the four cases considered, we have the following asymptotic formulas for the normalized eigenfunctions of the problem (67)–(68):

In case (a) ($\sin A = 0$ and $\sin B = 0$),

$$v_n(t) = \sqrt{\frac{2}{l}} \sin \frac{n\pi}{l} t + O\left(\frac{1}{n}\right).$$

In case (b) ($\sin A \neq 0$ and $\sin B \neq 0$),

$$v_n(t) = \sqrt{\frac{2}{l}} \cos \frac{n\pi}{l} t + O\left(\frac{1}{n}\right).$$

In case (c) ($\sin A = 0$ and $\sin B \neq 0$),

$$v_n(t) = \sqrt{\frac{2}{l}} \sin\left(n + \frac{1}{2}\right)\frac{\pi}{l}t + O\left(\frac{1}{n}\right).$$

In case (d) ($\sin A \neq 0$, $\sin B = 0$),

$$v_n(t) = \sqrt{\frac{2}{l}} \cos\left(n + \frac{1}{2}\right)\frac{\pi}{l}t + O\left(\frac{1}{n}\right). \tag{76}$$

Obviously, on the basis of (72), each of these functions satisfies the integral equation

$$v_n(t) = c_n\left[-\sin A \cos \mu_n t + \cos A \frac{\sin \mu_n t}{\mu_n}\right] - \int_0^t \frac{\sin \mu_n (t-\xi)}{\mu_n} h(\xi) v_n(\xi)\, d\xi, \tag{72'}$$

where c_n is a normalizing factor.* If we differentiate this equation with respect to t, we obtain

$$v_n'(t) = c_n[\mu_n \sin A \sin \mu_n t + \cos A \cos \mu_n t] - \int_0^t \cos \mu_n (t-\xi) h(\xi) v_n(\xi)\, d\xi. \tag{72''}$$

Since the function $v_n(t)$ is normalized, we have the following inequality for the last term on the right-hand side of Eq. (72″):

$$\left|\int_0^t \cos \mu_n (t-\xi) h(\xi) v_n(\xi)\, d\xi\right| < \sqrt{\int_0^t h^2(\xi)\, d\xi \int_0^t v_n^2(\xi)\, d\xi} = \sqrt{\int_0^t h^2(\xi)\, d\xi}$$

Consequently, the first term on the right-hand side of Eq. (72″) is the main term of $v_n(t)$. But this term coincides with the derivative of the main term of the function defined by Eq. (72′). Thus, we conclude that the asymptotic formulas for $v_n(t)$ are obtained from formulas (76) by differentiating the main terms with respect to t and replacing the terms $O(1/n)$ with quantities of the form $O(1)$.

* For $\sin A \neq 0$, the normalizing factor c_n is a bounded quantity, which is obvious from the asymptotic expressions for $w_n(t)$. For $\sin A = 0$, this factor is $O(n)$, which is obvious from the same asymptotic expressions.

From the differential equation (67), we have $v_n = -\lambda_n v_n + h v_n$. Therefore, to obtain asymptotic formulas for the second derivatives of the eigenfunctions, we need only multiply the right-hand members of formulas (72′) by $-\mu_n^2$. In other words, *to obtain asymptotic formulas for $v_n''(t)$, we need only differentiate the main terms in formulas (72′) twice with respect to t and replace the quantities $O(1/n)$ with quantities of the form $O(n)$.* This settles the question of the asymptotic behavior of the eigenfunctions and their derivatives in the case of the boundary-value problem (67)–(68).

To obtain asymptotic formulas for the eigenfunctions $u_n(x)$ of the problem (48)–(49) that are orthonormal with weight $\rho(x)$, we need only recall the transformation formulas (66) since the equation $\int_a^b \rho u_n^2\, dx = 1$ becomes, after the change of variables (66),

$$\int_a^b \rho u_n^2\, dx = \int_0^l \sqrt{\frac{\rho}{p}}\, v_n^2 \frac{dx}{dt}\, dt = \int_0^l v_n^2\, dt = 1.$$

In view of this, we obtain the following asymptotic formulas for the eigenfunctions $u_n(x)$ of the problem (48)–(49) that are normalized with weight $\rho(x)$ in (76):

In case (a) ($\sin\alpha = 0$, $\sin\beta = 0$),

$$u_n(x) = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi}{l} \int_a^x \sqrt{\frac{\rho}{p}}\, dx\right) + O\left(\frac{1}{n}\right).$$

In case (b) ($\sin\alpha \neq 0$, $\sin\beta \neq 0$),

$$u_n(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{n\pi}{l} \int_a^x \sqrt{\frac{\rho}{p}}\, dx\right) + O\left(\frac{1}{n}\right).$$

In case (c) ($\sin\alpha = 0$, $\sin\beta \neq 0$),

$$u_n(x) = \sqrt{\frac{2}{l}} \sin\left[\left(n + \frac{1}{2}\right)\frac{\pi}{l} \int_a^x \sqrt{\frac{\rho}{p}}\, dx\right] + O\left(\frac{1}{n}\right).$$

In case (d) ($\sin\alpha \neq 0$, $\sin\beta = 0$),

$$u_n(x) = \sqrt{\frac{2}{l}} \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi}{l} \int_a^x \sqrt{\frac{\rho}{p}}\, dx\right] + O\left(\frac{1}{n}\right). \quad (77)$$

The asymptotic formulas for the first and second derivatives of $u = u_n(x)$ with respect to the variable x are obtained by differentiating the main terms of formulas (77) with respect to x and replacing the quantities $O(1/n)$ with quantities of the form $O(1)$ in

the case of the first derivatives and with quantities of the form $O(n)$ in the case of second derivatives.

6. Theorems of the Fourier coefficients and termwise differentiation of the expansion in eigenfunctions. We introduce the operators

$$\Gamma^0(u) = u,\ \Gamma(u) = \frac{1}{\rho} L(u) = \frac{1}{\rho}\frac{d}{dx}\left(p\frac{du}{dx}\right) - \frac{q}{\rho}u,$$
$$\Gamma^2(u) = \Gamma(\Gamma(u)) = \frac{1}{\rho} L\left(\frac{1}{\rho} L(u)\right),\ \Gamma^m(u) = \Gamma(\Gamma^{m-1}(u)). \tag{78}$$

Obviously, the operator $\Gamma^m(u)$ (for $m \geqslant 1$) is meaningful in the interval (a, b) if in that interval the function u is $2m$ times continuously differentiable, the coefficient p is $2m-1$ times continuously differentiable, and if the coefficients q and $1/\rho$ are $2m-2$ times differentiable.

THEOREM 1. In the generalized Sturm–Liouville problem (48)–(50), suppose that the coefficient p is $2m-1$ times continuously differentiable and q and $1/\rho$ are $2m-2$ times continuously differentiable in the interval (a, b). Suppose that the function $f(x)$ is $2m$ times continuously differentiable in the interval (a, b) and that its operators $\Gamma^k(f)$ (for $k = 0, 1, \ldots, m-1$) are square integrable and satisfy boundary conditions (50); that is, suppose that

$$\sqrt{p(a)}\, R_0(\Gamma^k(f)) = 0,\ \sqrt{p(b)}\, R_1(\Gamma^k(f)) = 0,\ k = 0, 1, \ldots, m-1. \tag{79}$$

Then, the series

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^{2m} < \infty, \tag{80}$$

converges, where the c_n are the Fourier coefficients of the function $f(x)$ with respect to the orthonormal eigenfunctions of the generalized Sturm–Liouville problem and the λ_n are the eigenvalues of that problem.

Proof: Since $L(u_n) = -\lambda_n \rho u_n$, we have

$$c_n = \int_a^b \rho u_n(x) f(x)\, dx = -\frac{1}{\lambda_n}\int_a^b f(x) L(u_n)\, dx.$$

Therefore, in view of the boundary conditions (79) for $m = 1$, i.e., since $\sqrt{p(a)}\, R_0(f) = 0$ and $\sqrt{p(b)}\, R_1(f) = 0$, we have, on the basis of formula (47′),

$$c_n = -\frac{1}{\lambda_n}\int_a^b u_n L(f)\, dx = -\frac{1}{\lambda_n}\int_a^b \rho u_n \Gamma(f)\, dx. \tag{80′}$$

If we apply Bessel's inequality to the function $\Gamma(f)$, we obtain

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^2 \leqslant \int_a^b \rho (\Gamma (f))^2 dx < \infty.$$

This proves the lemma for $m=1$. If we now write Eq. (80′) in the form

$$c_n = \frac{1}{\lambda_n^2} \int_a^b L(u_n) \Gamma (f)\, dx = \frac{1}{\lambda_n^2} \int_a^b u_n L(\Gamma (f))\, dx =$$

$$= \frac{1}{\lambda_n^2} \int_a^b \rho u_n \Gamma^2 (f)\, dx$$

and apply Bessel's inequality to the function $\Gamma^2 (f)$, we obtain the assertion of the lemma for $m=2$:

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^4 \leqslant \int_a^b \rho [\Gamma^2 (f)]^2 dx < \infty.$$

In the same way, we prove the validity of the theorem in the general case.

Let us agree to say that the function $f(x)$ satisfies the "associated" boundary conditions to the boundary conditions

$$\sqrt{p(a)}\, R_0(u) = 0, \quad \sqrt{p(b)}\, R_1(u) = 0, \tag{50}$$

if, for every function $u(x)$ that satisfies boundary conditions (50), the following equation holds:*

$$[pfu']_a^b = p(a) f[a] u'(a) - p(b) f(b)\, u'(b) = 0. \tag{81}$$

Analogously, we shall speak of the associated boundary conditions to the boundary conditions (49). For example, if the boundary conditions (49) are of the form $u(a)=0$ and $u(b)=0$, the associated boundary conditions will be $f(a)=0$ and $f(b)=0$. If the boundary conditions (49) are of the form $u'(a)=0$ and $u'(b)=0$, the associated boundary conditions will be $f(a) \neq \infty$ and $f(b) \neq \infty$. Similarly, the associated boundary conditions of the boundary conditions (50) with $p(a)=0$ and $p(b)=0$ will be $f(a) \neq \infty$ and $f(b) \neq \infty$.

THEOREM 2. Suppose that the coefficients of the generalized Sturm–Liouville problem (48)–(50) satisfy the condition $q/p \geqslant -M$ *in the interval* (a, b), *where* M *is a sufficiently large positive number. If a continuous function* $f(x)$ *on the interval* $[a, b]$ *has a piecewise-*

* The introduction here of the new concept of associated boundary conditions and of the operators $\Gamma^k(u)$ defined in (78) that we have especially chosen is necessary in connection with the procedure that we shall adopt for studying the convergence of Fourier series in eigenfunctions of the Sturm-Liouville problem.

continuous derivative with points of discontinuity of the first kind and satisfies the associated boundary conditions of the boundary conditions of the original problem, then

$$\sum_{k=1}^{\infty} c_n^2 \lambda_n \leqslant I(f, f) + M \int_a^b \rho f^2 \, dx < \infty, \tag{82}$$

where the c_n are the Fourier coefficients of the function $f(x)$ in the expansion in orthonormal eigenfunctions with weight ρ of the generalized Sturm–Liouville problem, the λ_n are the eigenvalues of that problem, and $I(f, f)$ is the functional

$$I(f, f) = \int (pf'^2 + qf^2) \, dx. \tag{82'}$$

Proof: Indeed, we have

$$-\int_a^b fL(u) \, dx = \int_a^b f \left[\frac{d}{dx}\left(p \frac{du}{dx} \right) - qu \right] dx =$$
$$= -[pfu']_a^b + \int_a^b (pf'u' + qfu) \, dx.$$

From this it follows that if $u(x)$ satisfies the boundary conditions of the generalized Sturm–Liouville problem and the function f satisfies the associated boundary conditions, then

$$-\int_a^b fL(u) \, dx = I(f, u), \tag{83}$$

where

$$I(f, u) = \int_a^b (pf'u' + qfu) \, dx.$$

The Fourier coefficients c_n of the function $f(x)$ are given by

$$c_n = \int_a^b \rho u_n f \, dx = -\frac{1}{\lambda_n} \int_a^b fL(u_n) \, dx = \frac{1}{\lambda_n} I(f, u_n),$$

i.e.,

$$I(f, u_n) = \lambda_n c_n \tag{83'}$$

and, in particular,

$$I(u_n, u_m) = \begin{cases} \lambda_n, & m = n, \\ 0, & m \neq n. \end{cases} \tag{83''}$$

Let $\sigma_N = \sum_{n=1}^{N} c_n u_n(x)$ be a partial sum of the Fourier series of the function $f(x)$. We have

$$I(f-\sigma_N, f-\sigma_N) = \int_a^b [p(f'-\sigma_N')^2 + q(f-\sigma_N)^2]\,dx =$$

$$= I(f, f) + I(\sigma_N, \sigma_N) - 2I(f, \sigma_N).$$

The second and third terms of the right-hand member of the last equation are, on the basis of (83′) and (83″), respectively, equal to $\sum_{n=1}^{N} c_n^2 \lambda_n$ and $-2\sum_{n=1}^{n} c_n^2 \lambda_n$. Thus, we obtain

$$I(f-\sigma_N, f-\sigma_N) = I(f, f) - \sum_{n=1}^{N} c_n^2 \lambda_n. \qquad (83''')$$

On the other hand, we have

$$I(f-\sigma_N, f-\sigma_N) > \int_a^b q(f-\sigma_N)^2\,dx > -M\int_a^b \rho(f-\sigma_N)^2\,dx =$$

$$= -M\left(\int_a^b \rho f^2\,dx - \sum_{n=1}^{N} c_n^2\right) > -M\int_a^b \rho f^2\,dx.$$

When we substitute this into (83‴), we obtain inequality (82). This completes the proof of the theorem.

THEOREM 3. Suppose that the coefficient p in the generalized Sturm–Liouville problem (48)–(50) has continuous derivatives of the first 2m—1 orders in the interval (a, b) and that the coefficients q and 1/ρ have continuous derivatives of the first 2m—2 orders. Suppose that $q/\rho > -M \neq -\infty$. *Suppose that the function f(x) is 2m times continuously differentiable in the interval (a, b) and that its operators* $\Gamma^k(f)$ *(for* $k = 0, 1, \ldots, m-1$*) are square integrable and satisfy the boundary conditions of the given problem (50). Then, if the (2m+1)st derivative of f(x) on the interval [a, b] is piecewise continuous with points of discontinuity of the first kind and if the operator* $\Gamma^m(f)$ *satisfies the associated boundary conditions to conditions (50), then the series*

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^{2m+1} < \infty \qquad (82'')$$

converges. *

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*In other words, we may say that if in the interval (a, b) the function $f(x)$ and the coefficients in the generalized Sturm–Liouville problem satisfy the conditions of Theorem 1 and if, in addition, the derivative $d^{2m+1}f(x)/dx^{2m+1}$ is piecewise continuous on the closed interval $[a, b]$ with points of discontinuity of the first kind, if $q/\rho > -M \neq \infty$, and if the operator $\Gamma^m(f)$ satisfies the associated boundary conditions to the boundary conditions (50), then the series (82″) converges.

Proof: Since the function $\Gamma^0(f)=f(x)$ satisfies the boundary condition (50), we have

$$c_n=\int_a^b \rho u_n f\,dx=-\frac{1}{\lambda_n}\int_a^b L(u_n)\,f\,dx=-\frac{1}{\lambda_n}\int_a^b \rho u_n\frac{L(f)}{\rho}\,dx.$$

If $m=1$, then by hypothesis $\Gamma(f)$ satisfies the associated boundary conditions to the boundary conditions (50). Therefore, if we apply inequality (82) to the function $\Gamma(f)$, we obtain

$$\sum_{n=1}^{\infty}(c_n\lambda_n)^2\lambda_n<I(\Gamma,\Gamma)+M\int_a^b \rho[\Gamma(f)^2]\,dx<\infty.$$

This proves the theorem for $m=1$. Since $\Gamma(f)$ satisfies the boundary conditions (50), we have for $m=2$,

$$c_n=\frac{1}{\lambda_n^2}\int_a^b L(u_n)\Gamma(f)\,dx=\frac{1}{\lambda_n^2}\int_a^b \rho u_n\frac{L(\Gamma)}{\rho}\,dx.$$

Since $\Gamma^2(f)=L(\Gamma)/\rho$ satisfies the associated boundary conditions to the boundary conditions (50), we have

$$\sum_{n=1}^{\infty}(c_n\lambda_n)^2\lambda_n<I(\Gamma^2,\Gamma^2)+M\int_a^b \rho[\Gamma^2(f)]^2\,dx<\infty.$$

This proves the theorem for $m=2$. In the same way, we see that the theorem is valid in the general case.

Under suitable assumptions on the coefficients in the generalized Sturm–Liouville problem (48)–(50), Theorems 1, 2, and 3 answer the question of the convergence of series of the form $\sum_{n=1}^{\infty}|c_n|^2|\lambda_n|^k$, where k is a positive integer.

If we use this result and the asymptotic formulas for the eigenvalues (75), we can easily solve the problem of the convergence of series of the form $\sum_{n=1}^{\infty}|c_n|\,|\lambda_n|^{\frac{k}{2}}$. We have the following theorems:

THEOREM 4. Suppose that the coefficients of the Sturm–Liouville problem (48)–(49) satisfy the following conditions on the closed interval [a, b]: p and ρ are twice continuously differentiable positive functions and q is continuous. Then, if the function f(x) has a piecewise-continuous derivative on the closed interval [a, b] with points of discontinuity of the first kind and if it satisfies the associated boundary conditions to the boundary conditions (49), then

$$\sum_{n=1}^{\infty}|c_n|<\infty. \tag{84}$$

Proof: We have

$$\sum_{n=1}^{\infty}|c_n| = \sum_{n=1}^{\infty}|c_n|\sqrt{|\lambda_n|}\,\frac{1}{\sqrt{|\lambda_n|}} \leqslant \frac{1}{2}\sum_{n=1}^{\infty}|c_n|^2|\lambda_n| + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{|\lambda_n|}.$$

But, under the assumptions made on the coefficients p, ρ, and q, the asymptotic formulas $\sqrt{|\lambda_n|}=O(n)$ hold. Therefore, the second series in the right-hand member of the last inequality converges. The first series in the right-hand member converges by virtue of Theorem 2. Thus, the series (84) converges.

COROLLARY. The Fourier series of the function $f(x)$ converges absolutely and uniformly on the interval $[a, b]$ under the conditions of Theorem 4.

This assertion follows immediately from the convergence of the series (84) and the asymptotic formulas for orthonormal eigenfunctions (77).

This corollary is the content of a well-known theorem of Steklov.*

THEOREM 5. Suppose that the coefficients of the Sturm–Liouville problem (48)–(49) satisfy the following conditions on the closed interval $[a, b]$: p and ρ are twice continuously differentiable positive functions and q is continuous. If the function $f(x)$ is twice continuously differentiable on the interval $[a, b]$ and satisfies the boundary conditions (49), then

$$\sum_{n=1}^{\infty}|c_n|\sqrt{|\lambda_n|}<\infty. \tag{84'}$$

Proof: We have

$$\sum_{n=1}^{\infty}|c_n|\sqrt{|\lambda_n|} = \sum_{n=1}^{\infty}|c_n||\lambda_n|\frac{1}{\sqrt{|\lambda_n|}} \leqslant \frac{1}{2}\sum_{n=1}^{\infty}\left(c_n^2|\lambda_n|^2+\frac{1}{|\lambda_n|}\right).$$

But the series in the right-hand member of this inequality converges by virtue of Theorem 1 (with $m=1$) and the asymptotic formulas $\sqrt{|\lambda_n|}=O(n)$. Consequently, the series (84') also converges.

THEOREM 6. Suppose that the coefficients of the Sturm–Liouville problem (48)–(49) satisfy the following conditions on the closed

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*See V. I. Smirnov, *A Course of Higher Mathematics*, p. 841, Vol. IV, Reading, Massachusetts, Addison-Wesley, 1964; Sansone, G., *Equazioni differenziali nel campo reale*, 2nd ed., Zanichelli, Bologna, 1948-49.

interval $[a, b]$: p *and* ρ *are twice continuously differentiable positive functions and* q *is continuously differentiable. Suppose also that a function* $f(x)$ *is twice continuously differentiable on the interval* $[a, b]$ *and satisfies the boundary conditions (49) and that its third derivative is piecewise continuous with points of discontinuity of the first kind. Suppose finally that the operator* $\Gamma(f)$ *satisfies the associated boundary conditions to the boundary conditions (49). Then,*

$$\sum_{n=1}^{\infty} |c_n||\lambda_n| < \infty. \tag{84''}$$

Proof: We have

$$\sum_{n=1}^{\infty} |c_n||\lambda_n| = \sum_{n=1}^{\infty} |c_n||\lambda_n|^{3/2} \frac{1}{\sqrt{|\lambda_n|}} \leqslant \frac{1}{2} \sum_{n=1}^{\infty} \left(c_n^2 |\lambda_n|^3 + \frac{1}{|\lambda_n|} \right),$$

but the series in the right-hand member of this inequality converges by virtue of the asymptotic formulas $\sqrt{|\lambda_n|} = O(n)$ and Theorem 3 (with $m=1$). Consequently, the series (84″) also converges.

THEOREM 7. Suppose that the coefficients of the Sturm–Liouville problem (48)–(49) satisfy the following conditions on the closed interval $[a, b]$: p *has continuous derivatives of the first three orders and* ρ *and* q *are twice continuously differentiable positive functions. If a function* $f(x)$ *has continuous derivatives of the first four orders on the interval* $[a, b]$ *and satisfies the boundary conditions (49) and if the operator* $\Gamma(f)$ *satisfies these same boundary conditions, then*

$$\sum_{n=1}^{\infty} |c_n||\lambda_n|^{3/2} < \infty. \tag{84'''}$$

Proof: We have

$$\sum_{n=1}^{\infty} |c_n||\lambda_n|^{3/2} = \sum_{n=1}^{\infty} |c_n||\lambda_n|^2 \frac{1}{\sqrt{|\lambda_n|}} \leqslant \frac{1}{2} \sum_{n=1}^{\infty} \left(c_n^2 \lambda_n^4 + \frac{1}{|\lambda_n|} \right).$$

But the series in the right-hand member of this inequality converges by virtue of the asymptotic equations $\sqrt{|\lambda_n|} = O(n)$ and Theorem 1 (with $m=2$). Consequently, the series (84‴) also converges.

COROLLARY (on term-by-term differentiation of a Fourier series). *If a function* $f(x)$ *and the boundary-value problem (48)–(49) satisfy the conditions of Theorem 6 or Theorem 7, then the series obtained by differentiating twice the expansion of the function* $f(x)$ *in a series of eigenfunctions of the problem (48)–(49) will converge absolutely and uniformly on the closed interval* $[a, b]$.

Proof. From the asymptotic equations (75) and (77), we have $u_n''(x)=O(\lambda_n)$ and, consequently,

$$\sum_{n=1}^{\infty} |c_n u_n''(x)| \leqslant M \sum_{n=1}^{\infty} |c_n||\lambda_n| < \infty \qquad (M = \text{const}).$$

7. Solution of boundary-value problems for the general parabolic equation in two independent variables. Let us first consider boundary-value problems for a homogeneous parabolic equation on an interval $[a, b]$:

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) - qu - \rho\frac{\partial u}{\partial t} = 0, \tag{85}$$

$$u|_{t=0} = \psi(x), \tag{85'}$$

$$\varkappa u|_{x=a} = 0, \quad \varkappa u|_{x=b} = 0, \tag{85''}$$

where $p=p(x)$, $q=q(x)$, and $\rho=\rho(x)$ are functions of x and $\varkappa u$ is a linear boundary operator: $\varkappa u=u$ in the case of the first boundary-value problem, $\varkappa u = \partial u/\partial n$ in the case of the second boundary-value problem, and $\varkappa u = \partial u/\partial n - \alpha u$ (where $\alpha=\text{const}>0$) in the case of the third boundary-value problem:

$$\left.\frac{\partial u}{\partial n}\right|_{x=a} = \left.\frac{\partial u}{\partial x}\right|_{x=a} \qquad \left.\frac{\partial u}{\partial n}\right|_{x=b} = -\left.\frac{\partial u}{\partial x}\right|_{x=b}.$$

To be able to use Steklov's theorem (see subsection 6), let us assume that the functions $p(x)$ and $\rho(x)$ are twice continuously differentiable positive functions on the closed interval $[a, b]$, that the function q is continuous, and that the function $\psi(x)$ giving the initial condition has a piecewise-continuous derivative and satisfies the associated boundary conditions to the boundary conditions (85'').*

By using the method of separation of variables, i.e., by considering particular solutions of Eq. (85) of the form $T(t)\,X(x)$, we obtain a solution of Eq. (85) in the form of a series

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} X_n(x), \tag{86}$$

where the c_n are arbitrary constants, and the λ_n and $X_n(x)$ are, respectively, the eigenvalues and eigenfunctions of the Sturm–Liouville problem.

* For example, if $\varkappa u=u$, i.e., in the case of the first boundary-value problem, $\psi(a)=0$ and $\psi(b)=0$, and in the case of the second boundary-value problem, $\psi(a)\neq\infty$ and $\psi(b)\neq\infty$.

$$\frac{d}{dx}\left(p\frac{dX}{dx}\right)-qX+\lambda\rho X=0, \tag{87}$$

$$\varkappa X|_{x=a}=0, \quad \varkappa X|_{x=b}=0. \tag{87'}$$

Obviously, the series (86) satisfies the boundary conditions (85″) since the eigenfunctions $X_n(x)$ satisfy these boundary conditions. Furthermore, if the constants c_n are assumed equal to the Fourier coefficients of the function $\psi(x)$, i.e., if

$$c_n=\int_a^b \psi(x)\,X_n(x)\,\rho(x)\,dx,$$

then, on the basis of Steklov's theorem (see subsection 6),

$$\psi(x)=\sum_{n=1}^{\infty} c_n X_n(x).$$

Furthermore, this series converges absolutely and uniformly. Therefore, since only a finite number of the eigenvalues λ_n can be negative, we conclude that

$$\lim_{t\to+0} u(x,t)=\sum_{n=1}^{\infty} c_n X_n(x)=\psi(x).$$

Using the asymptotic formulas for the eigenfunctions and their first two derivatives (77) and also the fact that only a finite number of the eigenvalues λ_n can be negative, we see that, for $t>0$, the series (86) will be twice continuously differentiable with respect to x and continuously differentiable with respect to t.* This means that the series (86) represents a solution of the boundary-value problem (85)–(85′)–(85″) that is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

Let us turn now to the boundary-value problem for a nonhomogeneous parabolic equation with homogeneous initial conditions:

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)-qu-\rho\frac{\partial u}{\partial t}=-\rho f(x,t), \tag{88}$$

$$u|_{t=0}=0, \tag{88'}$$

$$\varkappa u|_{x=a}=0, \quad \varkappa u|_{x=b}=0. \tag{88''}$$

*It is easy to see that, for $t>0$, the series (86) will also have continuous derivatives of all orders with respect to t.

Let us seek a solution of this problem in the form of a series that can be differentiated twice with respect to x and once with respect to t:

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x), \tag{89}$$

where the $X_n(x)$ are the orthonormal eigenfunctions of the Sturm–Liouville problem (87)–(87′). In view of the relation

$$L(X_n) = \frac{d}{dx}\left(p \frac{dX_n}{dx}\right) - qX_n = -\lambda_n \rho X_n$$

substitution of the series (89) into Eq. (88) yields

$$\sum_{n=1}^{\infty} -[T_n \lambda_n + T_n'] X_n(x) \rho(x) = -\rho(x) f(x, t).$$

Therefore, if we denote by $f_n(t)$ the Fourier coefficients of the function $f(x, t)$ as a function of x, we obtain

$$T_n' + \lambda_n T = f_n(t). \tag{90}$$

In view of the homogeneous initial conditions (88′), $T_n(0) = 0$ and, from Eq. (90), we obtain

$$T_n(t) = \int_0^t f_n(t') e^{-\lambda_n (t-t')} dt'. \tag{91}$$

When we substitute this expression into Eq. (89), we conclude that the desired solution of the problem (88)–(88′)–(88″) must be of the form

$$u(x, t) = \int_0^t \sum_{n=1}^{\infty} f_n(t') e^{-\lambda_n (t-t')} X_n(x) dt' \tag{92}$$

or

$$u(x, t) = \int_0^t \int_a^b f(x', t') \sum_{n=1}^{\infty} X_n(x) X_n(x') e^{-\lambda_n (t-t')} dx' dt'. \tag{93}$$

Conversely, when the series obtained by differentiating the series (92) termwise twice with respect to x and once with respect to t converges uniformly, this series is obviously a solution of the problem (88)–(88′)–(88″), which is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

Now consider the general boundary-value problem

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) - qu - \rho\frac{\partial u}{\partial t} = -\rho f(x, t), \tag{94}$$

$$u|_{t=0} = \psi(x), \tag{94'}$$

$$\varkappa u|_{x=a} = \Phi_1(t), \quad \varkappa u|_{x=} = \Phi_2(t). \tag{94''}$$

By subtracting any function satisfying boundary conditions (94″) from the desired solution, we can reduce this problem to the case of the corresponding problem with homogeneous boundary conditions and then consider the solution of the last problem as the sum of solutions of the problem (85)–(85′)–(85″) and the problem (88)–(88′)–(88″).

We note that the conditions under which we have succeeded in proving the existence of twice continuously differentiable solutions of the boundary-value problems (85)–(85′)–(85″) and (88)–(88′)–(88″) and then finding these solutions are quite restricted. This is partly explained by the fact that we have sought such solutions by differentiating the corresponding Fourier series term by term. However, it can happen that the function itself has continuous derivatives of every order and yet its Fourier series cannot be differentiated term by term. For example, in the interval $[0, \pi]$,

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx,$$

and the function $u(x)=x$ has derivatives of all orders in that interval. Yet, we cannot obtain these derivatives by differentiating its Fourier series term by term.

One may ask what the method of separation of variables yields whenever the conditions under which termwise differentiation of the Fourier series obtained are not satisfied.

In connection with this, let us look once again at the boundary-value problems (85)–(85′)–(85″) and (88)–(88′)–(88″). This time, we do not make the same assumptions on p, q, and ρ. Instead, we assume that these functions are such that the conditions of the theorem on the completeness of a system of eigenfunctions are satisfied for the Sturm–Liouville problem (87)–(87′); that is, p, dp/dx, q and ρ are continuous functions of x on the closed interval $[a, b]$ and that p and ρ are positive. We assume that the function $\psi(x)$ is square integrable on the interval $[a, b]$.

Suppose that $u_n(x, t)$ is a solution of the boundary-value problem (85)–(85′)–(85″) except that the function $\psi(x)$ giving the initial conditions is replaced with a partial sum of its Fourier series:

$$\psi_n(x) = \sum_{k=1}^{n} c_k X_k(x).$$

Obviously, the function $u_n(x, t)$ can be written in the form

$$u_n(x, t) = \sum_{k=1}^{n} c_k e^{-\lambda_k t} X_k(x) \qquad (86')$$

and is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

Let us look at the square of the mean-square deviation of the two functions $u_n(x, t)$ and $u_m(x, t)$, where n and m are arbitrary positive integers with $(m>n)$

$$\rho[u_n, u_m] = \int_a^b [u_n(x, t) - u_m(x, t)]^2 \rho(x)\, dx = \sum_{k=n+1}^{m} c_k^2 e^{-\lambda_k t}.$$

From this, it is clear that, if ε is any positive number, $\rho[u_n, u_m]<\varepsilon$ whenever n and m are sufficiently large. In other words, the functions $u_n(x, t)$ form a Cauchy sequence in the sense of convergence in the mean (with weight ρ). On the basis of the well-known Riesz–Fischer theorem, it follows from this that there exists a function $u(x, t)$ which is square integrable on the interval $[a, b]$ and to which the sequence $u_n(x, t)$ converges in the mean with respect to t, i.e., a function $u(x, t)$ exists such that

$$\int_a^b [u(x, t) - u_n(x, t)]^2 \rho(x)\, dx < \varepsilon,$$

whenever n is sufficiently great. Here, ε is any positive number (and is independent of t). We can call the function $u(x, t)$ obtained in this way a generalized solution of the boundary-value problem (85)–(85′)–(85″). The Fourier series of this generalized solution coincides with the right-hand member of Eq. (86) and it converges in the mean to the given generalized solution uniformly with respect to t.

Analogously, we shall denote by $u_n(x, t)$ the solution of the boundary-value problem (88)–(88′)–(88″) with the function $f(x, t)$ replaced by a partial sum of its Fourier series:

$$f_n(x, t) = \sum_{k=1}^{n} f_k(t) X_k(x).$$

Corresponding to (92), we have

$$u_n(x, t) = \sum_{k=1}^{n} \int_0^t f_k(t') e^{-\lambda_k (t-t')} X_k(x)\, dt'.$$

For the square of the mean-square deviation with weight ρ of two solutions $u_n(x, t)$ and $u_m(x, t)$, where $m>n$, we have

$$\rho[u_n, u_m] = \sum_{k=n+1}^{m} \left[\int_0^t f_k(t')\, e^{-\lambda_k(t-t')}\, dt'\right]^2 \leqslant$$

$$\leqslant \sum_{k=n+1}^{m} \int_0^t f_k^2(t')\, dt' \int_0^t e^{-2\lambda_k(t-t')}\, dt' =$$

$$= \sum_{k=n+1}^{m} \frac{1}{2\lambda_k}(1 - e^{-2\lambda_k t}) \int_0^t f_k^2(t')\, dt' \leqslant \int_0^t \sum_{k=n+1}^{m} f_k^2(t')\, dt'.$$

Thus, if

$$\int_0^T dt \int_a^b f^2(x, t)\, dx \leqslant M < \infty,$$

on an interval $0 \leqslant t \leqslant T < \infty$, the sequence $u_n(x, t)$ will be a Cauchy sequence in the sense of convergence in the mean. It will converge in the mean uniformly with respect to t in the interval $0 \leqslant t \leqslant T$ to some function $u(x, t)$ that is square integrable on the interval $[a, b]$. Taking this function as a generalized solution of the problem (88)–(88′)–(88″), we can assert that the Fourier series in the right-hand member of Eq. (92) converges in the mean uniformly with respect to t (where $0 \leqslant t \leqslant T$) to the generalized solution $u(x, t)$ of the problem (88)–(88′)–(88″).

8. Solution of boundary-value problems for the general hyperbolic equation with two independent variables. Let us first consider the following boundary-value problem for a homogeneous hyperbolic equation on the interval $[a, b]$:

$$\frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) - qu - \rho \frac{\partial^2 u}{\partial t^2} = 0, \tag{95}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x), \tag{95′}$$

$$\varkappa u|_{x=a} = 0, \quad \varkappa u|_{x=b} = 0, \tag{95″}$$

where p, q, $\psi(x)$, and $\psi_1(x)$ are functions of x satisfying certain restrictions to be specified later, the remaining notations being the same as in subsection 7.

Using the method of separation of variables, we obtain a solution of Eq. (95) in the form of a series

$$u(x, t) = \sum_{n=1}^{m}\left(a_n \cosh \mu_n t + \frac{b_n}{\mu_n} \sinh \mu_n t\right) X_n(x) +$$

$$+ \sum_{n=m+1}^{\infty}\left(a_n \cos \mu_n t + \frac{b_n}{\mu_n} \sin \mu_n t\right) X_n(x), \tag{96}$$

where $\mu_n = \sqrt{|\lambda_n|}$, λ_n, and $X_n(x)$ are the eigenvalues and orthonormal eigenfunctions with weight ρ of the problem (87)–(87′), m is the number of nonpositive eigenvalues,* a_n are the Fourier coefficients of the function $\psi(x)$, and where b_n are the Fourier coefficients of the function $\psi_1(x)$. Here, if $\lambda_m = 0$, then $\cosh\mu_m(t)$ and $\mu_m \sinh \mu_m t$ in Eq. (96) are replaced with 1 and t, respecspectively.

It now remains for us to show under what conditions the series (96) can be differentiated twice with respect to x and t without destroying the uniform convergence. In such cases, we can assert that the series (96) is a twice continuously differentiable solution of the problem (95)–(95′)–(95″). We shall give two illustrations of an affirmative answer to this question.

1. Suppose that the coefficients of Eq. (95) satisfy the following requirements on the interval $[a, b]$: p and ρ are positive twice continuously differentiable functions, and q is a continuously differentiable function. Then, for the series (96) to be a twice continuously differentiable solution of the problem (95)–(95′)–(95″), it will be sufficient if the functions $\psi(x)$ and $\psi_1(x)$ satisfy the following conditions on the interval $[a, b]$: The function $\psi(x)$ is twice continuously differentiable, satisfies the boundary conditions (95″), and has a piecewise-continuous third derivative with points of discontinuity of the first kind; and the operator $\Gamma(\psi)$ satisfies associate boundary conditions to the boundary conditions (95″). The function $\psi_1(x)$ is twice continuously differentiable and satisfies the boundary conditions (95″).

Proof: By virtue of the asymptotic formulas for the eigenvalues (75) and the eigenfunctions (77), the series that are obtained by differentiating the series

$$\sum_{n=m+1}^{\infty} a_n \cos \mu_n t X_n(x), \quad \sum_{n=m+1}^{\infty} \frac{b_n}{\mu_n} \sin \mu_n t\, X_n(x)$$

term by term twice with respect to either x or t are majorized by the series of numbers

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*The problem (87)–(87′) can have nonpositive eigenvalues, as can be shown by an example. If $p=\rho=1$, $q=-m^2$ (where m is an integer) $a=0$, $b=\pi$ and $\varkappa u = u$, the problem (87)–(87′) takes the form

$$X'' + m^2 X + \lambda X = 0, \quad X\big|_{x=0} = 0, \quad X\big|_{x=\pi} = 0.$$

The eigenvalues of this problem can be found from the condition $\sin \sqrt{m^2+\lambda}\, x\big|_{x=\pi} = 0$, which means that $\lambda_k = k^2 - m^2$ (for $k = 1, 2, \ldots$) so that $\lambda_1, \lambda_2, \ldots \lambda_m$ are nonpositive. In this case, the problem (95)–(95′)–(95″) can have a solution which becomes unbounded as $t \to \infty$. This is only to be expected since in this case Eq. (95) coincides with the equation $\partial^2 u/\partial x^2 - \partial^2 u/\partial t^2 = -m^2 u$, in which the term $m^2 u$ plays the role of a body force (cf. the derivation of the vibrating-string equation in Chapter I).

$$M\sum_{m+1}^{\infty}|a_n|\lambda_n,\quad M\sum_{m+1}^{\infty}|b_n|\mu_n,\qquad M=\text{const}.$$

But, on the basis of Theorems 6 and 5 (see subsection 6), these last series converge.

2. Suppose that the coefficients of Eq. (95) satisfy the following requirements on the interval $[a, b]$: p has continuous derivatives of the first three orders, q and ρ have continuous derivatives of the first two orders, and p and ρ are positive. Then, for the series (96) to be a twice continuously differentiable solution of the problem (95)–(95′)–(95″), it is sufficient that the functions $\psi(x)$ and $\psi_1(x)$ giving the initial conditions on the interval $[a, b]$ satisfy the following conditions: The function $\psi(x)$ has continuous derivatives of the first four orders and satisfies the boundary conditions (95″), and the operator $\Gamma(\psi)$ satisfies the same boundary conditions. The function $\psi_1(x)$ is twice continuously differentiable and satisfies boundary conditions (95″).

Proof: In this case, the series $\sum_{m+1}^{\infty}|b_n|\,|\mu_n|$ obviously converges just as in the preceding case and the series $\sum_{m+1}^{\infty}|a_n|\,|\lambda_n|$ converges on the basis of Theorem 7 (see subsection 6). Consequently, the series obtained by differentiating the series (96) twice is majorized by a convergent series of numbers. This proves the assertion made.

Let us turn now to the solution of a problem for a nonhomogeneous equation with homogeneous initial conditions:

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)-qu-\rho\frac{\partial^2 u}{\partial t^2}=-\rho f(x,t),\tag{97}$$

$$u|_{t=0}=0,\quad \frac{\partial u}{\partial t}\Big|_{t=0}=0,\tag{97′}$$

$$\varkappa u|_{x=a}=0,\quad \varkappa u|_{x=b}=0.\tag{97″}$$

We seek a solution of this problem in the form of a series that can be differentiated twice term by term with respect to x and t:

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x),\tag{98}$$

where the $X_n(x)$ are orthonormal eigenfunctions with weight ρ of the problem (87)–(87′). In view of the equation $L(X_n)=-\lambda_n\rho X_n(x)$, substitution of this series into Eq. (97) yields

$$\sum_{n=1}^{\infty}-(T_n\lambda_n+T_n'')\,\rho X_n(x)=-\rho f(x,t).$$

From this, we obtain

$$T_n'' + \lambda_n T = f_n(t), \tag{98'}$$

where the $f_n(t)$ are the Fourier coefficients of $f(x, t)$ treated as a function of x. On the basis of the initial conditions (97'), we have $T_n(0)=0$ and $T_n'(0)=0$. From Eq. (98'), we obtain

$$T_n(t) = \begin{cases} \dfrac{1}{\mu_n}\displaystyle\int_0^t \sin \mu_n (t-t') f_n(t')\, dt', & n \geqslant m+1, \\ \dfrac{1}{\mu_n}\displaystyle\int_0^t \sinh \mu_n (t-t') f_n(t')\, dt', & n \leqslant m, \quad \mu_n = \sqrt{|\lambda_n|}. \end{cases}$$

After substituting this expression into Eq. (98), we conclude that the desired solution of the problem (97)–(97')–(97'') must be of the form

$$u(x, t) = \int_0^t \sum_{n=1}^{m} \frac{1}{\mu_n} \sinh \mu_n (t-t') f_n(t') X_n(x)\, dt' + \int_0^t \sum_{n=m+1}^{\infty} \frac{1}{\mu_n} \sin \mu_n (t-t') f_n(t') X_n(x)\, dt'. \tag{99}$$

Obviously, the converse is also true: If the series that are obtained by differentiating the series (99) twice with respect to x and t converge uniformly, the series (99) is a twice continuously differentiable solution of the problem (97)–(97')–(97'').

Obviously, the most general case of the boundary-value problem

$$\frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) - qu - \rho \frac{\partial^2 u}{\partial t^2} = -\rho f(x, t),$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x),$$

$$\varkappa u|_{x=a} = \Phi_1(t), \quad \varkappa u|_{x=b} = \Phi_2(t)$$

can, by subtracting any function satisfying the given boundary conditions from the desired solution, be reduced to the case of the corresponding boundary-value problem with homogeneous boundary conditions. Then, the solution of this last boundary-value problem can be obtained as the sum of the solution of the boundary-value problem (95)–(95')–(95'') and the solution of the boundary-value problem (97)–(97')–(97'').

In analogy with subsection 7, let us look at the boundary-value problem (95)–(95')–(95'') from the point of view of constructing

generalized solutions for it in the class of functions that are square integrable on the interval $[a, b]$. Here, we shall assume that p, q, and ρ satisfy only the conditions of the theorem on the completeness of the system of eigenfunctions of the problem (87)–(87′) (see subsection 4).

Let n be a sufficiently large number and let $u_n(x, t)$ be a solution of the problem (95)–(95′)–(95″) with the functions $\psi(x)$ and $\psi_1(x)$ representing the initial conditions replaced with partial sums of their Fourier series

$$\psi^{(n)}(x) = \sum_{k=1}^{n} a_k X_k(x), \quad \psi_1^{(n)}(x) = \sum_{k=1}^{n} b_k X_k(x).$$

Then, in accordance with (96), we have

$$u_n(x, t) = \sum_{k=1}^{m} \left(a_k \cosh \mu_k t + \frac{b_k}{\mu_k} \sinh \mu_k t\right) X_k(x) + \\ + \sum_{k=m+1}^{n} \left(a_k \cos \mu_k t + \frac{b_k}{\mu_k} \sin \mu_k t\right) X_k(x). \qquad (96')$$

For the square of the mean-square deviation with weight ρ of the functions $u_n(x, t)$ and $u_{m'}(x, t)$ (where $m' > n$), we have

$$\rho[u_n, u_{m'}] = \int_a^b [u_n(x, t) - u_{m'}(x, t)]^2 \rho(x)\, dx = \\ = \sum_{k=n+1}^{m'} \left(a_k \cos \mu_k t + \frac{b_k}{\mu_k} \sin \mu_k t\right)^2 \leqslant \\ \leqslant 2 \sum_{k=n+1}^{m'} \left(a_k^2 + \frac{b_k^2}{\mu_k^2}\right) < 2 \sum_{k=n}^{m'} (a_k^2 + b_k^2).$$

From this it is clear that if the functions $\psi(x)$ and $\psi_1(x)$ are square integrable on the interval $[a, b]$, then the functions $u_n(x, t)$ (for $n = 1, 2, \ldots$) constitute a Cauchy sequence in the sense of convergence in the mean. Therefore, by virtue of the Riesz–Fischer theorem, there exists a function $u(x, t)$ that is square integrable on $[a, b]$ to which $u_n(x, t)$ converges in the mean uniformly with respect to t as $n \to \infty$. Let us take this function $u(x, t)$ as a generalized solution of the problem (95)–(95′)–(95″). The Fourier series of the generalized solution $u(x, t)$ coincides with the right-hand member of Eq. (96) and converges in the mean to the given generalized solution uniformly with respect to t.

In the case of the boundary-value problem (97)–(97′)–(97″), we denote by $u_n(x, t)$ the solution of this problem with the right-hand member of the equation for $f(x, t)$ replaced with a partial sum of its Fourier series

$$f_n(x, t)=\sum_{k=1}^{n} f_n(t) X_n(x).$$

Then, in accordance, with (99), $u_n(x, t)$ can be written in the form

$$u_n(x, t) = \sum_{k=1}^{m} \int_0^t \frac{1}{\mu_k} \sinh \mu_k (t-t') f_k(t') X_k(x)\, dt' +$$

$$+ \sum_{k=m+1}^{n} \int_0^t \frac{1}{\mu_k} \sin \mu_k (t-t') f_k(t') X_k(x)\, dt'.$$

For $m'>n$, we have

$$\rho[u_n, u_{m'}] = \int_a^b [u_n(x, t) - u_{m'}(x, t)]^2 \rho(x)\, dx =$$

$$= \sum_{k=n+1}^{m'} \left[\int_0^t \frac{1}{\mu_k} \sin \mu_k (t-t') f_k(t')\, dt'\right]^2 \leqslant$$

$$\leqslant \sum_{k=n+1}^{m'} \frac{1}{\mu_k^2} \int_0^t \sin^2 \mu_k (t-t')\, dt' \int_0^t f_k^2(t')\, dt' \leqslant$$

$$\leqslant \int_0^t t \sum_{k=n+1}^{m'} \frac{1}{\mu_k^2} f_k^2(t')\, dt' \leqslant t \int_0^t \sum_{k=n+1}^{m'} f_k^2(t')\, dt'.$$

From this it is clear that if

$$\int_0^T dt \int_a^b f^2(x, t)\, dx \leqslant M < \infty \qquad (T = \text{const}),$$

then the solutions $u_n(x, t)$ where $n = 1, 2, \ldots$, constitute a Cauchy sequence of functions in the sense of convergence in the mean. As $n \to \infty$, the sequence of the $u_n(x, t)$ converges in the mean to some function $u(x, t)$, that is square integrable on the interval $[a, b]$, and this convergence is uniform with respect to t when $0 \leqslant t \leqslant T$. Let us take this function $u(x, t)$ as the generalized solution of the problem (97)–(97′)–(97″). The Fourier series of the generalized solution of

$u(x, t)$ coincides with the right-hand member of Eq. (99). This Fourier series converges in the mean to the generalized solution $u(x, t)$ uniformly with respect to t in every finite interval of variation of t $(0 \leqslant t \leqslant T)$.

9. Approximate determination of the eigenvalues and eigenfunctions and their variational properties. The Ritz method. In many branches of pure and applied mathematics, the problem of finding the eigenvalues and eigenfunctions of the Sturm–Liouville problem is quite important. However, only in certain special cases can we, by various methods including the so-called *factorization method*,* get an exact answer to this question. Therefore, all types of approximate methods of finding the eigenvalues and eigenfunctions are of great significance. Let us now look at one of these approximate methods, namely, the *Ritz method.*

Let us first establish certain variational properties of the eigenvalues and eigenfunctions of the Sturm–Liouville problem**

$$L(u) = \lambda \rho u = 0 \quad \left(L(u) = \frac{d}{dx}\left(p \frac{du}{dx}\right) - qu\right), \tag{48}$$

$$R_0(u) = u(a)\cos\alpha + u'(a)\sin\alpha = 0,$$

$$R_1(u) = u(b)\cos\beta + u'(b)\sin\beta = 0. \tag{49}$$

These properties are of considerable interest since they enable us to approach the question of approximate determination of the eigenvalues and eigenfunctions of the problem (48)–(49) in the simplest manner.

In what follows, we shall, as usual, denote by $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$ the eigenvalues of the problem (48)–(49) and we shall denote by $u_1(x), u_2(x), \ldots, u_n(x), \ldots$ the corresponding eigenfunctions normalized with weight $\rho(x)$.

We shall call the expression

$$(\rho u, v) = \int_a^b \rho u v \, dx, \tag{100}$$

where u and v are arbitrary square-integrable functions on the segment $[a, b]$, the *scalar product* of the functions u and v in the case in which $\rho(x) \equiv 1$, and we shall call the scalar product of u and v with weight $\rho(x)$ in the case in which $\rho(x) \not\equiv 1$. We shall denote by M_0 the set of functions that are twice continuously differentiable

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*The reader can familiarize himself with the method of factorization, for example, in the book by F. M. Morse and G. Feshbach, *Methods of Theoretical Physics*, New York, McGraw-Hill, 1953.

**Here and in all that follows, unless the contrary is explicitly stated, the assumption regarding the problem (48)–(49) are the usual ones; that is, on the interval $[a, b]$, the functions p and ρ are positive, p is a continuously differentiable function, ρ and q are continuous functions, and α and β are constants such that $0 \leqslant \alpha, \beta < 2\pi$.

on the interval $[a, b]$ and that satisfy the boundary conditions (49). We shall denote by M the set of functions that are twice continuously differentiable on the interval $[a, b]$ (irrespective of any boundary conditions).

*THEOREM 1. The eigenvalue λ_n of the problem (48)–(49) is equal to the minimum of the functional**

$$(-Lu, u) = \int_a^b -L(u)\,u\,dx = -[puu']_a^b + \int_a^b (pu'^2 + qu^2)\,dx \qquad (101)$$

on the set of functions M_0 under the supplementary conditions of normalization

$$(\rho u, u) = \int_a^b \rho u^2\,dx = 1 \qquad (101')$$

*and the additional conditions of orthogonality***

$$(\rho u, u_l) = \int_a^b \rho u u_l\,dx = 0, \quad i = 1, 2, \ldots, n-1. \qquad (101'')$$

Also,

$$\lambda_n = (-Lu_n, u_n). \qquad (102)$$

Proof: The functions $u_1(x)$, $u_2(x)$, ... form a complete orthonormal system of functions with weight ρ. From the completeness relation

$$\int_a^b \rho\,[U + V]^2\,dx = \sum_{n=1}^{\infty} (U_n + V_n)^2,$$

where U and V are arbitrary piecewise-continuous square-integrable functions of x on the interval $[a, b]$ and U_n and V_n are their Fourier coefficients, we have

$$(\rho U, V) = \int_a^b \rho UV\,dx = \sum_{n=1}^{\infty} U_n V_n. \qquad (103)$$

From this equation, we obtain

$$(-Lu, u) = \int_a^b -uL(u)\,dx = \sum_{l=1}^{\infty} c_l \int_a^b -u_l \frac{L(u)}{\rho}\,\rho\,dx, \qquad (103')$$

*The expression on the right in (101) is obtained by integrating by parts.

** For $n = 1$, i.e., in the case of the first eigenvalue, there are no orthogonality conditions.

where the c_i are the Fourier coefficients of the function $u(x)$:

$$c_i = \int_a^b \rho u u_i \, dx.$$

Since the function $u(x)$ belongs to the set M_0, i.e., since it is twice continuously differentiable and satisfies the boundary conditions (49), we obtain

$$\int_a^b -L(u)\, u_i \, dx = \int_a^b -uL(u_i)\, dx = \lambda_i \int_a^b u u_i \rho dx = \lambda_i c_i.$$

If we substitute this expression into Eq. (103′) and note that (by virtue of (101″)) $c_1 = c_2 = \ldots = c_n = 0$, we obtain

$$(-Lu, u) = \sum_{i=n}^{\infty} \lambda_i c_i^2. \tag{103″}$$

The normalization conditions (101′) takes the form

$$(\rho u, u) = \int_a^b \rho u^2 \, dx = \sum_{i=n}^{\infty} c_i^2 = 1. \tag{103‴}$$

Noting that $\lambda_n < \lambda_{n+1} < \ldots$, we obtain from Eqs. (103″) and (103‴)

$$(-Lu, u) \geqslant \lambda_n.$$

Here, equality holds if and only if the function $u(x)$ coincides with the eigenfunction $u_n(x)$ and, in particular, belongs to the set of functions M_0. This completes the proof of the theorem.*

Remark (on natural boundary conditions): In particular, if the boundary conditions (49) take the form

$$u'(a) = 0, \quad u'(b) = 0 \tag{49′}$$

(such conditions are called *natural boundary conditions*), then, in accordance with the theorem just proved, the nth eigenvalue λ_n of the problem (48)–(49′) is equal to the minimum of the functional

$$I(u) = I(u, u) = \int_a^b (pu'^2 + qu^2) \, dx \tag{104}$$

under the additional conditions (101′)–(101″) on the set of functions M_0.

*Another proof of this theorem, based on the calculus of variations, can be found in the book by V. I. Smirnov, *A Course of Higher Mathematics*, Vol. IV, Reading, Massachusetts, Addison Wesley, 1964, pp. 552–556.

We shall show that here the class of admissible functions can be somewhat broadened; specifically, the set of functions M_0 can be replaced with the larger set of functions M. Let $u(x)$ be a function belonging to the set M that minimizes the functional (104) in the class of functions M under the supplementary conditions (101′)–(101″). Our assertion will be proven if we can show that the function $u(x)$ belongs to the set of functions M_0, i.e., that it satisfies the boundary conditions (49′). To do this, we note that, for every function $\eta(x)$ belonging to the set M and satisfying the orthogonality conditions (101″), we have

$$\int_a^b [p(u' + \alpha\eta')^2 + q(u + \alpha\eta)^2]\,dx \geqslant \lambda \int_a^b \rho(u + \alpha\eta)^2\,dx, \qquad (104')$$

where λ is the smallest value of the functional (104) and α is a real parameter. Since, by hypothesis,

$$\int_a^b (pu'^2 + qu^2)\,dx = \lambda \int_a^b \rho u^2\,dx,$$

we obtain from Eq. (104′)

$$2\alpha \int_a^b (pu'\eta' + qu\eta - \lambda\rho u\eta)\,dx + \alpha^2 \int_a^b (p\eta'^2 + q\eta^2 - \lambda\rho\eta^2)\,dx \geqslant 0. \qquad (104'')$$

From this, we obtain the equation

$$\int_a^b (pu'\eta' + qu\eta - \lambda\rho u\eta)\,dx = 0. \qquad (105)$$

Integration of this equation by parts yields

$$p\eta u'|_{x=b} - p\eta u'|_{x=a} - \int_a^b (L(u) + \lambda\rho u)\,\eta\,dx = 0. \qquad (105')$$

As the functions $\eta(x)$, we can take a function $\eta_0(x)$ satisfying the following conditions: (a) $\eta_0(b) = 1$; (b) $|\eta_0(x)| < \varepsilon$ on the interval $[a, b]$ outside a sufficiently small interval of length δ with right end point at $x=b$, where ε is an arbitrarily small positive number. The question of the existence of such a function $\eta_0(x)$ when $n = 1$ does not arise since in this case there are no orthogonality conditions. For $n \geqslant 2$, we may proceed as follows: Let $\eta^*(x)$ denote a twice continuously differentiable function on the interval $[a, b]$ that satisfies conditions (a) and (b). Since ε and δ are arbitrarily small, the Fourier coefficients $a_1, a_2, \ldots, a_{n-1}$ of the function $\eta^*(x)$ corresponding to the eigenfunctions $u_1, u_2, \ldots, u_{n-1}$ may be assumed arbitrarily small. If we now set

$$\eta_0(x) = A[\eta^*(x) - a_1u_1(x) - a_2u_2(x) - \ldots - a_{n-1}u_{n-1}(x)],$$

where A is a constant sufficiently close to unity, we obtain a function $\eta_0(x)$ belonging to the set M and satisfying the orthogonality conditions (101″) and the conditions (a) and (b).

If we now assume that $u'(b) \neq 0$ and set $\eta(x) = \eta_0(x)$ in Eq. (105′), we arrive at a contradiction since the second and third terms of the left-hand member of Eq. (105′) are, by the way in which the function $\eta_0(x)$ was constructed, arbitrarily small in absolute value. An analogous procedure at the point $x=a$ leads to the conclusion that $u'(a)=0$. This proves the assertion made.

THEOREM 2 (R. Courant). *Let* $m(v_1, \ldots, v_{n-1})$ *be the minimum of the functional*

$$\frac{(-Lu, u)}{(\rho u, u)} = \frac{1}{\int_a^b \rho u^2\, dx}\left[-[puu']_a^b + \int_a^b (pu'^2 + qu^2)\, dx\right] \tag{106}$$

on the set of functions M_0 *under the supplementary orthogonality conditions*

$$(\rho u, v_i) = \int_a^b \rho u v_i\, dx = 0, \quad i = 1, 2, \ldots, n-1, \tag{106′}$$

where $v_1(x), \ldots, v_{n-1}(x)$ *are piecewise-continuous square integrable functions on the interval* $[a, b]$. *Then, the* n*th eigenvalue* λ_n *of the problem (48)–(49) coincides with the maximum value of the quantity* $m(v_1, v_2, \ldots, v_{n-1})$ *for all possible choices of systems of functions* $v_1, v_2, \ldots, v_{n-1}$ *and is given by*

$$\lambda_n = m(u_1, u_2, \ldots, u_{n-1}). \tag{106″}$$

Proof: In the case in which $v_1 = u_1(x), \ldots, v_{n-1} = u_{n-1}(x)$, we have, in accordance with Theorem 1,

$$m(u_1, \ldots, u_{n-1}) = \lambda_n.$$

Therefore, we only need to show that, for every system of functions $v_1, \ldots, v_{n-1}$, there exists at least one function $u(x)$ belonging to the set M_0 such that

$$\frac{(-Lu, u)}{(\rho u, u)} \leqslant \lambda_n. \tag{107}$$

To construct such a function $u(x)$, we set

$$u(x) = \sum_{k=1}^{n} c_k u_k(x), \tag{107′}$$

where $c_1, \ldots, c_n$ are constants chosen so that the orthogonality condition (106′) will be satisfied, i.e.,

$$(\rho u, v_i) = \sum_{k=1}^{n} c_k (\rho u_k, v_i) = 0, \quad i = 1, 2, \ldots, n-1. \tag{107″}$$

The system of $n-1$ equations (107″) with unknowns $c_1, \ldots, c_n$ has nontrivial solutions which can always be made to satisfy the condition

$$\sum_{k=1}^{n} c_k^2 = 1. \tag{107‴}$$

Keeping this in mind, let us evaluate the functional (106) for the function $u(x)$ defined by Eq. (107′). We have

$$\frac{(-Lu, u)}{(\rho u, u)} = (-Lu, u) = \left(-L\left(\sum_{k=1}^{n} c_k u_k\right), \sum_{m=1}^{n} c_m u_m\right) =$$

$$= \sum_{k,m=1}^{n} c_k c_m (-Lu_k, u_m).$$

But since the eigenfunctions $u_1, \ldots, u_n$ of the problem (48)–(49) are orthonormal, we have

$$(-Lu_k, u_m) = \int_a^b -L(u_k)\, u_m\, dx = \int_a^b \lambda_k \rho u_k u_m\, dx = \begin{cases} \lambda_k, & k = m, \\ 0, & k \neq m. \end{cases}$$

Consequently,

$$\frac{(-Lu, u)}{(\rho u, u)} = \sum_{k=1}^{n} \lambda_k c_k^2.$$

Therefore, since $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$, we have on the basis of (107‴),

$$\frac{(-Lu, u)}{(\rho u, u)} \leqslant \lambda_n,$$

i.e., inequality (107) is valid. Therefore, $m(v_1, \ldots, v_{n-1})$ is also less than λ_n. This completes the proof of the theorem.

Remark (on natural boundary conditions): If the boundary conditions (49) are of the form

$$u'(a) = 0, \quad u'(b) = 0, \tag{49′}$$

then, in accordance with the theorem just proved, the nth eigenvalue λ_n of the problem (48)–(49′) is equal to the maximum of the minimum values $m(v_1, \ldots, v_{n-1})$ of the functional

$$\frac{1}{(\rho u,\ u)}\int_a^b (pu'^2 + qu^2)\,dx \tag{108}$$

defined on the set of functions M_0 under the supplementary orthogonality conditons (106′). Let us show that in this case the set of functions M_0 can be replaced with the larger set M. By virtue of the remark of Theorem 1, if $u(x)$ belongs to the set M and minimizes the functional (108) when $v_1 = u_1(x), \ldots, v_{n-1} = u_{n-1}(x)$, then $u(x)$ belongs to the set M_0 and hence coincides with $u_n(x)$, and Eq. (106″) holds. On the other hand, it is obvious that broadening of the class of admissible functions can only decrease the value of $m(v_1, \ldots, v_{n-1})$. Therefore, λ_n is equal to the maximum of the minimum values of the functional (108) on the set of functions M under the supplementary conditions (106′).

We shall now present a few very simple corollaries of Theorems 1 and 2, assuming that the boundary conditions (49) are of the form $\varkappa(u)|_{x=a} = 0$, $\varkappa u|_{x=b} = 0$, where $\varkappa u$ is a linear operator of the first, second, or third boundary-value problem.

COROLLARY 1. The nth eigenvalue λ_n of the problem

$$L(u) + \lambda\rho u = 0 \left(L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) - qu\right), \tag{48}$$

$$\left(\frac{\partial u}{\partial n} - \alpha_1 u\right)_{x=a} = 0, \quad \left(\frac{\partial u}{\partial n} - \alpha_2 u\right)_{x=b} = 0, \quad (\alpha_1, \alpha_2 \geqslant 0) \tag{49″}$$

does not decrease if the coefficients $p(x)$ and $q(x)$ are increased or if the coefficient $\rho(x)$ is decreased.

Proof: By virtue of the boundary conditions (49″), the functional (106) can be written in the form

$$\frac{(-Lu,\ u)}{(\rho u,\ u)} = \frac{1}{\int_a^b \rho u^2\,dx}\left[p(b)u^2(b)\alpha_1 + p(a)u^2(a)\alpha_2 + \int_a^b (pu'^2 + qu^2)\,dx\right]. \tag{109}$$

It is clear from this equation that, for fixed $\rho(x)$, increasing the coefficients $p(x)$ and $q(x)$ increases the value of the functional for every function $u(x)$ belonging to M_0 which satisfies the orthogonality conditions (106′). Therefore, in accordance with Theorem 2, the eigenvalue λ_n will increase, or at least will not decrease under these circumstances. Suppose now that the coefficients $p(x)$ and $q(x)$ remain fixed and the coefficient $\rho(x)$ is replaced with a coefficient $\rho^*(x) \leqslant \rho(x)$. Let $u(x)$ denote a function belonging to the set M_0 that satisfies the orthogonality conditions (106′). If we set

$$v_i^*(x) = \frac{\rho(x)}{\rho^*(x)} v_i(x),$$

we see that this function $u(x)$ satisfies the orthogonality conditions with weight $\rho^*(x)$:

$$(\rho^* u, v_i^*) = \int_a^b \rho^* u v_i^* dx = 0.$$

But, obviously,

$$\frac{(-Lu, u)}{(\rho^* u, u)} \geqslant \frac{(-Lu, u)}{(\rho u, u)}.$$

And, since to an arbitrary choice of the systems of functions $v_1, v_2, \ldots, v_{n-1}$, there corresponds an arbitrary choice of systems of functions $v_1^*, v_2^*, \ldots, v_{n-1}^*$, it follows that $\lambda_n^* \geqslant \lambda_n$, where λ_n^* is the nth eigenvalue of the problem (48)–(49″) with coefficients $\rho^*(x)$ and λ_n is the nth eigenvalue of this problem with coefficient $\rho(x)$. This completes the proof.

COROLLARY 2. If the constants α_1 and α_2 in the boundary conditions (49″) increase, the nth eigenvalue λ_n of the problem (48)–(49″) does not decrease.

The validity of this assertion follows from Eq. (109) on the basis of Theorem 2 because, for every function $u(x)$ belonging to the set M_0 and satisfying conditions (106′), the right-hand member of Eq. (109) increases, or at least does not decrease, with increasing α_1 and α_2.

COROLLARY 3. If the boundary conditions (49) are of the form

$$u(a) = 0, \quad u(b) = 0,$$

the nth eigenvalue λ_n cannot decrease with decreasing length of the interval $[a, b]$.

Proof: On the basis of Theorem 2, the nth eigenvalue λ_n corresponding to the interval $[a, b]$ is the maximum of the minimum values $m(v_1, \ldots, v_{n-1})$ of the functional

$$\frac{1}{(\rho u, u)} \int_a^b (pu'^2 + qu^2) dx \tag{110}$$

defined on the set of functions M_0 under the supplementary orthogonality conditions (106′). Now, if $[a', b']$ is an interval such that $a \leqslant a'$ and $b' \leqslant b$, the problem of finding the nth eigenvalue λ'_n corresponding to the interval $[a', b']$ can be treated as the same variational problem as for the λ_n case except that we now have the supplementary conditions that each of the admissible functions $u(x)$ must vanish identically on the intervals $[a, a']$ and $[b', b]$. But, as a result of this additional condition, each of the minimum

values $m(v_1, \ldots, v_{n-1})$ can only increase or remain constant. Therefore, on the basis of Theorem 2, we have $\lambda'_n \geqslant \lambda_n$.

These corollaries to Theorems 1 and 2 make it possible to determine approximately the eigenvalues of the problem (48)–(49) by means of estimates in terms of the eigenvalues of the same problem but with changed coefficients and boundary conditions.

Theorem 1 enables us to apply the so-called Ritz method to obtain an approximate determination of the eigenvalues and eigenfunctions of the problem (48)–(49). The basic idea of this method as applied to this problem consists in seeking the minimum value of the functional (101) under the normalization condition (101′) and the orthogonality conditions (101″) over some sufficiently restricted auxiliary set of functions that are twice continuously differentiable on the interval $[a, b]$ and that satisfy the boundary conditions (49). This last property is not essential in the case of natural boundary conditions, i.e., when (49) coincides with (49′). The Ritz method is particularly convenient for finding eigenvalues.

The general plan of the Ritz method as applied to the eigenvalue problem (48)–(49) consists in the following.

Let

$$\varphi_1(x), \ \varphi_2(x), \ \ldots, \ \varphi_n(x), \ \ldots \tag{111}$$

be an arbitrary system of linearly independent functions that are twice continuously differentiable on the interval $[a, b]$ and satisfy the boundary conditions (49), unless these are natural boundary conditions, in which case they need not be satisfied. Such a system of functions is called a *system of coordinate functions*.

Let us introduce the auxiliary set of functions M_N

$$\sigma_N(x) = \sum_{i=1}^{N} c_i \varphi_i(x), \tag{112}$$

where N is a fixed integer and the c_i are arbitrary real constants. We denote by $U_n(x)$ (for $1 \leqslant n \leqslant N$) that function in M_N which minimizes the functional

$$(-Lu, u) = \int_a^b -uLu\,dx = [puu']_a^b + \int_a^b (pu'^2 + qu^2)\,dx \tag{101}$$

with the normalization condition

$$(\rho u, u) = \int_a^b \rho u^2\,dx = 1 \tag{101'}$$

and the orthogonality conditions

$$(\rho u, U_i) = \int_a^b \rho u U_i\,dx = 0, \quad i = 1, 2, \ldots, n-1. \tag{101''}$$

We denote by Λ_n the value of this minimum. In the general plan of the Ritz method it is necessary to show the method of finding the numbers Λ_n and the functions $U_n(x)$ and to show that the numbers Λ_n converge to the corresponding eigenvalues λ_n of the Sturm–Liouville problem (48)–(49) as $N \to \infty$ and that the functions $U_n(x)$ converge in some sense or other (with respect to the metric of a certain well-defined function space) to the corresponding eigenfunctions $u_n(x)$ of that problem. When this last equation has an affirmative answer, the numbers Λ_n are usually taken as approximate values of the eigenvalues λ_n and the functions $U_n(x)$ are taken as approximate values of the eigenfunctions $u_n(x)$.

To obtain a method of finding the numbers Λ_n and the functions $U_n(x)$, we note that the functional (101) defined on the set of functions M_N can be converted into a function of N independent variables of the following form:

$$F_N(c_1, \ldots, c_N) = (-L\sigma_N, \sigma_N) = \left(-L\left(\sum_{i=1}^{N} c_i\varphi_i\right), \sum_{k=1}^{N} c_k\varphi_k\right) =$$

$$= \sum_{i,k=1}^{N} c_i c_k (-L\varphi_i, \varphi_k).$$

The normalization conditions (101′) and the orthogonality conditions (101″) take the following forms, respectively:

$$(\rho\sigma_N, \sigma_N) = \sum_{i,k=1}^{N} c_i c_k (\rho\varphi_i, \varphi_k) = 1,$$

$$(\rho\sigma_N, U_i) = \sum_{k=1}^{N} c_k (\rho\varphi_k, U_i) = 0, \quad (i = 1, 2, \ldots, n-1).$$

Thus, the problem of finding the numbers Λ_n and the functions $U_n(x)$ is reduced to the following minimum problem with side conditions for functions of N variables.

Find the minimum Λ_n of the function of N variables

$$F_N(c_1, \ldots, c_N) = \sum_{i,k=1}^{N} c_i c_k (-L\varphi_i, \varphi_k) \tag{113}$$

under the normalization condition

$$\sum_{i,k=1}^{N} c_i c_k (\rho\varphi_i, \varphi_k) = 1 \tag{113′}$$

and the orthogonality conditions

$$\sum_{k=1}^{N} c_k (\rho\varphi_k, U_i) = 0, \quad i = 1, 2, \ldots, n-1. \tag{113''}$$

Here, if $n=1$, there are no orthogonality conditions and the function $U_1(x)$ can be represented in the form

$$U_1(x) = \sum_{k=1}^{N} c_k^1 \varphi_k(x), \tag{114}$$

where the c_k^1 are values of c_k minimizing the function (113) under the conditions (113'). For $n=2$, the orthogonality conditions (113'') amount to a single equation, which, by virtue of (113'') and (114), takes the form

$$\sum_{i,k=1}^{N} c_i c_k^1 (\rho\varphi_i, \varphi_k) = 0. \tag{115}$$

The function $U_2(x)$ is determined by the equation

$$U_2(x) = \sum_{k=1}^{N} c_k^2 \varphi_k(x), \tag{116}$$

where the c_k^2 are the values of the c_k which minimize the function (113) under the normalization conditions (113') and the orthogonality conditions (115). Analogously, for $n \leqslant N$, the orthogonality conditions constitute $n-1$ equations of the following form:

$$\sum_{i,k=1}^{N} c_i c_k^m (\rho\varphi_i, \varphi_k) = 0, \quad m = 1, 2, \ldots, n-1, \tag{117}$$

and the function $U_n(x)$ is determined from the equation

$$U_n(x) = \sum_{k=1}^{N} c_k^n \varphi_k(x), \tag{118}$$

where the c_k^n are the values of the c_k that minimize the function (113) under the normalization condition (113') and the orthogonality conditions (113'') or, what amounts to the same thing, (117).

Let us first examine in detail the case in which $n=1$; that is, let us find the number Λ_1 of the function $U_1(x)$. Here, we shall use Lagrange's method of undetermined multipliers. For $c_i = c_i^1$, the function (113) under the conditions (113') must have a minimum. Therefore, for $c_i = c_i^1$, the derivatives of the function

$$\Phi_N = \sum_{i,k=1}^{N} c_i c_k (-L\varphi_i, \varphi_k) - \Lambda \left(\sum_{i,k=1}^{N} c_i c_k (\rho\varphi_i, \varphi_k) - 1 \right), \tag{119}$$

where Λ is an undetermined parameter, with respect to all the c_i must vanish. When we find expressions for these derivatives and set them equal to zero, we obtain the following system of linear algebraic equations:

$$\sum_{k=1}^{N} c_k [(-L\varphi_i, \varphi_k) - \Lambda(\rho\varphi_i, \varphi_k)] = 0, \quad i = 1, 2, \ldots, n. \tag{120}$$

Let us agree to call this system of equations the determining system of equations. For this system to have nontrivial solutions, it is necessary and sufficient for its determinant to vanish. Thus, we have the so-called *characteristic equation* for determining Λ:

$$\begin{vmatrix} (-L\varphi_1, \varphi_1) - \Lambda(\rho\varphi_1, \varphi_1) \ldots (-L\varphi_1\varphi_N) - \Lambda(\rho\varphi_1, \varphi_N) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (-L\varphi_N, \varphi_1) - \Lambda(\rho\varphi_N, \varphi_1) \ldots (-L\varphi_N, \varphi_N) - \Lambda(\rho\varphi_N, \varphi_N) \end{vmatrix} = 0. \tag{121}$$

Obviously, in the case of natural boundary conditions, we may set

$$(-L\varphi_i, \varphi_k) = I(\varphi_i, \varphi_k)$$

in the characteristic determinant, i.e., in the determinant in the left-hand member of Eq. (121).

The characteristic equation (121) is an nth-degree algebraic equation in Λ. In fact, if we divide the left-hand member of Eq. (121) by $(-\Lambda)^N$ and then let ν approach infinity, we attain in the limit the coefficient of $(-\Lambda)^N$ in the form of the so-called Gramm determinant

$$\Delta = \begin{vmatrix} (\rho\varphi_1, \varphi_1) & (\rho\varphi_1, \varphi_2) \ldots (\rho\varphi_1, \varphi_N) \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ (\rho\varphi_N, \varphi_1) & (\rho\varphi_N, \varphi_2) \ldots (\rho\varphi_N, \varphi_N) \end{vmatrix}. \tag{122}$$

But, as we know, for the functions $\varphi_1, \ldots, \varphi_N$ to be linearly independent, it is necessary and sufficient that their Gramm determinant (122) not vanish.* Since under our hypotheses the coordinate functions

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* To see this, suppose that the functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ are linearly dependent. Then,

$$\sum_{k=1}^{N} a_k\varphi_k = 0,$$

where the a_k are constants not all of which are zero. From this we obtain the system of equations

$\varphi_1, \varphi_2, \ldots, \varphi_N$ are linearly independent, the determinant (122) is not zero and the characteristic equation (121) is an nth-degree algebraic equation in Λ.

On the basis of what has been said, the characteristic equation (121) has exactly N roots. Let Λ^0 denote any one of these roots. When we substitute this root for Λ into the determining system of equations (120), we obtain a nontrivial solution of that system, $c_1^0, \ldots, c_N^0$, which obviously, we can always assume to satisfy the normalization condition (113′). If the root Λ^0 were complex, its conjugate would also be a root of Eq. (121). To this root $\bar{\Lambda}^0$ would correspond the nontrivial solution $\bar{c}_i^0$ (for $i = 1, 2, \ldots, N$) of the determining system of equations (120). Thus, we would have the identities

$$\sum_{k=1}^{N} c_k^0(-L\varphi_i, \varphi_k) = \Lambda^0 \sum_{k=1}^{N} c_k^0(\rho\varphi_i, \varphi_k), \tag{123}$$

$$\sum_{k=1}^{N} \bar{c}_k^0(-L\varphi_i, \varphi_k) = \bar{\Lambda}^0 \sum_{k=1}^{N} \bar{c}_k^0(\rho\varphi_i, \varphi_k). \tag{123′}$$

If we multiply Eq. (123) by $\bar{c}_i^0$ and Eqs. (123′) by c_i^0, sum both these sets of equations over i, and then subtract the second of the resulting equations from the first, we obtain

$$\sum_{k,i=1}^{N} \bar{c}_i^0 c_k^0(-L\varphi_i, \varphi_k) - \sum_{k,i=1}^{N} c_i^0 \bar{c}_k^0(-L\varphi_i, \varphi_k) =$$
$$= \Lambda^0 \sum_{i,k=1}^{N} c_k^0 \bar{c}_i^0(\rho\varphi_i, \varphi_k) - \bar{\Lambda}^0 \sum_{i,k=1}^{N} \bar{c}_k c_i^0(\rho\varphi_i\varphi_k).$$

$$\sum_{k=1}^{N} a_k(\rho\varphi_k\varphi_i) = 0, \quad i = 1, 2, \ldots, N. \tag{α}$$

Since this system has a nonzero solution, the Gramm determinant (122) must be equal to zero.

Suppose now that the Gramm determinant (122) is equal to zero. This means that the system of equations (α) has a non trivial solution. If we set $\sum_{k=1}^{N} a_k\varphi_k = \varphi$, the system of equations (α) can be written in the form $(\rho\varphi, \varphi_i) = 0$ for $i = 1, \ldots, N$. If we multiply this last equation by a_i and sum over the i, we obtain

$$\sum_{i=1}^{N} a_i(\rho\varphi, \varphi_i) = 0$$

or $(\rho\varphi, \varphi) = 0$. But this means that

$$\varphi = \sum_{k=1}^{N} a_k\varphi_k \equiv 0,$$

i.e., that the functions $\varphi_1, \ldots, \varphi_N$ are linearly dependent.

But the left-hand side of this last equation is equal to zero since $(-L\varphi_i, \varphi_k) = (-L\varphi_k, \varphi_i)$. Therefore,

$$(\Lambda^0 - \bar{\Lambda}^0) \sum_{i,k=1}^{N} c_k^0 \bar{c}_i^0 (\rho\varphi_i, \varphi_k) = 0.$$

If we now set $\varphi = \sum_{k=1}^{N} c_k \varphi_k$, we obtain from the last equation

$$(\rho\varphi, \bar{\varphi}) = \int_a^b \rho |\varphi|^2 dx = 0.$$

This means that $\varphi \equiv 0$ and, hence, $c_k^0 = 0$ since the coordinate functions φ_i are linearly independent. This proves that all the roots of the characteristic equation (121) are real.

If we multiply Eqs. (123) by c_i^0 and sum over i, we obtain

$$\sum_{i,k=1}^{N} c_i^0 c_k^0 (-L\varphi_i, \varphi_k) = \Lambda^0 \sum_{i,k=1}^{N} c_i^0 c_k^0 (\rho\varphi_i, \varphi_k).$$

On the basis of the normalization condition (113′) and the definition of the function $F_N(c_1, \ldots, c_N)$, this equation means that

$$F_N(c_1^0, c_2^0, \ldots, c_N^0) = \Lambda^0. \tag{124}$$

Thus, we conclude that the characteristic equation (121) has N roots, all of them real. To each of these roots Λ^0 there corresponds a nontrivial solution, $c_1^0, \ldots, c_N^0$, of the determining systems of equations (120), and Eq. (124) holds.

From this, we conclude that to find the number Λ_1, we need to choose the smallest of all the roots of the characteristic equation (121) and to take the corresponding nontrivial solution, $c_1^0, \ldots, c_N^0$, of the determining system of equations (120) as the coefficients $c_1^1, \ldots, c_N^1$ of the linear combination defining the function $U_1(x)$.

Let us now seek to find the value Λ_2 and the function $U_2(x)$. Here, making note of both the normalization condition (113′) and the orthogonality condition (115), we see that, for $c_i = c_i^2$, the derivatives with respect to c_i of the function

$$\Phi_N = \sum_{i,k=1}^{N} c_i c_k (-L\varphi_i, \varphi_k) - \Lambda \left(\sum_{i,k=1}^{N} c_i c_k (\rho\varphi_i, \varphi_k) - 1 \right) - \\ - \mu_1 \sum_{i,k=1}^{N} c_i c_k^1 (\rho\varphi_i, \varphi_k), \tag{125}$$

where Λ and μ_1 are undetermined parameters, must vanish. This means that the equations

$$\sum_{k=1}^{N} c_k [(-L\varphi_i, \varphi_k) - \Lambda(\rho\varphi_i, \varphi_k)] - \mu_1 \sum_{k=1}^{N} c_k^1 (\rho\varphi_i, \varphi_k) = 0,$$
$$i = 1, 2, \ldots, N \tag{126}$$

must be satisfied. If we multiply these equations by c_i^1 and sum over i, we obtain, keeping the normalization condition in mind,

$$\mu_1 = \sum_{i,k=1}^{N} c_i^1 c_k [(-L\varphi_i, \varphi_k) - \Lambda(\rho\varphi_i, \varphi_k)] =$$
$$= \sum_{k=1}^{N} c_k \sum_{i=1}^{N} c_i^1 [(-L\varphi_i, \varphi_k) - \Lambda_1(\rho\varphi_i, \varphi_k) + (\Lambda_1 - \Lambda)(\rho\varphi_i, \varphi_k)].$$

Therefore, since the c_i^1 must satisfy the determining system (120), we have

$$\mu_1 = (\Lambda_1 - \Lambda) \sum_{i,k=1}^{N} c_k c_i^1 (\rho\varphi_i, \varphi_k) = (\Lambda_1 - \Lambda)(\rho u, U_1).$$

From this it follows that $\mu_1 = 0$ since, by hypothesis, the unknown function $u(x) = U_2(x)$ and the function $U_1(x)$ are orthogonal with weight $\rho(x)$. If we substitute $\mu_1 = 0$ into (126), we conclude that the unknown constants $c_i = c_i^2$ corresponding to the function $U_2(x)$ satisfy the same determining system of equations (120). Similarly, we conclude that the constants $c_i = c_i^n$ corresponding to the function $U_n(x)$ (for $n \leqslant N$) satisfy the same determining system of equations (120).

Our final result is as follows:

The numbers Λ_n (for $1 \leqslant n \leqslant N$) *are roots of the characteristic equation (121) arranged in increasing order of magnitude, and each of the functions* $U_n(x)$ (*for* $1 \leqslant n \leqslant N$) *is a linear combination of the form (118), where* c_k^n (*for* $k = 1, 2, \ldots, N$) *is that solution of the determining system (120) corresponding to the root* Λ_n *of the characteristic equation (121).*

Let us now look at the question of the limiting values of the numbers Λ_n as $N \to \infty$, i.e., as we broaden indefinitely the set of functions M_N.

Here, without loss of generality, we can assume that all the eigenvalues λ_n of the Sturm–Liouville problem (48)–(49) are positive. (In the opposite case, since only a finite number of the eigenvalues can be negative [see subsection 5], we need only replace the operator

$$L(u) = \frac{d}{dx}\left(p \frac{du}{dx}\right) - qu$$

with the operator

$$L(u) = \frac{d}{dx}\left(p\,\frac{du}{dx}\right) - (q + k\rho)\,u,$$

where k is a sufficiently large positive number.)

On the basis of Theorem 1, the eigenvalue λ_n of the problem (48)–(49) is the minimum value of the functional

$$I(u) = \frac{(-Lu, u)}{(\rho u, u)} \tag{127}$$

under the supplementary orthogonality conditions (101″), over the set of functions Ω which are twice continuously differentiable and satisfy the boundary conditions (49) (satisfaction of these boundary conditions not being required if they are of the form (49′)). In particular, under these conditions, the following important inequality holds:*

$$\frac{(-Lu, u)}{(\rho u, u)} \geqslant \lambda_n > 0 \quad * \tag{128}$$

or

$$(\rho u, u) \leqslant \frac{1}{\lambda_n}(-Lu, u). \tag{129}$$

Let us now assume that the system of coordinate functions (111) possesses the so-called property of completeness in the sense of the norm

$$\|u\| = \sqrt{(-Lu, u)}. \tag{130}$$

This means that if $u(x)$ is a function belonging to the set Ω, then, for any positive number ε, there exists a function

$$\sigma_N(x) = \sum_{k=1}^{N} c_k \varphi_k(x),$$

where the c_k are constants and N is a sufficiently large number that

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*In those cases in which

$$\frac{(-Lu, u)}{(\rho u, u)} \geqslant \text{const} > 0,$$

we say that the operator Lu is positive definite (see, for example, V. I. Smirnov, *A Course of Higher Mathematics*, Vol. V, Reading, Massachusetts, Addison-Wesley, 1964).

$$\|u - \sigma_N\| \leqslant \varepsilon.$$

We note the following property of the norm (130):*

$$\|\varphi + \psi\| \leqslant \|\varphi\| + \|\psi\|. \tag{131}$$

This is true because, for every real value of a parameter θ, we have

$$0 \leqslant \|\varphi + \theta\psi\|^2 = (-L(\varphi + \theta\psi),\ \varphi + \theta\psi) = \|\varphi\|^2 + \theta^2\|\psi\|^2 + \\ + \theta[(-L\varphi,\ \psi) + (-L\psi,\ \varphi)]. \tag{132}$$

It follows from this that

$$(-L\varphi,\ \psi) + (-L\psi,\ \varphi) \leqslant 2\|\varphi\| \cdot \|\psi\|. \tag{132'}$$

If we substitute this into (132) with $\theta = 1$, we obtain inequality (131).

THEOREM. If the system of coordinate functions (111) is complete in the sense of the norm (130), then the numbers Λ_n *converge to the eigenvalue* λ_n *of the Sturm–Liouville problem (48)–(49) as* $N \to \infty$.

Proof: On the basis of (128) there exists by virtue of the definition of a lower bound, a function $U(x)$ belonging to the set Ω and satisfying the normalization condition (101′) and the orthogonality conditions (101″) such that

$$-\lambda_n \leqslant (-LU,\ U) < \lambda_n + \varepsilon,$$

where ε is an arbitrarily small positive number. This last inequality can also be written in the form

$$\sqrt{\lambda_n} \leqslant \|U\| \leqslant \sqrt{\lambda_n + \varepsilon}. \tag{133}$$

By virtue of the completeness of the coordinate functions in the sense of the norm (130), there exists a function

$$T_N(x) = \sum_{k=1}^{N} a_k \varphi_k(x),$$

where the a_k are constants and N is a sufficiently large number that

$$\|U - T_N\| < \sqrt{\varepsilon}.$$

*It is easy to show that the norm (130) possesses all the properties of a norm in a general Hilbert space.

Let us now construct a function satisfying the orthogonality conditions (101‴)

$$\sigma_N(x) = \sum_{k=1}^{N} c_k \varphi_k(x) = T_N(x) - \sum_{i=1}^{n-1} \alpha_i u_i(x),$$

where $\alpha_i = (\rho T_N, u_i)$. Since the functions U and the u_i (for $i = 1, 2, \ldots, n-1$) are orthonormal, we obtain, by using the Cauchy–Bunyakovskiy inequality

$$|(\rho\varphi, \psi)| \leqslant \sqrt{(\rho\varphi, \varphi)}\sqrt{(\rho\psi, \psi)}$$

and inequality (129), we obtain

$$\alpha_i = (\rho(T_N - U), u_i) \leqslant \sqrt{(\rho(T_N - U), (T_N - U))}\sqrt{(\rho u_i, u_i)} \leqslant \leqslant \frac{\|T_N - U\|}{\sqrt{\lambda_n}} \leqslant \frac{\sqrt{\varepsilon}}{\sqrt{\lambda_n}}.$$

In accordance with (133′), we obtain from this

$$\|U - \sigma_N\| \leqslant \|U - T_N\| + \left\| \sum_{i=1}^{n-1} \alpha_i u_i(x) \right\|,$$

$$\|U - \sigma_N\| < A\sqrt{\varepsilon}, \tag{134}$$

where A is a constant independent of ε.

By virtue of property (131) of the norm, we have

$$\|\sigma_N\| \leqslant \|U\| + A\sqrt{\varepsilon} \leqslant \sqrt{\lambda_n + \varepsilon} + A\sqrt{\varepsilon}$$

or

$$(-L\sigma_N, \sigma_N) \leqslant (\sqrt{\lambda_n + \varepsilon} + A\sqrt{\varepsilon})^2. \tag{134′}$$

It follows from inequalities (129) and (134) that

$$(\rho(\sigma_N - U), (\sigma_N - U)) \leqslant \frac{1}{\lambda_n} A^2 \varepsilon.$$

Recalling that inequality* (131) also holds for the norm defined by the equation

$$\|u\| = \sqrt{(\rho u, u)},$$

*Proof of this inequality is the same as for the case of the norm defined by Eq. (130).

we obtain

$$\sqrt{(\rho\sigma_N, \sigma_N)} - \sqrt{(\rho U, U)} \geqslant -\sqrt{(\rho(\sigma_N - U), (\sigma_N - U))} > -A\sqrt{\frac{\varepsilon}{\lambda_n}}$$

or, since $(\rho U, U) = 1$,

$$\sqrt{(\rho\sigma_N, \sigma_N)} \geqslant 1 - A\sqrt{\frac{\varepsilon}{\lambda_n}}. \tag{134''}$$

When we substitute (134′) and (134″) into inequality (128), we obtain

$$\lambda_n \leqslant \frac{(-L\sigma_N, \sigma_N)}{(\rho\sigma_N, \sigma_N)} \leqslant \frac{\left(\sqrt{\lambda_n + \varepsilon} + A\sqrt{\frac{\varepsilon}{\lambda_n}}\right)^2}{\left(1 - A\sqrt{\frac{\varepsilon}{\lambda_n}}\right)^2} \leqslant \lambda_n + \eta,$$

where η is an arbitrarily small positive number. The last inequality means that

$$\Lambda_n = \min \frac{(-L\sigma_N, \sigma_N)}{(\rho\sigma_N, \sigma_N)} \to \lambda_n$$

as $N \to \infty$. This completes the proof of the theorem.*

3. The Simplest Properties of Bessel Functions and Their Applications in the Solution of Boundary-Value Problems

1. **Definition and simplest properties of Bessel functions.** The ordinary differential equation

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0, \tag{135}$$

where ν is a real or complex constant, is called *Bessel's equation* of order ν. Starting with this equation, we can construct a whole class of special functions known as *Bessel functions* of order ν. Bessel functions are also known as "cylindrical functions".

If ν is not an integer, it is easy to show by direct substitution that the two power series

* For a more detailed exposition of the Ritz method, see, for example, the book by S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Oxford and New York, Pergamon, 1964.

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}, \tag{136}$$

and

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+1-\nu)} \left(\frac{x}{2}\right)^{2n-\nu}, \tag{136'}$$

where $\Gamma(s)$ is the gamma function

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, dx, \qquad (\operatorname{Re} s > 0).$$

are linearly independent solutions of Bessel's equation of order ν. The power series (136) and (136') are called *Bessel functions of the first kind* (or orders ν and $-\nu$, respectively).

If ν is equal to a positive integer, we see, remembering that $\Gamma(s)$ has a pole at zero and at every negative integer, that the first $\nu - 1$ in the power series (136') vanish and, hence, that

$$J_\nu(x) = (-1)^\nu J_{-\nu}(x). \tag{137}$$

This means that for integral values of ν, the two Bessel functions of the first kind shown above are linearly dependent. Therefore, to obtain two linearly independent solutions of Eq. (135) for all possible values of ν, we take as our second solution, not the function $J_{-\nu}(x)$ but the so-called *Bessel function of the second kind,* which, for nonintegral ν, is defined by

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}, \tag{138}$$

and for integral values of ν by taking the limit (as the argument approaches the integer in question) in Eq. (138).

The functions

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iN_\nu(x), \\ H_\nu^{(2)}(x) &= J_\nu(x) - iN_\nu(x) \end{aligned} \tag{139}$$

are called *Bessel functions of the third kind.* The first of these is also known as the *Hankel function of the first kind,* and the second as the *Hankel function of the second kind.*

In addition to these Bessel functions, Bessel functions of imaginary argument are of considerable importance. These functions are defined by the equations

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu},$$

$$K_\nu(x) = \frac{\pi}{2} \frac{1}{\sin \nu\pi} (I_{-\nu}(x) - I_\nu(x)). \tag{140}$$

Obviously, these functions are solutions of the equation

$$x^2u'' + xu' - (x^2 + \nu^2)u = 0, \tag{135}$$

which is obtained from Eq. (135) by replacing x with ix.

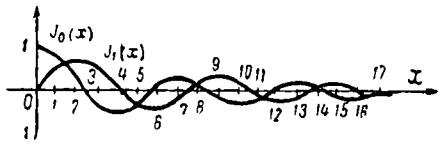

Fig. 38.

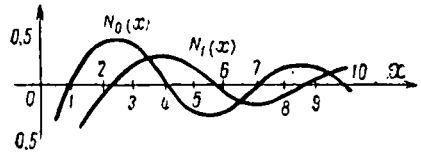

Fig. 39.

Consideration of Bessel functions of the first and second kinds for various different complex arguments leads to a large number of other Bessel functions, usually denoted by

$$\mathrm{ber}_\nu(x),\ \mathrm{bei}_\nu(x),\ \mathrm{ker}_\nu(x),\ \mathrm{kei}_\nu(x),\ \mathrm{her}_\nu(x),\ \mathrm{hei}_\nu(x).$$

We shall not give the definitions of these functions but merely point out that they are analogous to the functions that we would obtain (particularly combinations of the trigonometric and exponential functions) if we considered e^z on all possible rays issuing from the coordinate origin in the complex z-plane.

Very precise tables and graphs have been constructed for Bessel functions of all kinds and many orders. Figures 38 and 39 show the graphs of functions of the first and second kinds of orders 0 and 1.

We shall now give some of the simpler properties common to Bessel functions, confining ourselves to functions of the first kind since all Bessel functions can be expressed in terms of them rather simply.

(1) Differentiation formulas and recursion formulas.

We have the following formulas:

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x), \tag{141}$$

$$J'_\nu(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x) \tag{141'}$$

and, in particular,

$$J_0'(x) = -J_1(x), \quad [xJ_1(x)]' = xJ_0(x). \tag{141''}$$

Proof: Simple calculation yields

$$\frac{d}{dx}\frac{J_\nu(x)}{x^\nu} = \frac{1}{2^\nu}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+1+\nu)}\left(\frac{x}{2}\right)^{2n-1} n =$$

$$= \frac{1}{2^\nu}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!\,\Gamma(k+1+\nu+1)}\left(\frac{x}{2}\right)^{2k+1} =$$

$$= -\frac{1}{x^\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+1+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu+1},$$

i.e.,

$$\frac{d}{dx}\frac{J_\nu(x)}{x^\nu} = -\frac{1}{x^\nu}J_{\nu+1}(x).$$

From this, it follows that formula (141) is valid. Similarly, we can show that formula (141′) is valid. If we set $\nu=0$ in formula (141), we obtain the first of formulas (141″). If we set $\nu=1$ in (141′), we obtain the second of formulas (141″).

By subtracting and adding Eqs. (141) and (141″), we obtain the following important formulas:

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x), \tag{142}$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x). \tag{142'}$$

These formulas enable us to express the values of $J_\nu(x)$ and $J'_\nu(x)$ for all integral values of ν in terms of the values of $J_0(x)$ and $J_1(x)$, which are to be bound in all tables of Bessel functions.

(2) Functions of half-integral order. For half-integral ν, the functions $J_\nu(x)$ can be expressed in terms of elementary functions, specifically, in terms of trigonometric functions and polynomials. Let us first find expressions for $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$. We have

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(n+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{2}},$$

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2n-\frac{1}{2}}.$$

But, by virtue of the properties of the gamma function,

$$\Gamma\left(\frac{3}{2}+n\right)=\frac{1\cdot 3\ldots(2n+1)}{2^{n+1}}\Gamma\left(\frac{1}{2}\right),$$

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{1\cdot 3\ldots(2n-1)}{2^{n}}\Gamma\left(\frac{1}{2}\right),\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Therefore,

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!},\quad J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!},$$

that is,

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x,\quad J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x. \tag{143}$$

If we set $\nu = 1/2$ in the recursion formula (142), we obtain

$$J_{\frac{3}{2}}(x)=\frac{1}{x}J_{\frac{1}{2}}(x)-J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(-\cos x+\frac{1}{x}\sin x\right)$$

or

$$J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\pi}{2}\right)+\frac{1}{x}\cos\left(x-\frac{\pi}{2}\right)\right]. \tag{143'}$$

In the same way, we can show that, for arbitrary integral ν,

$$J_{\nu+\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\frac{\nu\pi}{2}\right)P_\nu\left(\frac{1}{x}\right)+\right.$$
$$\left.+\cos\left(x-\frac{\nu\pi}{2}\right)Q_\nu\left(\frac{1}{x}\right)\right], \tag{143''}$$

where $P_\nu(1/x)$ and $Q_\nu(1/x)$ are specific polynomials in $1/x$.

(3) The generating function and integral representation of integral order. Consider the analytic function of a complex variable z depending on a parameter x

$$f(x)=e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}. \tag{144}$$

This function is a regular analytic function everywhere except at the points $z=0$ and $z=\infty$. Therefore, it can be expanded in a Laurent series

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}=\sum_{-\infty}^{\infty}c_n(x)\,z^n, \tag{144'}$$

where

$$c_n(x) = \frac{1}{2\pi i}\int_C e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\frac{dt}{t^{n+1}},$$

C being a closed contour encircling the point $z=0$. If we introduce a new variable of integration ζ defined by $t = 2\zeta/x$, we obtain

$$c_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{2\pi i}\int_{C'} e^{\zeta-\frac{x^2}{4\zeta}}\frac{d\zeta}{\zeta^{n+1}},$$

where C' is a closed contour encircling the point $\zeta=0$. But

$$e^{-\frac{x^2}{4\zeta}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^{2k}\frac{1}{\zeta^k}.$$

Therefore,

$$c_n(x) = \left(\frac{x}{2}\right)^n \frac{1}{2\pi i}\int_{C'}\frac{e^{\zeta}}{\zeta^{n+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^{2k}\frac{1}{\zeta^k}\,d\zeta =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{x}{2}\right)^{2k+n}\frac{1}{2\pi i}\int_{C'} e^{\zeta}\frac{d\zeta}{\zeta^{k+n+1}}.$$

According to the theory of residues, for n a positive integer, we have

$$\frac{1}{2\pi i}\int_{C'} e^{\zeta}\frac{d\zeta}{\zeta^{k+n+1}} = \frac{1}{(k+n)!}.$$

Thus, we obtain

$$c_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{2k+n} = J_n(x) \qquad (n \geqslant 0).$$

If we now replace z in Eq. (144′) with $-1/z$, the left-hand member of this equation will not be changed. Consequently,

$$c_{-n}(x) = (-1)^n J_n(x) = J_{-n}(x).$$

Thus, we conclude that the function (144) is a "generating" function for Bessel functions of the first kind of integral orders, i.e., that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \sum_{-\infty}^{\infty} J_n(x)\cdot z^n. \tag{145}$$

If we assume x real in this equation and set $z=e^{i\theta}$, we obtain

$$e^{ix\sin\theta}=\sum_{-\infty}^{\infty}J_n(x)\,e^{in\theta}.$$

If we multiply both sides of this equation by $e^{-in\theta}$ and integrate from $-\pi$ to π, we obtain*

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-n\theta)}\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta-n\theta)\,d\theta=$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\cos(x\sin\theta-n\theta)\,d\theta\ *. \tag{146}$$

These equations are called the integral representations for Bessel functions of the first kind of integral order. These integral representations, which we have established for real values of x, remain valid for complete values by virtue of the property of analytic continuation of functions.

We note that in the case of nonintegral orders, we have, as can be shown, instead of formula (146), the equation

$$J_\nu(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\theta-\nu\theta)}\,d\theta-\frac{\sin\nu\pi}{\pi}\int_{0}^{\infty}e^{-x\sinh\xi-\nu\xi}\,d\xi. \tag{147}$$

(4) Asymptotic formulas for functions of integral orders and their zeros. If we set $\theta=\psi-\pi/2$ in Bessel's integral (146), we obtain

$$J_n(x)=\frac{(-i)^n}{2\pi}\int_{-\frac{\pi}{2}}^{\pi+\frac{\pi}{2}}e^{i(x\cos\psi+n\psi)}\,d\psi$$

or, by virtue of the periodicity of the integrand,

$$J_n(x)=\frac{(-i)^n}{2\pi}\int_{-\pi}^{\pi}e^{i(x\cos\psi+n\psi)}\,d\psi=\frac{(-i)^n}{2\pi}\int_{-\pi}^{\pi}e^{ix\cos\psi}\cos n\psi\,d\psi=$$

$$=\frac{(-i)^n}{\pi}\int_{0}^{\pi}e^{ix\cos\psi}\cos n\psi\,d\psi.$$

*The last two expressions in Eq. (146) are obtained from the preceding one by virtue of the oddness of the sine and the evenness of the cosine.

If we now set $\cos\psi=\xi$, we obtain

$$J_n(x)=\frac{(-i)^n}{\pi}\int_{-1}^{1}e^{ix\xi}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}d\xi, \tag{148}$$

where

$$T_n(\xi)=\cos n\arccos\xi=\operatorname{Re}(\cos\psi+i\sin\psi)^n=\operatorname{Re}(\xi+i\sqrt{1-\xi^2})^n$$

is Chebyshev's polynomial of nth degree. It is clear from the above expression of this polynomial that

$$T_n(-\xi)=(-1)^nT_n(\xi).$$

Therefore, Eq. (148) can be written in the form

$$J_n(x)=\frac{(-i)^n}{\pi}\int_{0}^{1}e^{ix\xi}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}d\xi+\frac{i^n}{\pi}\int_{0}^{1}e^{-ix\xi}\frac{T_n(\xi)}{\sqrt{1-\xi^2}}d\xi. \tag{148'}$$

Consider an integral of the form

$$Q(x)=\int_{0}^{1}e^{\pm ix\xi}\frac{f(\xi)}{\sqrt{1-\xi}}d\xi, \tag{149}$$

where $f(\xi)$ is an arbitrary function that is twice continuously differentiable on the interval [0, 1]. Let us show that, for large positive values of x, the following asymptotic equation holds:

$$Q(x)=\sqrt{\frac{\pi}{x}}e^{\pm i\left(x-\frac{\pi}{4}\right)}[f(1)+\delta], \tag{150}$$

where δ is a quantity that approaches zero as $x\to\infty$. If we set $\eta=1-\xi$, we obtain

$$Q(x)=e^{\pm ix}\left\{f(1)\int_{0}^{1}\frac{e^{\mp ix\eta}}{\sqrt{\eta}}d\eta+\int_{0}^{1}e^{\mp ix\eta}g(\eta)\sqrt{\eta}\,d\eta\right\}, \tag{150'}$$

where $g(\eta)=[f(1-\eta)-f(1)]/\eta$ is a continuously differentiable function on the interval $0\leqslant\eta\leqslant 1$. Integration by parts yields

$$\int_{0}^{1}e^{\mp ix\eta}g(\eta)\sqrt{\eta}\,d\eta=\frac{1}{\mp ix}e^{\mp ix\eta}g(\eta)\sqrt{\eta}\Big|_{0}^{1}-$$

$$-\frac{1}{\mp ix}\int_{0}^{1}e^{\mp ix\eta}(g(\eta)\sqrt{\eta})'\,d\eta=O\left(\frac{1}{x}\right). \tag{150''}$$

Furthermore, if we set $\sqrt{\eta x}=t$, we have

$$\int_0^1 e^{\mp ix\eta}\frac{d\eta}{\sqrt{\eta}}=\frac{2}{\sqrt{x}}\int_0^{\sqrt{x}} e^{\mp it^2}\,dt=\frac{2}{\sqrt{x}}\left[\int_0^{\infty} e^{\mp it^2}\,dt+\delta\right],$$

where δ is a quantity that approaches zero as $x\to\infty$. From the known values of the Fresnel integrals

$$\int_0^{\infty}\cos t^2\,dt=\int_0^{\infty}\sin t^2\,dt=\frac{\sqrt{\pi}}{2\sqrt{2}},$$

we obtain

$$\int_0^1 e^{\mp ix\eta}\frac{d\eta}{\sqrt{\eta}}=\sqrt{\frac{\pi}{x}}\,\left(e^{\mp i\frac{\pi}{4}}+\delta\right). \qquad (150''')$$

If we substitute (150″) and (150‴) into Eq. (150′), we obtain

$$Q(x)=\sqrt{\frac{\pi}{x}}\;e^{\pm i\left(x-\frac{\pi}{4}\right)}[f(1)+\delta]. \qquad (150)$$

If we now substitute (150) into Eq. (148′) noting that

$$\frac{1}{\sqrt{1+\xi}}T_n(\xi)\Big|_{\xi=1}=\frac{1}{2},$$

we obtain

$$J_n(x)=\frac{1}{\sqrt{2\pi x}}\left[e^{i\left(x-\frac{\pi}{4}\right)-in\frac{\pi}{2}}+e^{-i\left(x-\frac{\pi}{4}\right)+in\frac{\pi}{2}}+\delta\right]$$

or

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right)+\delta\right], \qquad (151)$$

where δ is a quantity that approaches zero as $x\to\infty$. Formula (151) indicates the asymptotic behavior of Bessel functions of the first kind of integral orders for sufficiently large positive values of x.

We point out that, in the case of integral orders, there is an asymptotic formula for the function $N_n(x)$ analogous to formula (151):

$$N_n(x)=\sqrt{\frac{2}{\pi x}}\left[\sin\left(x-\left(n+\frac{1}{2}\right)\frac{\pi}{2}\right)+\delta\right]. \qquad (151')$$

In Eqs. (151) and (151′), $\delta = O(x^{-\frac{3}{2}})$. Furthermore, formulas (151), (151′), and analogous formulas for the functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are also valid for nonintegral orders.*

From the asymptotic formulas for Bessel functions, we can easily obtain asymptotic formulas for their zeros. On the basis of (151), the zeros $\mu_k^{(n)}$ of the functions $J_n(x)$ are determined from the equations

$$\mu_k^{(n)} - \frac{\pi}{2}\left(n + \frac{1}{2}\right) \simeq (k-1)\pi + \frac{\pi}{2},$$

where the number k is a sufficiently large number. Thus, we obtain

$$\mu_k^{(n)} \simeq k\pi + \left(n - \frac{1}{2}\right)\frac{\pi}{2}. \tag{152}$$

2. The problem of a vibrating circular membrane with fixed edges. As a typical example of the application of Bessel functions to the solution of boundary-value problems by the method of separation of variables, let us consider the problem of free vibrations of a circular membrane with fixed edges.

If, for definiteness, we assume the radius of the membrane to be unity, this problem reduces to solving the following boundary-value problem for the circles $x^2+y^2<1$:

$$u_{xx} + u_{yy} - u_{tt} = 0, \tag{153}$$

$$u\Big|_{t=0} = \psi(x, y), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x, y), \tag{153′}$$

$$u\Big|_{x^2+y^2=1} = 0. \tag{153″}$$

In the cylindrical coordinate system $x=\rho\cos\theta$, $y=\rho\sin\theta$, the Laplacian operator takes the form

$$u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}$$

(see Chapter III, Sect. 3, subsection 1). Therefore, in cylindrical coordinates, the boundary-value problem (153)–(153′)–(153″) becomes

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* For more details on this and other properties of Bessel functions, consult textbooks on special functions, for example, R. O. Kuz'man, *Besselevy funktsii* (Bessel functions), ONTI, 1935, or G. N. Watson, *Bessel Functions*, New York, MacMillan, 1945.

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{\partial^2 u}{\partial t^2} = 0, \tag{154}$$

$$u\Big|_{t=0} = \psi(\rho, \theta), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(\rho, \theta), \tag{154'}$$

$$u\Big|_{\rho=1} = 0. \tag{154''}$$

Let us apply the method of separation of variables to the problem (154)–(154′)–(154″). We set

$$u = v(\rho, \theta)\, T(t), \tag{155}$$

where v is a function of ρ and θ only and T is a function of t only. When we substitute this expression into Eq. (154), we find

$$T\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho}T + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2}T - vT''(t) = 0.$$

If we divide both members of this equation by vT, we obtain the conditions of separability of the variables:

$$\frac{1}{v}\left(\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2}\right) = \frac{T''}{T} = -\lambda = \text{const}. \tag{156}$$

Keeping the boundary conditions (154′) in mind, we obtain from the above equation the following two-dimensional Sturm–Liouville problem for determining the function $v(\rho, \theta)$:

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2} + \lambda v = 0, \tag{157}$$

$$v\Big|_{\rho=1} = 0. \tag{157'}$$

To solve this problem, we again use the method of separation of variables. If we set

$$v = R(\rho)\,\Phi(\theta), \tag{158}$$

where R is a function of ρ only and Φ is a function of θ only, we get

$$R''\Phi + \frac{1}{\rho}R'\Phi + \frac{1}{\rho^2}R\Phi'' + \lambda R\Phi = 0.$$

If we divide both members of this equation by $R\Phi$ and multiply by ρ^2, we obtain the condition of separability of the variables:

$$\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \lambda\rho^2 = -\frac{\Phi''}{\Phi} = \alpha = \text{const}. \tag{159}$$

The function $\Phi(\theta)$ must be periodic with period 2π because, otherwise, the function $v=R(\rho)\,\Phi(\theta)$ would not be single-valued. Keeping this in mind, we obtain the following Sturm–Liouville problem for determining the function $\Phi(\theta)$:

$$\Phi'' + \alpha\Phi = 0, \tag{160}$$

$$\Phi(2\pi) = \Phi(0). \tag{160'}$$

Obviously, nontrivial solutions of the problem (160)–(160′) exist only for $\alpha=0,\ 1^2,\ \ldots,\ n^2,\ \ldots$. These solutions may be written in the form

$$\varphi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta,$$

where the A_n and the B_n are arbitrary constants. The functions

$$\Phi_0(\theta) = \frac{1}{\sqrt{2\pi}},\quad \Phi_n(\theta) = \sqrt{\frac{2}{\pi}}\cos n\theta,\quad \Phi_n^{\bullet}(\theta) = \sqrt{\frac{2}{\pi}}\sin n\theta \tag{161}$$

are orthogonal eigenfunctions of the problem (160)–(160′). These functions constitute a complete set of functions on the interval $[0,\ 2\pi]$.

From the condition of separability of the variables (159), we have

$$\rho^2 R'' + \rho R' + (\lambda\rho^2 - n^2)R = 0$$

or

$$\frac{d}{d\rho}(\rho R') - \frac{n^2}{\rho}R + \lambda\rho R = 0.$$

If we now remember that the function $R(\rho)$ must vanish at $\rho=1$ and must be finite at $\rho=0$, we arrive at the following generalized Sturm–Liouville problem (see Sect. 2, subsection 2):

$$\frac{d}{d\rho}(\rho R') - \frac{n^2}{\rho}R + \lambda\rho R = 0, \tag{162}$$

$$R\big|_{\rho=0} \neq \infty,\quad R\big|_{\rho=1} = 0. \tag{162'}$$

Let us find the eigenvalues and eigenfunctions of this problem, denoting them, respectively, by λ_{nm} and R_{nm}. If we introduce the new variable $x=\rho\sqrt{\lambda}$, then problem (162)–(162′) takes the form

$$x^2R'' + xR' + (x^2 - n^2)R = 0, \tag{163}$$

$$R\big|_{x=0} \neq \infty, \quad R\big|_{x=\sqrt{\lambda}} = 0. \tag{163'}$$

Equation (163) is Bessel's equation of nth order. The only solution (up to a constant factor) of this equation that is bounded at $x=0$ is $J_n(x)$.* This function has infinitely many positive zeros, the values of which as $m \to \infty$ are given by the asymptotic equations

$$\mu_m^{(n)} = m\pi + \left(n - \frac{1}{2}\right)\frac{\pi}{2} \tag{152}$$

(see subsection 1). Thus, we conclude that the eigenvalues and eigenfunctions of the problem (162)–(162′) that are orthonormal with weight ρ are given by the equations

$$\lambda_{nm} = [\mu_m^{(n)}]^2, \quad R_{nm}(\rho) = \frac{J_n(\rho\mu_m^{(n)})}{\sqrt{\int_0^1 J_n^2(\rho\mu_m^{(n)})\,\rho\,d\rho}} \quad (m = 1, 2, \ldots). \tag{164}$$

The eigenvalues and eigenfunctions of the two-dimensional Sturm–Liouville problem (157)–(157′) are given by the equations

$$\lambda_{nm} = [\mu_m^{(n)}]^2, \quad R_{nm}(\rho)\,\Phi_n(\theta), \quad R_{nm}(\rho)\,\Phi_n^*(\theta) \tag{165}$$
$$(n = 0, 1, \ldots, m = 1, 2, \ldots).$$

For every fixed value of n, the functions $R_{nm}(\rho)$ (for $m = 1, 2, \ldots$) constitute a system of eigenfunctions of the generalized Sturm–Liouville problem (162)–(162′) that is orthonormal with weight ρ (see Sect. 2, subsection 2). Without giving the complete proof, we point out that this orthonormal system of eigenfunctions, like the system of eigenfunctions of the Sturm–Liouville problem (48)–(49), constitutes, for every fixed n, a complete system of functions on the interval [0, 1]. In view of this and the lemma on

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*That the second solution $N_n(x)$ of Eq. (163) is unbounded in a neighborhood of $x = 0$ follows from the fact that if we write Eq. (163) in the form

$$\frac{d}{dx}(xR') - x\left(1 - \frac{n^2}{x}\right)R = 0, \text{ then}$$

$$J_n\frac{d}{dx}(xN_n') - N_n\frac{d}{dx}(xJ_n') = \frac{d}{dx}[x(J_nN_n' - N_nJ_n')] \equiv 0.$$

$$\frac{d}{dx} = \frac{N_n}{J_n} = \frac{c}{xJ_n^2}, \quad c = \text{const}, \quad N_n(x) = J_n\left[\int_0^x \frac{c}{xJ_n^2}\,dx + \text{const}\right],$$

we see that $N_n(0) = \infty$.

the completeness of the system of functions of two variables (see Sect. 2, subsection 4), we see that the system of eigenfunctions of the two-dimensional Sturm–Liouville problem (157)–(157′) that are defined by Eqs. (165) constitute a system of functions that is complete in the closed rectangle $0 \leqslant \rho \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, and that is orthonormal with weight ρ.

If we turn now to the condition of separability of the variables (156), we obtain, for each eigenvalue λ_{nm}, an equation for finding the function $T = T_{nm}$:

$$T''_{nm} + \lambda_{nm} T_{nm} = 0. \tag{166}$$

Linearly independent solutions of this equation are the functions

$$T_{nm}(t) = \cos \mu_m^{(n)} t, \; T_{nm}\big|_{t=0} = 1, \; T'_{nm}\big|_{t=0} = 0 \tag{167}$$

$$T^*_{nm}(t) = \frac{1}{\mu_m^{(n)}} \sin \mu_m^{(n)} t, \; T^*_{nm}\big|_{t=0} = 0, \; T^{*\prime}_{nm}\big|_{t=0} = 1. \tag{167′}$$

Suppose now that the functions $\psi(\rho, \theta)$ and $\psi_1(\rho, \theta)$ giving the initial conditions are expanded in a Fourier series of the eigenfunctions (165):

$$\psi(\rho, \theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [a_{nm} \Phi_n(\theta) + a^*_{nm} \Phi^*_n(\theta)] R_{nm}(\rho), \tag{168}$$

$$\psi_1(\rho, \theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [b_{nm} \Phi_n(\theta) + b^*_{nm} \Phi^*_n(\theta)] R_{nm}(\rho). \tag{168′}$$

If we multiply the first of these by T_{nm} and the second by T^*_{nm}, we obtain the series

$$u(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{[a_{nm} \Phi_n(\theta) + a^*_{nm} \Phi^*_{nm}(\theta)] R_{nm}(\rho) T_{nm}(t) + [b_{nm} \Phi_n(\theta) + b^*_{nm} \Phi^*_{nm}(\theta)] R_{nm}(\rho) T^*_{nm}(t)\}. \tag{169}$$

By virtue of (155) and (158), each of the terms in this series is a solution of Eq. (174) and satisfies the boundary conditions (154″). If we assume that the series (169) can be twice differentiated termwise with respect to the variables ρ, θ, and t, then this series is a solution of the problem of the vibrations of a circular membrane (154)–(154′)–(154″) since the initial conditions (154′) are, by virtue of (167), (167′), (168), and (168′), satisfied.

Thus, we have solved the problem of the vibrations of a circular membrane (154)–(154′)–(154″) under the assumption that the series (169) can be twice differentiated termwise or, what amounts to the same thing, under the assumption that the Fourier-series

expansions (168) and (168′) of the functions giving the initial conditions converge sufficiently rapidly.

4. The Simplest Properties of Spherical Harmonics and Their Application to the Solution of Boundary-Value Problems

1. **Legendre polynomials.** Legendre polynomials are closely associated with the unit source function for the three-dimensional Laplace equation

$$\delta(P, P_0) = \frac{1}{4\pi r}, \tag{170}$$

where r is the distance from a point P to a fixed point P_0. Let ρ and ρ_0 be the radius vectors of the points P and P_0 and let θ be the angle between them (see Fig. 40). Then, we can write

$$\delta(P, P_0) = \frac{1}{4\pi} \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\theta}} =$$

$$= \frac{1}{4\pi} \begin{cases} \frac{1}{\rho_0} \frac{1}{\sqrt{1 + R^2 - 2R\cos\theta}}, & R = \frac{\rho}{\rho_0} < 1, \\ \frac{1}{\rho} \frac{1}{\sqrt{1 + R^2 - 2R\cos\theta}}, & R = \frac{\rho_0}{\rho} < 1. \end{cases} \tag{171}$$

The expansion of the function

$$\psi(R, x) = \frac{1}{\sqrt{1 + R^2 - 2Rx}} \tag{172}$$

in a series of powers R yields

$$(1 + R^2 - 2Rx)^{-\frac{1}{2}} = 1 + \frac{\left(-\frac{1}{2}\right)}{1!}(R^2 - 2Rx) +$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(R^2 - 2Rx)^2 + \ldots, \tag{173}$$

that is,

$$\psi(R, x) = \sum_{n=0}^{\infty} P_n(x) R^n, \tag{173′}$$

where the $P_n(x)$ are the nth-degree polynomials in x. For example,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x). \tag{173''}$$

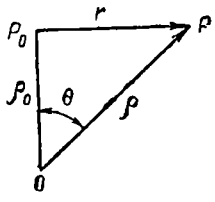

Fig. 40.

These polynomials $P_n(x)$ are called *Legendre polynomials* and the function $\psi(R, x)$ is called the *generating function* of the Legendre polynomials.

It follows immediately from the definition given for Legendre polynomials that they possess quite a number of remarkable properties. It is clear directly from the expansion (173) that for even powers of R we have even powers of x and for odd powers of R we have odd powers of x. This means that $P_n(x)$ is an even function when n is even and an odd function when n is odd. Thus,

$$P_n(-x) = (-1)^n P_n(x). \tag{174}$$

At $x = 1$, the expansion (173′) takes the form

$$\frac{1}{1-R} = 1 + R + R^2 + \ldots = \sum_{n=0}^{\infty} P_n(1) R^n.$$

It follows from this that

$$P_n(1) = 1. \tag{175}$$

If we differentiate $\psi(R, x)$ with respect to R, we obtain

$$(1 + R^2 - 2Rx)\frac{\partial \psi}{\partial R} = (x - R)\psi.$$

If we substitute into the left-hand member of this equation the series obtained by differentiating Eq. (173′), i.e.,

$$\frac{\partial \psi}{\partial R} = P_1(x) + 2P_2(x) R + \ldots + nP_n(x) R^{n-1} + \ldots,$$

and into the right-hand member equation (173′) itself and then equate like powers of R, we obtain a *recursion formula for Legendre polynomials:*

$$(n+1) P_{n+1} - x(2n+1) P_n + nP_{n-1} = 0. \tag{176}$$

Let us evaluate the coefficient a_n of the highest power of x in the polynomial $P_n(x)$. From formula (176), it is clear that

$$a_{n+1} = \frac{2n+1}{n+1} a_n. \tag{176'}$$

Setting $a_0 = 1$ in this equation, we obtain successively $a_2 = 3/2$, $a_3 = 5/2$, and, in the general case,

$$a_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!}. \tag{176''}$$

In spherical coordinates ρ, θ, φ (where $z = \rho \cos\theta$, $x = \rho \sin\theta \cos\varphi$, and $y = \rho \sin\theta \sin\varphi$), the three-dimensional Laplace equation becomes

$$\frac{1}{\rho} \frac{\partial^2}{\partial \rho^2} (\rho u) + \frac{1}{\rho^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\rho^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2} = 0.$$

If we substitute into this equation the unit source function

$$\delta(P, P_0) = \frac{1}{4\pi} \begin{cases} \frac{1}{\rho_0} \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_0} \right)^n P_n(\cos\theta) \text{ for } \rho < \rho_0, \\ \frac{1}{\rho} \sum_{n=0}^{\infty} \left(\frac{\rho_0}{\rho} \right)^n P_n(\cos\theta) \text{ for } \rho_0 < \rho, \end{cases} \tag{171'}$$

let us say, for $\rho < \rho_0$, and equate the coefficients of like powers of ρ, we obtain

$$n(n+1) P_n(\cos\theta) + \frac{1}{\sin\theta} \frac{d}{d\theta} \left[\sin\theta \frac{dP_n(\cos\theta)}{d\theta} \right] = 0.$$

If we set $\cos\theta = x$ and $\sin\theta = \sqrt{1-x^2}$, remembering that

$$-\sin\theta \, d\theta = dx, \quad \sin\theta \frac{d}{d\theta} = \sin^2\theta \frac{d}{\sin\theta \, d\theta} = (1-x^2) \frac{d}{dx},$$

we obtain the differential equation for the *Legendre polynomials*

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx} \right] + n(n+1) P_n(x) = 0. \tag{177}$$

We conclude from this differential equation that the Legendre polynomials can be regarded as the eigenfunctions of the generalized Sturm–Liouville problem

$$\frac{d}{dx} \left[(1-x^2) \frac{du}{dx} \right] + \lambda u = 0, \tag{178}$$

$$u\big|_{x=-1} \neq \infty, \quad u\big|_{x=1} \neq \infty,$$

and that the numbers $\lambda_n = n(n+1)$ can be regarded as the eigenvalues of that problem. From the general properties of the eigenfunctions of the generalized Sturm–Liouville problem, we concluded that the *Legendre polynomials of different powers are orthogonal:*

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad (m \neq n).$$

Let us evaluate the square of the norm of the polynomial $P_n(x)$. We have

$$N_n^{(0)} = \int_{-1}^{1} P_n^2(x)\, dx = \int_{-1}^{1} P_n(x) \left[\frac{a_n}{a_{n-1}} x P_{n-1}(x) + Q(x) \right] dx,$$

where $Q(x)$ is a polynomial of degree lower than n and, consequently, orthogonal to $P_n(x)$ since it is a linear combination of $P_0(x)$, $P_1(x)$, ..., $P_{n-1}(x)$. On the basis of the recursion formula (176), we have

$$xP_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x)$$

and hence

$$N_n^0 = \frac{a_n}{a_{n-1}} \int_{-1}^{1} xP_nP_{n-1}\, dx = \frac{a_n}{a_{n-1}} \frac{n}{2n+1} \int_{-1}^{1} P_{n-1}^2\, dx.$$

Remembering that $a_n / a_{n-1} = 2n-1/n$, we have

$$N_n^{(0)} = \frac{2n-1}{2n+1} N_{n-1}^0.$$

Since $P_0(x) = 1$ and $N_0^{(0)} = 2$, we find $N_1^{(0)} = 2N_2^{(0)}/3 = 2/5$, and, in general,

$$N_n^{(0)} = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}. \qquad (178')$$

Obviously, an arbitrary polynomial $\sigma_N(x) = \sum_{n=0}^{N} c_n x^n$ can be represented as a linear combination of the Legendre polynomials $P_0(x)$, $P_1(x)$, ..., $P_N(x)$. But, by a theorem of Weierstrass, any function which is continuous on the interval $[-1, 1]$ can be uniformly approximated by means of polynomials $\sigma_N(x)$. Consequently,

this uniform approximation can be made by means of linear combinations of Legendre polynomials. But the possibility of uniform approximation implies the possibility of approximation to an arbitrary degree of accuracy in the sense of mean square. This last means that the Legendre polynomials constitute a complete set of functions on the interval $[-1, 1]$. It follows from this property of the Legendre polynomials that the generalized Sturm–Liouville problem (178) has no eigenvalues other than $\lambda_n = n(n+1)$ (for $n = 0, 1, \ldots$). This is true because if there were another eigenvalue, the eigenfunction $u(x)$ corresponding to it would have to be orthogonal to all the Legendre polynomials and, since the Legendre polynomials constitute a complete system, this would mean that

$$\int_{-1}^{1} u^2(x)\,dx = 0,$$

i.e., $u \equiv 0$. In other words, we can say that *the Legendre polynomials completely exhaust the system of eigenfunctions of the problem (178).*

Let us show that the Legendre polynomial $P_n(x)$ can be represented in the form

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n]. \tag{179}$$

This differential formula is often called Rodrigues' formula. If we set $u = (x^2-1)^n$, we have

$$u' = 2nx(x^2-1)^{n-1}, \quad (x^2-1)u' - 2nxu = 0.$$

If we differentiate this equation $m+1$ times, we obtain

$$u^{(m+2)}(x^2-1) + (m+1)u^{(m+1)}\,2x + m(m+1)u^{(m)} - u^{(m+1)}\,2nx - \\ -(m+1)u^{(m)}\,2n = 0.$$

If we now set $m=n$, we have

$$(x^2-1)u^{(n+2)} + 2xu^{(n+1)} - n(n+1)u^{(n)} = 0$$

or

$$(1-x^2)u^{(n+2)} - 2xu^{(n+1)} + n(n+1)u^{(n)} = 0.$$

This means that the function

$$\widetilde{P}_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n]$$

satisfies the equation for the Legendre polynomials (177). The coefficient of the highest power x^n of the polynomial $\widetilde{P}_n(x)$ is equal to

$$\frac{2n(2n-1)\cdots[2n-(n+1)]}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!},$$

which is the same as the coefficient of the highest power of x in the Legendre polynomial $P_n(x)$. Consequently, $P_n(x) \equiv \widetilde{P}_n(x)$. This proves formula (179).

It follows from formula (179) that the Legendre polynomial $P_n(x)$ has n zeros on the interval $[-1, 1]$. This is true because the function $d[(x^2-1)^n]/dx$ has zeros at the end points of the interval $[-1, 1]$ and one zero in the interior of that interval. The function $d^2[(x^2-1)^n]/dx^2$ has zeros at the end points of the interval and two zeros in the interior of that interval. Continuing this process, we conclude that the function $d^n[(x^2-1)^n]/dx^n$ has no fewer than n zeros in the interval $[-1, 1]$. Since this function is an nth-degree polynomial and hence cannot have more than n zeros, this proves our assertion.

Let us show that the following integral representation is valid for the Legendre polynomial $P_n(x)$:

$$P_n(x) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x + i\sqrt{1-x^2}\sin\theta)^n d\theta \quad (-1 \leqslant x \leqslant 1). \tag{180}$$

Since $\sin\theta$ is an odd function, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a - ib\sin\theta} = a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a^2 + b^2\sin^2\theta}. \tag{180'}$$

Using the known value of the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a^2 + b^2\sin^2\theta} = \frac{\pi}{a\sqrt{a^2+b^2}}$$

and setting $a = 1 - Rx$ and $b = R\sqrt{1-x^2}$, we obtain

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{1 - (Rx + i\sqrt{1-x^2}\,R\sin\theta)} = \frac{1}{\sqrt{1+R^2-2Rx}}. \tag{180''}$$

If we expand both members of Eq. (180″) in powers of R, we obtain

$$\sum_{n=0}^{\infty} R^n \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x + i\sqrt{1-x^2}\sin\theta)^n \, d\theta = \sum_{n=0}^{\infty} P_n(x) R^n.$$

If we equate the coefficients of like powers of R, we obtain Eq. (180).

For $-1 \leqslant x \leqslant 1$, we have

$$|x + i\sqrt{1+x^2}\sin\theta| = \sqrt{x^2 + (1-x^2)\sin^2\theta} = \\ = \sqrt{x^2\cos^2\theta + \sin^2\theta} \leqslant 1$$

and, consequently,

$$|P_n(x)| < \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |x + i\sqrt{1-x^2}\sin\theta|^n \, d\theta \leqslant \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta,$$

that is, the Legendre polynomials are bounded on the interval $[-1, 1]$:

$$|P_n(x)| \leqslant 1. \tag{181}$$

2. Associated Legendre functions. *Associated Legendre functions* of order m are functions of the form

$$P_n^{(m)}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x), \tag{181′}$$

where m is a positive integer not exceeding n.

Let us show that the associated Legendre functions (181′) are eigenfunctions of the generalized Sturm–Liouville problem

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] - \frac{m^2}{1-x^2}u + \lambda u = 0, \tag{182}$$

$$u\Big|_{x=-1} \neq \infty, \quad u\Big|_{x=1} \neq \infty.$$

If we make the substitution

$$u = (1-x^2)^{\frac{m}{2}} Y(x) \tag{182′}$$

the differential equation in problem (182) takes the form

$$(1-x^2)Y''-2(m+1)xY'+[\lambda-m(m+1)]Y=0. \qquad (182'')$$

On the other hand, if we differentiate the equation of the Legendre polynomials

$$(1-x^2)P_n''-2xP_n'+n(n+1)P_n=0, \qquad (177)$$

m times, we obtain

$$(1-x^2)P_n^{(m+2)}-2(m+1)xP^{(m+1)}+$$
$$+[n(n+1)-m(m+1)]P_n^{(m)}=0. \qquad (177')$$

Comparing this with (182''), we see that Eq. (177') has nontrivial solutions for $\lambda=n(n+1)$.

$$Y_n=\frac{d^m}{dx^m}P_n(x).$$

Therefore, for $\lambda=n(n+1)$, the problem (182) has the eigenfunctions

$$P_n^{(m)}(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x).$$

Therefore, by virtue of the general properties of eigenfunctions of the generalized Sturm–Liouville problem, it follows that the associated functions of mth order constitute an orthogonal system of functions, i.e., that

$$\int_{-1}^{1}P_n^{(m)}(x)P_{n'}^{(m)}(x)\,dx=0 \qquad (n\neq n').$$

Let us evaluate the square of the norm of $P_n^{(m)}(x)$. Integrating by parts, we obtain

$$N_n^{(m)}=\int_{-1}^{1}[P_n^{(m)}(x)]^2\,dx=\int_{-1}^{1}(1-x^2)^m\left[\frac{d^m}{dx^m}P_n\right]^2dx=$$
$$=-\int_{-1}^{1}\frac{d^{m-1}P_n}{dx^{m-1}}\frac{d}{dx}\left[(1-x^2)^m\frac{d^mP_n}{dx^m}\right]dx.$$

Multiplying the differential equation (177') by $(1-x^2)^m$, we obtain

$$\frac{d}{dx}\left[(1-x^2)^{m+1}P_n^{(m+1)}\right] = -[n(n+1)-m(m+1)](1-x^2)^m P_n^{(m)}.$$

In view of this, we obtain

$$N_n^{(m)} = [n(n+1)-m(m+1)]\int_{-1}^{1}(1-x^2)^{m-1}\left[P_n^{(m-1)}\right]^2 dx =$$

$$= (n+m)(n-m+1)N_n^{(m-1)}.$$

Therefore, on the basis of (178′), we obtain

$$N_n^{(m)} = (n+m)(n+m-1)\ldots(n+1)(n-m+1)\times$$

$$\times(n-m+2)\ldots nN_n^{(0)} = \frac{(n+m)!}{n!}\,\frac{n!}{(n-m)!}N_n^{(0)} = \frac{(n+m)!}{(n-m)!}\,\frac{2}{2n+1}.$$

Thus, we may write

$$N_n^{(m)} = \int_{-1}^{1} P_n^{(m)}(x)P_n^{(m)}(x)\,dx = \begin{cases} \dfrac{2}{2n+1}, & m=0, \\ \dfrac{2}{2n+1}\cdot\dfrac{(n+m)!}{(n-m)!}, & m\neq 0. \end{cases} \tag{183}$$

Let us show that the mth-order associated functions $P_n^{(m)}(x)$ constitute a complete system of function on the interval $[-1, 1]$. For every function $f(x)$ that is piecewise continuous and square integrable on the interval $[-1, 1]$ and for every positive ε, there exists a continuous function $\varphi(x)$ that is identically zero on sufficiently small intervals $[-1, -1+\eta]$ and $[1-\eta, 1]$, where $\eta>0$, and that satisfies the inequality

$$\int_{-1}^{1}[f(x)-\varphi(x)]^2\,dx < \varepsilon. \tag{184}$$

Since an arbitrary polynomial in x can be represented as a linear combination of the polynomials $d^m P_n(x)/dx^m$, it follows on the basis of Weierstrass' theorem that, for given $\varepsilon>0$, the inequality

$$\left|\frac{\varphi(x)}{(1-x^2)^{\frac{m}{2}}} - \sum_{n=m}^{N} c_n \frac{d^m}{dx^m}P_n(x)\right| < \varepsilon, \tag{184′}$$

where N is a sufficiently large number and the c_n are constants, is valid on the interval $[-1, 1]$. If we multiply both sides of this inequality by $\left|(1-x^2)^{\frac{m}{2}}\right|$, we obtain

$$\left|\varphi(x) - \sum c_n P_n^{(m)}(x)\right| < \varepsilon.$$

It follows from this that

$$\int_{-1}^{1} \left[\varphi(x) - \sum_{n=m}^{N} c_n P_n^{(m)}(x)\right]^2 dx < \varepsilon', \qquad (184'')$$

where ε' is an arbitrarily small positive number. From (184) and (184′), we obtain

$$\int_{-1}^{1} \left[f(x) - \sum_{n=m}^{N} c_n P_n^{(m)}(x)\right]^2 dx \leqslant 2\int_{-1}^{1} (f-\varphi)^2 dx + 2\int_{-1}^{1} \left(\varphi - \sum_{n=m}^{N} c_n P_n^{(m)}\right)^2 dx < 2\varepsilon + 2\varepsilon'.$$

This inequality means that the mth-order associated functions $P_n^{(m)}(x)$ constitute a complete system of functions on the interval $[-1, 1]$.

It follows from the completeness of the functions $P_n^{(m)}(x)$ that they completely exhaust all the eigenfunctions of the problem (182). This is true because, if the problem (182) had an eigenfunction $u(x)$ for some eigenvalue $\lambda \neq n(n+1)$ (for $n = 0, 1, \ldots$) it would be orthonormal to all the associated functions $P_n^{(m)}(x)$ and, consequently, would be identically equal to zero.

3. Spherical harmonics. The $2n+1$ functions defined on the unit sphere $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$ by

$$Y^{(0)} = P_n(\cos\theta), \quad (n=0,1,\ldots), \qquad (185)$$

$$Y_n^{(m)} = P_n^{(m)}(\cos\theta)\cos m\varphi,\ Y_n^{(-m)} = P_n^{(m)}(\cos\theta)\sin n\varphi \quad (m = 1, 2, \ldots, n) \qquad (185')$$

are called spherical harmonics (or surface harmonics) of nth order. The function (185) of any order n is also called a *zonal harmonic* since the nodal lines of this function partition the sphere into "horizontal" zones in each of which the sign of the function remains constant. The nodal lines of each of the functions (185′) partition the sphere into curvilinear quadrilaterals bounded by meridians and parallels. These functions are for this reason called *tesseral harmonics*.

Let us show that the spherical functions (185) and (185′) constitute a complete system of functions that are orthogonal with weight $\sin\theta$ on the unit sphere or, what amounts to the same thing, in the closed rectangle $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$. The system of functions 1, $\cos\varphi$, $\sin\varphi$, ..., $\cos m\varphi$, $\sin m\varphi$, ... is a complete orthogonal system of functions on the interval $0 \leqslant \varphi \leqslant 2\pi$. The square of the norm on the first of these functions is equal to 2π and the square of the norm of each of the others is equal to π.

Furthermore, for every fixed m, the functions $P_n^{(m)}(\cos\theta)$ (for $n = 0, 1, \ldots$) constitute a complete orthogonal system with weight $\sin\theta$ on the interval $0 \leqslant \theta \leqslant \pi$. Here, in accordance with (183), the squares of the norms of these functions are given by the equations

$$\int_0^\pi P_n^{(m)}(\cos\theta)\, P_n^{(m)}(\cos\theta)\sin\theta\, d\theta = \begin{cases} \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}, & n \neq 0, \\ 2, & n = 0. \end{cases}$$

Therefore, on the basis of the lemma on the completeness of a system of functions of two variables (see Sect. 2, subsection 4), the functions

$$Y_n^{(0)} = P_n(\cos\theta) \quad (n = 0, 1, 2, \ldots),$$

$$Y_n^{(m)} = P_n^{(m)}(\cos\theta)\cos m\varphi,\ Y_n^{(-m)} = P_n^{(m)}(\cos\theta)\sin m\varphi \ (m = 1, 2, \ldots, n)$$

constitute a complete orthogonal system of functions with weight $\sin\theta$ in the closed rectangle $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$. The squares of the norms of these functions are defined as the products of the squares of the norms of their coefficients; that is,

$$M_n^{(m)} = \int_0^{2\pi}\int_0^{\pi} [Y_n^{(\pm m)}(\theta, \varphi)]^2 \sin\theta\, d\theta\, d\varphi = \begin{cases} \dfrac{4\pi}{2n+1}, & m = 0, \\ \dfrac{2\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!}, & m \neq 0. \end{cases} \tag{186}$$

Obviously, the system

$$\widetilde{Y}_n^{(0)} = \frac{1}{\sqrt{M_n^{(0)}}} P_n(\cos\theta) \quad (n = 0, 1, 2, \ldots), \tag{187}$$

$$\widetilde{Y}_n^{(m)} = \frac{1}{\sqrt{M_n^{(m)}}} P_n^{(m)}(\cos\theta)\cos m\varphi,\ \widetilde{Y}_n^{(-m)} = \frac{1}{\sqrt{M_n^{(m)}}} P_n^{(m)}(\cos\theta)\sin m\varphi$$

$$(m = 1, 2, \ldots, n)$$

is a complete orthonormal set of functions with weight $\sin\theta$ on the unit sphere. This means that every function $f(\theta, \varphi)$ which is continuous on the unit sphere can be approximated in the mean-square sense with weight $\sin\theta$ by means of the Fourier series

$$\sum_{n=0}^{\infty}\left\{a_n^{(0)}\widetilde{Y}_n^{(0)}(\theta) + \sum_{m=-n}^{n} a_n^{(m)}\widetilde{Y}_n^{(m)}(\theta, \varphi)\right\}, \tag{188}$$

where the $a_n^{(m)}$ are the Fourier coefficients of the function $f(\theta, \varphi)$:

$$a_n^{(m)} = \int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)\, \widetilde{Y}_n^{(m)}(\theta, \varphi)\sin\theta\, d\theta\, d\varphi. \tag{188'}$$

It can be shown that if the function $f(\theta, \varphi)$ is twice continuously differentiable on the unit sphere, its Fourier series (188) converges absolutely and uniformly. We shall not give a proof of this assertion but refer the reader to texts on special functions.*

4. Examples of problems that lead to spherical harmonics. The simplest examples of problems leading to the consideration of spherical harmonics are probably the boundary-value problems associated with the three-dimensional Laplace's equation in spherical coordinates

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0, \quad (189)$$

and also those for the three-dimensional equation $\Delta u + k^2 u = 0$.

Using the method of separation of variables, if we set

$$u(\rho, \theta, \varphi) = R(\rho) Y(\theta, \varphi) \quad (189')$$

in Eq. (189), we obtain the condition of separability

$$\frac{\Delta_{\theta\varphi} Y}{Y} = -\frac{\rho^2 R'' + 2\rho R'}{R} = -\lambda = \text{const}, \quad (190)$$

$$\Delta_{\theta\varphi} Y = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}. \quad (190')$$

To determine the function $Y(\theta, \varphi)$ in the rectangle $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$, we have the following two-dimensional Sturm–Liouville problem:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2} + \lambda Y = 0, \quad (191)$$

$$Y(\theta, \varphi + 2\pi) = Y(\theta, \varphi), \quad (191')$$

$$Y(0, \varphi) \neq \infty, \; Y(\pi, \varphi) \neq \infty. \quad (191'')$$

To solve this problem, we again apply the method of separation of variables. If we set

$$Y(\theta, \varphi) = P(\theta)\Phi(\varphi), \quad (192)$$

we obtain the condition of separability

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*See, for example, E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, 1931, and New York, Chelsea, 1955.

$$\frac{\sin^2\theta}{P}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\lambda\right\}=-\frac{\Phi''}{\Phi}=\mu=\text{const}. \tag{193}$$

From this, we get the Sturm–Liouville problem

$$\Phi''+\mu\Phi=0,\quad \Phi(\varphi+2\pi)=\Phi(\varphi). \tag{194}$$

for determining $\Phi(\varphi)$. The eigenvalues of this problem are the numbers m^2 (for $m = 0, 1, \ldots$) and the eigenfunctions are of the form

$$1, \cos\varphi, \sin\varphi, \ldots, \cos m\varphi, \sin m\varphi, \ldots . \tag{194'}$$

From (193), we obtain, for every value of m, the generalized Sturm–Liouville problem

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}P+\lambda P=0, \tag{195}$$

$$P|_{\theta=0}\neq\infty,\quad P|_{\theta=\pi}\neq\infty \tag{195'}$$

for determining the function $P(\theta)$. If we set $\cos\theta=x$ and remember that

$$-\sin\theta\,d\theta=dx,\quad \sin\theta\frac{d}{d\theta}=\sin^2\theta\frac{d}{\sin\theta\,d\theta}=(1-x^2)\frac{d}{dx},$$

then the problem (195)–(195′) takes the form

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]-\frac{m^2}{1-x^2}P+\lambda P=0; \tag{196}$$

$$P|_{x=-1}\neq\infty,\quad P|_{x=1}\neq\infty.$$

But this problem coincides with the Sturm–Liouville problem (182). Therefore, the eigenvalues of this problem are the numbers $n(n+1)$ (for $n = 0, 1, \ldots$), and the eigenfunctions for $m = 0$ are the Legendre polynomials $P_n(x)$. For $m\neq0$, they are the associated functions $P_n^{(m)}(x)$. Thus, keeping in mind the eigenfunctions (194′) of the problem (194), we conclude that the eigenfunctions of the two-dimensional Sturm–Liouville problem (191)–(191′)–(191″) are the spherical harmonics

$$Y_n^{(0)}=P_n(\cos\theta),\quad (n=0,1,\ldots) \tag{185}$$

$$Y_n^{(m)}=P_n^{(m)}(\cos\theta)\cos m\varphi,\quad Y_n^{(-m)}=P_n^{(m)}(\cos\theta)\sin m\varphi\quad (m=1,2,\ldots,n), \tag{185'}$$

and that the eigenvalues are the numbers $\lambda_n=n(n+1)$. Here, each of these eigenvalues is of rank $2n+1$ since the number of linearly

independent eigenfunctions corresponding to each eigenvalue λ_n is $2n+1$. Since the spherical harmonics (185)–(185′) constitute a complete system of orthogonal functions with weight $\sin\theta$ in the rectangle $0\leqslant\theta\leqslant\pi$, $0\leqslant\varphi\leqslant 2\pi$, they completely exhaust all the eigenfunctions of the two-dimensional Sturm–Liouville problem (191)–(191′)–(191″).

Let us turn now to the condition of separability (190) for determining the functions $R(\rho)$. We obtain the equation

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0. \tag{197}$$

Linearly independent solutions of this equation are ρ^n and ρ^{-n}. Consequently, solutions of the three-dimensional Laplace's equation (189) are the functions

$$\rho^n Y_n^{(0)}(\theta, \varphi) \quad (n = 0, 1, \ldots), \tag{198}$$

$$\rho^n Y_n^{(m)}(\theta, \varphi),\ \rho^n Y_n^{(-m)}(\theta, \varphi) \quad (m = 1, 2, \ldots, n). \tag{198′}$$

Any homogeneous nth degree of polynomials in x, y, and z that satisfies the three-dimensional Laplace's equation is called a solid harmonic.

Obviously, if an nth order solid harmonic is represented in the form $\rho^n F(\theta, \varphi)$, then the function $F(\theta, \varphi)$ must be a solution of the two-dimensional Sturm–Liouville problem (191)–(191′)–(191″) for $\lambda = n(n+1)$. Therefore, the function $F(\theta, \varphi)$ must be a linear combination of the $2n+1$ nth-order spherical harmonics and the nth-order solid harmonic itself must be a linear combination of the $2n+1$ functions defined by Eqs. (198) and (198′). From this, it follows in particular that the number of linearly independent nth-order solid harmonics is* $2n+1$. For these linearly independent functions, we may choose the functions defined by Eqs. (198) and (198′). The functions defined by Eqs. (198) and (198′) are sometimes called zonal and tesseral nth-order solid harmonics, respectively. If we replace the factors ρ^n in the functions (198) and (198′) with the factors ρ^{-n-1} (for $n\geqslant 0$), we obtain functions that are harmonic throughout all space except at the coordinate origin. These functions are called *exterior solid harmonics*.

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*This can be seen immediately since, by setting $u = \sum a_{rst} x^r y^s z^t$ (where $r + s + t = n$), we have

$$a_{rst} = \frac{\partial^n u}{\partial x^r\, \partial y^s\, \partial z^t},$$

If we use the equation $u_{xx} = -u_{yy} - u_{zz}$ to express the derivatives involving more than one differentiation with respect to x in terms of derivatives involving only one differentiation with respect to x, we conclude that all the coefficients a_{rst} are linear combinations of $2n + 1$ of these.

As a simple example of the application of spherical and solid harmonics to the solution of specific problems, we shall now give the solution of the Dirichlet problem for a sphere in the form of a series of solid harmonics.

Let us assume for simplicity that the boundary function $\Phi(\theta, \varphi)$ of the Dirichlet problem for the unit sphere

$$\Delta u = 0, \quad u|_{\rho=1} = \Phi(\theta, \varphi) \tag{199}$$

can be expanded in a uniformly and absolutely convergent Fourier series of spherical harmonics

$$\Phi(\theta, \varphi) = \sum_{n=0}^{\infty} \left\{ c_n^0 P_n(\cos\theta) + \sum_{m=1}^{n} (c_n^{(m)} \cos m\varphi + c_n^{(-m)} \sin m\varphi) P_n^{(m)}(\cos\theta) \right\}. \tag{200}$$

Then, for $\rho < 1$, the series of solid harmonics

$$u(\rho, \theta, \varphi) = \sum_{n=0}^{\infty} \rho^n \left\{ c_n^0 P_n(\cos\theta) + \right.$$

$$\left. + \sum_{m=1}^{\infty} (c_n^{(m)} \cos m\varphi + c_n^{(-m)} \sin m\varphi) P_n^{(m)}(\cos\theta) \right\} \tag{201}$$

can be differentiated twice and hence is a harmonic function. Furthermore, it is easy to see that the series (201) converges uniformly to the values of the sum of the series (200) as $\rho \to 1$. Therefore, the series (201) is the desired solution of the Dirichlet problem for the unit sphere.

We note that the series (201) can be summed rather simply. For this it is convenient to set $\theta = 0$ and $\varphi = 0$ initially. Remembering that $P_n(1) = 1$ and that $P_n^{(m)}(1) = 0$ (for $m = 1, 2, \ldots, n$), we have

$$u(\rho, 0, 0) = \sum_{n=0}^{\infty} c_n^0 \rho^n.$$

Noting that

$$c_n^0 = \frac{2n+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \Phi(\theta', \varphi') P_n(\cos\theta') \sin\theta' \, d\theta' \, d\varphi',$$

we obtain

$$u(\rho, 0, 0) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left\{ \sum_{n=0}^{\infty} (2n+1) \rho^n P_n(\cos\theta') \right\} \Phi(\theta', \varphi') \sin\theta' \, d\theta' \, d\varphi'.$$

But, as we saw above (see Chapter III, Sect. 3, subsection 11),

$$\sum_{n=0}^{\infty}(2n+1)\rho^n P_n(\cos\theta') = \frac{1-\rho^2}{(1+\rho^2-2\rho\cos\theta')^{3/2}}$$

and, consequently,

$$u(\rho, 0, 0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Phi(\theta', \varphi')\frac{1-\rho^2}{(1+\rho^2-2\rho\cos\theta')^{3/2}}\sin\theta'\,d\theta'\,d\varphi'.$$

If we now assume that $\theta \neq 0$, $\varphi \neq 0$, which corresponds to a displacement of the north pole of the sphere, we can write the sum of the series (201) in the general case

$$u(\rho, \theta, \varphi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\Phi(\theta', \varphi')\frac{1-\rho^2}{(1+\rho^2-2\rho\cos\gamma)^{3/2}}\sin\theta'\,d\theta'\,d\varphi', \qquad (202)$$

where γ is the angle between the radius vectors of the points (ρ, θ, φ) and $(1, \theta', \varphi')$. Equation (202) is Poisson's integral, which we obtained above (see Chapter III, Sect. 2, subsection 4) by means of the Green's function.*

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*For more details on the applicability of the method of separation of variables in different systems of curvilinear coordinates, we recommend the following literature: A. Wangerin, "Uber die Reduction der Gleichung

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$$

auf gewöhnliche Differentialgleichungen," *Monatsbericht der Preussischen Acad. der Wissenschaften zu Berlin*, 21, II, 1878; V. S. Dyachenko and K. A. Breus, "Zastosuvannya tsyklidychnykh koordynat do rivnyannya Laplasa" (Application of cyclical coordinates to Laplace's equation), *Zhurnal instytutu matematyky*, Akad. Nauk, Ukr. SSR, No. 4, 1937; V. V. Stepanov, "Ob uravnenii Laplasa i nekotorykh trizhdy ortogonal'nykh sistemakh" (On Laplace's equation and certain three-way orthogonal systems), *Matematicheskiy sbornik*, 2, No. 3, 1942; I. M. Gel'fand and Z. Ya. Shapiro, "Predstavleniye gruppy vrashcheniy trekhmernogo prostranstva i ikh primeneniya" (Representation of the group of rotations of three-dimensional space and their applications), *Uspekhi matem. nauk*, 7, No. 1, 1952; M. H. Martin, "A Generalization of the Method of Separation of Variables," *J. Rational Mech. and Analysis*, II, No. 2, 1953, M. Bocher, *Über die Reihenentwicklungen der Potentialtheorie*, Leipzig, 1894.

CHAPTER 7

Methods of Integral Transformation

This chapter is devoted to methods of integral transformations. The Fourier integral method is applied to the solution of the Cauchy problem. The fundamentals of operational calculus are expounded and examples are given of its application to the solution of various problems. The methods of finite integral transformations are treated from a general point of view. Finite integral transformations are introduced with a given kernel and a given weight. Their inverse formulas are constructed and the corresponding boundary-value problems for partial differential equations are solved.

1. The Fourier Integral Method of Solving the Cauchy Problem

It is shown in mathematical analysis that if a function $f(x)$, defined in the interval $(-\infty, \infty)$, is piecewise continuous, of bounded variation in a neighborhood of a point x, and

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty, \tag{1}$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\alpha) \cos \lambda (\alpha - x)\, d\alpha. \tag{2}$$

In this formula, the left-hand member of this equation must be replaced at points of discontinuity of the function $f(x)$ with the arithmetic mean of the limiting values of the function from the left and from the right. Formula (2) is known as the Fourier integral formula.

Since $\sin \lambda (\alpha - x)$ is an odd function of λ, the Fourier integral (2) can be written in the form

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}f(\alpha)\,e^{i\lambda(\alpha-x)}\,d\alpha. \tag{3}$$

The function

$$f^{+}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\alpha)\,e^{i\alpha x}\,d\alpha \tag{4}$$

is called the *Fourier transform* of the function $f(x)$ and the function

$$f^{-}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\lambda)\,e^{-i\lambda x}\,d\lambda \tag{5}$$

is called the *inverse Fourier transform* of the function $f(x)$. In these notations, Fourier's formula (3) can be written in the form of the operational equation

$$[f^{+}(x)]^{-}=f(x). \tag{3'}$$

In particular, if the function $f(x)$ is an even function, its Fourier transform and inverse transform take the forms

$$f^{+}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(\alpha)\cos\alpha x\,d\alpha,\quad f^{-}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(\lambda)\cos\lambda x\,d\lambda. \tag{6}$$

If the function $f(x)$ is odd, then

$$f^{+}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(\alpha)\sin\alpha x\,d\alpha,\quad f^{-}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(\lambda)\sin\lambda x\,d\lambda. \tag{7}$$

Fourier's integral formula (3) is easily generalized to the case of functions of several variables. This generalization to the case of functions of two variables $f(x_1, x_2)$, for example, can be obtained as follows: Treating x_2 as a constant, we obtain from (3) under suitable hypotheses

$$f(x_1, x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda_1\int_{-\infty}^{\infty}f(\alpha_1, x_2)\cos\lambda_1(\alpha_1-x_1)\,d\alpha_1$$

and, analogously,

$$f(\alpha_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda_2 \int_{-\infty}^{\infty} f(\alpha_1, \alpha_2) \cos \lambda_2 (\alpha_2 - x_2)\, d\alpha_2.$$

If we substitute this last expression into the preceding equation, we obtain

$$f(x_1, x_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\lambda_1\, d\lambda_2 \iint_{-\infty}^{\infty} f(\alpha_1, \alpha_2) \cos \lambda_1 (\alpha_1 - x_1) \times$$

$$\times \cos \lambda_2 (\alpha_2 - x_2)\, d\alpha_1\, d\alpha_2 \tag{8}$$

or, in complex form,

$$f(x_1, x_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\lambda_1\, d\lambda_2 \iint_{-\infty}^{\infty} f(\alpha_1, \alpha_2)\, e^{i[\lambda_1(\alpha_1 - x_1) + \lambda_2(\alpha_2 - x_2)]} d\alpha_1\, d\alpha_2. \tag{9}$$

In the case of a function depending on n variables, the Fourier integral formula takes the form

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int d\lambda_1 \ldots d\lambda_n \int_{-\infty}^{\infty} \cdots \int f(\alpha_1, \ldots, \alpha_n) \times$$

$$\times e^{i \sum_{k=1}^{n} \lambda_k (\alpha_k - x_k)} d\alpha_1 \ldots d\alpha_n \tag{10}$$

or, in more compact form,

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int d\lambda_1 \ldots d\lambda_n \int_{-\infty}^{\infty} \cdots \int f(\alpha_1, \ldots, \alpha_n) \times$$

$$\times e^{i\vec{\lambda}(\vec{\alpha} - \vec{x})} d\alpha_1 \ldots d\alpha_n, \tag{10'}$$

where $\vec{x}$, $\vec{\alpha}$, and $\vec{\lambda}$ are vectors $\vec{x} = (x_1, \ldots, x_n)$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$.

If we carry out the Fourier transformation

$$f^+(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int f(\alpha_1, \ldots, \alpha_n)\, e^{i\vec{\alpha}\vec{x}}\, d\alpha_1 \ldots d\alpha_n \tag{11}$$

and the inverse Fourier transformation

$$f^-(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int f(\lambda_1, \ldots, \lambda_n)\, e^{-i\vec{\lambda}\vec{x}}\, d\lambda_1 \ldots d\lambda_n, \tag{12}$$

on the function $f(x_1, x_2, \ldots, x_n)$, then Fourier's formula (10′) can be written in the form of the operational equation

$$[f^+(x_1, \ldots, x_n)]^- = f(x_1, \ldots, x_n). \tag{10″}$$

Fourier's integral formula makes it possible to demonstrate a general method of solving the Cauchy problem for the wave equation and, in general, for partial differential equations with constant coefficients. In order to avoid questions of the legitimacy of certain transformations in the course of our calculations, for example, the reversal of the order of integration, etc., it will be convenient to use this method for heuristic explorations, and then always to check whether the result obtained in such a manner is actually a solution of the problem.

Consider a linear homogeneous mth-order equation with constant coefficients

$$L(u) = 0, \tag{13}$$

where u is an unknown function of the independent variables $x_1, \ldots, x_n, t$.

Let us assume that Eq. (13) has a solution in the form of plane waves

$$e^{i(\alpha_1 x_1 + \ldots + \alpha_n x_n - \beta t)} = e^{i\vec{\alpha}\vec{x}} e^{-i\beta t} \tag{14}$$

or, more precisely, let us make the following assumption: For every system of real numbers $\alpha_1, \ldots, \alpha_n$ (for every vector $\vec{\alpha}$), there exist m distinct values

$$\beta = \beta_k(\alpha_1, \ldots, \alpha_n), \quad k = 1, 2, \ldots, m, \tag{15}$$

which depend algebraically on $\alpha_1, \alpha_2, \ldots, \alpha_n$ and for which the expression (14) is a solution of Eq. (13).

If we denote by $w_1, w_2, \ldots, w_m$ arbitrary functions of $\lambda_1, \ldots, \lambda_n$, we can, by making a superposition of plane waves, construct the expression

$$u = \sum_{k=1}^{m} \int_{-\infty}^{\infty} \cdots \int w_k(\lambda_1, \ldots, \lambda_n) e^{i\vec{\lambda}\vec{x}} e^{-i\beta_k t} d\lambda_1 \ldots d\lambda_n. \tag{16}$$

This expression is also a solution of Eq. (13) if all the integration processes converge and if the operation L can be carried out under the integral sign.

Let us use the expression (16) to construct a solution of Eq. (13) satisfying the given initial conditions:

$$\left.\begin{aligned} u|_{t=0} &= \varphi_0(x_1, \ldots, x_n), \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= \varphi_1(x_1, \ldots, x_n), \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \left.\frac{\partial^{m-1} u}{\partial t^{m-1}}\right|_{t=0} &= \varphi_{m-1}(x_1, \ldots, x_n), \end{aligned}\right\}, \tag{13'}$$

where the $\varphi_k(x_1, \ldots, x_n)$ are functions defined at all points in the x-space. If we differentiate the expression (16) with respect to t under the integral sign and substitute the result into the initial conditions (13'), we obtain the the following system of integral equations for determining the functions $w_k(\lambda_1, \ldots \lambda_n)$:

$$\left.\begin{aligned} \varphi_0(x_1, \ldots, x_n) &= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int \sum_{k=1}^{m} w_k(\lambda_1, \ldots, \lambda_n)\, e^{i\vec{\lambda}\vec{x}}\, d\lambda_1 \ldots d\lambda_n, \\ \varphi_1(x_1, \ldots, x_n) &= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int \sum_{k=1}^{m} (-i\beta_k)\, w_k(\lambda_1, \ldots, \lambda_n) \times \\ &\quad \times e^{i\vec{\lambda}\vec{x}}\, d\lambda_1 \ldots d\lambda_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_{m-1}(x_1, \ldots, x_n) &= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int \sum_{k=1}^{m} (-i\beta_k)^{m-1} w_k(\lambda_1, \ldots, \lambda_n) \times \\ &\quad \times e^{i\vec{\lambda}\vec{x}} d\lambda_1 \ldots d\lambda_n. \end{aligned}\right\}, \tag{17}$$

Therefore, on the basis of Fourier's formula, we have

$$\sum_{k=1}^{m} (-i\beta_k)^{\nu} w_k(\lambda_1, \ldots, \lambda_n) = \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \cdots \int \varphi_{\nu}(\alpha_1, \ldots, \alpha_n) \times$$

$$\times e^{-i\vec{\lambda}\vec{\alpha}}\, d\alpha_1 \ldots d\alpha_n \qquad (\nu = 0, 1, \ldots, n-1), \tag{18}$$

The right-hand members of these equations contain the unknown expressions. Thus, to determine the m unknown functions $w_k(\lambda_1, \ldots, \lambda_n)$, we have obtained a system of linear algebraic equations, the determinant of which $|(-i\beta_k)^{\nu}|$ cannot vanish since all the β_k are assumed to have different values. Consequently, the functions $w_k(\lambda_1, \ldots, \lambda_n)$ are uniquely determined, and substitution of these functions into (16) yields an explicit expression for the desired solution of the Cauchy problem.

As an example of the application of this general method, let us consider the Cauchy problem for the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{19}$$

$$u|_{t=0} = \psi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi_1(x). \tag{19'}$$

Equation (19) has a solution in the form of plane waves

$$e^{i(\alpha_1 x - \alpha_1 a t)}, \quad e^{i(\alpha_1 x + \alpha_1 a t)}. \tag{20}$$

If, in accordance with (16), we take the desired solution of the Cauchy problem in the form

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_1(\lambda_1)\, e^{i\lambda_1 x} e^{-i\lambda_1 a t}\, d\lambda_1 + \\ + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_2(\lambda_1)\, e^{i\lambda_1 x} e^{i\lambda_1 a t} d\lambda_1 \tag{21}$$

we have, by virtue of the initial conditions (19'),

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_1(\lambda_1)\, e^{i\lambda_1 x}\, d\lambda_1 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w_2(\lambda_1)\, e^{i\lambda_1 x}\, d\lambda_1,$$

$$\psi_1(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-i\lambda_1 a)\, w_1(\lambda_1)\, e^{i\lambda_1 x}\, d\lambda_1 + \\ + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (i\lambda_1 a)\, w_2(\lambda_1)\, e^{i\lambda_1 x}\, d\lambda_1. \tag{22}$$

From this we obtain

$$w_1(\lambda_1) + w_2(\lambda_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(\alpha_1)\, e^{-i\lambda_1 \alpha_1}\, d\alpha_1 = \psi^-(\lambda_1),$$

$$(-i\lambda_1 a)\, w_1(\lambda_1) + (i\lambda_1 a)\, w_2(\lambda_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi_1(\alpha_1)\, e^{-i\lambda_1 \alpha_1}\, d\alpha_1 = \psi_1^-(\lambda_1).$$

If we solve this system of equations, we obtain

$$w_1(\lambda_1) = \frac{\psi^-(\lambda_1)}{2} - \frac{1}{2i\lambda_1 a}\psi_1^-(\lambda_1),$$

$$w_2(\lambda_1) = \frac{\psi^-(\lambda_1)}{2} + \frac{1}{2i\lambda_1 a}\psi_1^-(\lambda_1).$$

If we substitute this into the expression (21), we obtain the desired solution of the Cauchy problem in the form

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\psi^-(\lambda_1)}{2}e^{i\lambda_1(x-at)}\,d\lambda_1 +$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\psi^-(\lambda_1)}{2}e^{i\lambda_1(x+at)}\,d\lambda_1 -$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\psi_1^-(\lambda_1)}{2i\lambda_1 a}e^{i\lambda_1(x-at)}\,d\lambda_1 +$$

$$+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\psi_1^-(\lambda_1)}{2i\lambda_1 a}e^{i\lambda_1(x+at)}\,d\lambda_1. \qquad (23)$$

Fourier's formula shows that the first two integrals on the right coincide with the expression

$$\frac{\psi(x-at)+\psi(x+at)}{2}.$$

The last two integrals can be written in the form

$$\int_{x-at}^{x+at}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi_1^-(\lambda_1)\,\frac{1}{2a}\,e^{i\lambda_1 y}\,d\lambda_1\right\}dy = \frac{1}{2a}\int_{x-at}^{x+at}\psi_1(y)\,dy.$$

Thus, the desired solution of the Cauchy problem can be written in the form

$$u(x, t) = \frac{\psi(x-at)+\psi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi_1(y)\,dy.$$

This is the well-known d'Alembert formula. It gives the desired solution of the Cauchy problem under the assumption that the function $\psi_1(x)$ is continuously differentiable and that the function $\psi(x)$ is twice continuously differentiable.

We conclude this section by pointing out that this method of using the Fourier transform with respect to the variable t, which varies from $-\infty$ to $+\infty$, proves useful not only in solving the

the Cauchy problem but also in solving many other problems when the limit of variation of one or several independent variables are infinite.

It is easy to see that Fourier's formula (2) is at the basis of the method that we have just explained. However, it is possible to construct a large number of formulas analogous to Fourier's formula in which transformations with infinite limits of integration play a part. Corresponding to each of these formulas is a method analogous to the Fourier transform method, which is suitable for solving some class or other of boundary-value problems when the limits of variation of one or more independent variables are infinite. Let us give a few of these methods that are fairly frequently encountered and that are worked out in greatest detail.

1. The Fourier-Bessel integral method. If the integral $\int_0^\infty f(x)\,dx$ converges absolutely and if $f(x)$ is a function of bounded variation in a neighborhood of the point x, then the following so-called Fourier-Bessel formula holds for $\nu \leqslant -1/2$:

$$f(x) = \int_0^\infty J_\nu(x\lambda)\,\lambda\,d\lambda \int_0^\infty J_\nu(\lambda\xi)\,\xi f(\xi)\,d\xi, \tag{24}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν. At points of discontinuity of the function $f(x)$, we need to replace $f(x)$ with the function $[f(x+0)+f(x-0)]/2$ in the left-hand member of this equation.*

The expression

$$\tilde{f}(\lambda) = \int_0^\infty J_\nu(\lambda\xi)\,\xi f(\xi)\,d\xi \tag{25}$$

is called the *Hankel transform* of order ν of the function $f(x)$. This transformation plays the same role as does the Fourier transformation in solving the boundary-value problems. The inverse formula for the Hankel transformation (25) corresponding to (24) is written in the form

$$f(x) = \int_0^\infty J_\nu(x\lambda)\,\lambda\,\tilde{f}(\lambda)\,d\lambda. \tag{25'}$$

2. The Mellin transform method. If the integral $\int_0^\infty x^{k-1}|f(x)|\,dx$ is bounded for some constant $k>0$, where $f(x)$ is a function of bounded variation in a neighborhood of the point x, and if

*See, for example, I. Sneddon, *Fourier Transforms*, New York, McGraw Hill, 1951.

$$\tilde{f}(\lambda) = \int_0^\infty x^{\lambda-1} f(x)\, dx, \tag{26}$$

then

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} x^{-\lambda} \tilde{f}(\lambda)\, d\lambda, \tag{26'}$$

where γ is a constant greater than k. At points of discontinuity of the function $f(x)$, we need to replace $f(x)$ with $[f(x+0)+f(x-0)]/2$ in the left-hand member of this equation.*

The expression (26) is called the *Mellin transform* of the function $f(x)$, and it plays the same role as does the Fourier transform in the solution of boundary-value problems.

3. The Laplace transform method. The Laplace transform of a function $f(x)$ is defined by the equation

$$F(p) = \int_0^\infty e^{-px} f(x)\, dx,$$

where $p=\sigma+i\tau$ is a complex variable.

The method of employing the Laplace transformation will be examined in detail in the following section.

2. The Laplace Transformation. The Foundations of Operational Calculus

Suppose that we have a set M of functions $f(t)$ that are defined and piecewise continuous for $0 \leqslant t < \infty$ and that satisfy the condition $f(t) = O(e^{\sigma_0 t})$, where σ_0 is a positive number, as $t \to \infty$. Then the Laplace transform of the function $f(t)$

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \qquad (p = \sigma + i\tau) \tag{27}$$

is an analytic function of the complex variable $p=\sigma+i\tau$ for $\sigma>\sigma_0$. This follows from the fact that the real and imaginary parts of the Laplace integral have continuous derivatives with respect to σ and τ, which are obtained by differentiating (27) under the integral sign, and these derivatives satisfy the Cauchy-Riemann conditions.

The function (of a complex variable)

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*See, for example, E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, 1937.

$$\tilde{f}(p) = p\int_0^\infty e^{-pt} f(t)\,dt \tag{28}$$

is called the *image* of the function $f(t)$, and the function $f(t)$ is called the original function* with respect to the function $\tilde{f}(p)$.

Obviously, each of the functions $\tilde{f}(p)$ is an analytic function of $p=\sigma+i\tau$ in the half-plane $\operatorname{Re} p>\sigma_0$.

THEOREM (on the inverse of the Carson-Laplace integral). *If the initial function* $f(t)$ *is of bounded variation in every finite interval* $0 \leqslant t \leqslant T < \infty$, *then*

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{pt}}{p}\,\tilde{f}(p)\,dp, \tag{29}$$

where the integral is taken along an arbitrary vertical straight line $\operatorname{Re} p=\sigma=\text{const}>\sigma_0$.**

Proof: The function $\psi(t)=e^{-\sigma t} f(t)$ is of bounded variation in the finite interval $0\leqslant t\leqslant T<\infty$, and

$$\int_0^\infty |\psi(t)|\,dt < \infty.$$

Therefore, the function $\psi(t)$ can be represented in the form of a Fourier integral

$$\psi(t) = \frac{1}{\pi}\int_0^\infty d\tau \int_{-\infty}^\infty \psi(\xi)\cos\tau(t-\xi)\,d\xi.$$

Since $f(t)\equiv 0$, the last equation can be written in the form

$$\psi(t) = \frac{1}{\pi}\int_0^\infty d\tau \int_0^\infty \psi(\xi)\cos\tau(t-\xi)\,d\xi.$$

Since $\cos\tau(t-\xi)$ is an even function and $\sin\tau(t-\xi)$ is an odd function, we can write

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^\infty d\tau \int_0^\infty \psi(\xi)\cos\tau(t-\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^\infty d\tau \int_0^\infty \psi(\xi)\,e^{i\tau(t-\xi)}\,d\xi.$$

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*The expression in the right-hand member of Eq. (28) is sometimes called the Carson-Laplace integral.

**At points at which the function $f(t)$ has discontinuities of the first kind, we need to take the arithmetic mean of the limits of the function $f(t)$ from the left and from the right in the left-hand member of Eq. (29).

Multiplying both sides of this equation by $e^{\sigma t}$, we obtain

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\sigma+i\tau)t}\, d\tau \int_{0}^{\infty} f(\xi)\, e^{-(\sigma+i\tau)\xi}\, d\xi =$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{pt}}{p} \tilde{f}(p)\, dp.$$

This completes the proof of the theorem.

It follows from Eq. (28) and the theorem that we have just proved that a one-to-one correspondence exists between the functions $f(t)$ in the set M and the functions $\tilde{f}(p)$ (the set of which we shall denote by $\tilde{M}$). To indicate this correspondence between any particular original function $f(t)$ and its image $\tilde{f}(p)$, it is convenient to use the notation $f(t) \longleftrightarrow \tilde{f}(p)$.

Let us give some of the simpler examples of the correspondence between the original functions and their images.

Suppose that

$$f(t) = \begin{cases} 1 & \text{for } t \geqslant 0, \\ 0 & \text{for } t < 0. \end{cases} \tag{30}$$

Then, it follows immediately from (28) that $\tilde{f}(p) = 1$. The function defined by Eq. (30) is called the unit function and is denoted by $\eta(t)$. Thus, we can write

$$\eta(t) \longleftrightarrow 1. \tag{31}$$

Similarly, by direct calculation from Eq. (28), we obtain

$$e^{\alpha t} \longleftrightarrow \frac{p}{p-\alpha}, \qquad \alpha = \text{const}. \tag{32}$$

From the definition of the function $\Gamma(s)$ for $\operatorname{Re} s > 0$, namely,

$$\Gamma(s) = \int_{0}^{\infty} e^{-x} x^{s-1}\, dx,$$

we obtain, by setting $x = pt$,

$$\Gamma(s) = p^s \int_{0}^{\infty} e^{-pt} t^{s-1}\, dt.$$

It follows from this and Eq. (28) that

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)} \longleftrightarrow \frac{1}{p^{\alpha}}, \tag{33}$$

where α is an arbitrary constant satisfying the condition $\operatorname{Re}\alpha > -1$.

In particular, for α equal to a positive integer, we have, from (33)

$$\frac{t^{n}}{n!} \longleftrightarrow \frac{1}{p^{n}}. \tag{34}$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we have, for $\alpha = -1/2$,

$$\frac{1}{\sqrt{\pi t}} \longleftrightarrow \sqrt{p}. \tag{35}$$

Since $\Gamma(3/2) = \sqrt{\pi}/2$, we have, for $\alpha = 1/2$,

$$2\sqrt{\frac{t}{\pi}} \longleftrightarrow \frac{1}{\sqrt{p}}. \tag{36}$$

If we differentiate (33) with respect to the parameter α, we obtain

$$\frac{t^{\alpha}\ln t}{\Gamma(\alpha+1)} - t^{\alpha}\frac{\Gamma'(\alpha+1)}{\Gamma^{2}(\alpha+1)} \longleftrightarrow \frac{1}{p^{\alpha}}\ln p.$$

If we set $\alpha=0$ in this relation and remember that $\Gamma(1) = 1$, we obtain

$$\ln t \longleftrightarrow \ln\frac{1}{p} + \Gamma'(1). \tag{37}$$

If we set

$$f(t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{r^{2}}{4t}},$$

where r^2 is an arbitary constant, then Eq. (29) yields

$$\frac{\tilde{f}(p)}{p} = \int_{0}^{\infty} e^{-\left(pt+\frac{r^{2}}{4t}\right)} \frac{1}{2\sqrt{\pi t}}\,dt = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-\left(px^{2}+\frac{r^{2}}{4x^{2}}\right)}\,dx.$$

But this last integral is the familiar Laplace integral, which is equal to $\frac{1}{2\sqrt{p}} e^{-r\sqrt{p}}$.

Thus, we obtain

$$\frac{1}{2\sqrt{\pi t}} e^{-\frac{r^{2}}{4t}} \longleftrightarrow \frac{\sqrt{p}}{2} e^{-r\sqrt{p}}. \tag{38}$$

We summarize these examples in the form of the following table.

$f(t)$	$\eta(t)$	$e^{\alpha t}$	$\frac{t^{\alpha}}{\Gamma(\alpha+1)}$, $\operatorname{Re}\alpha > -1$	$\frac{t^n}{n!}$	$\frac{1}{\sqrt{\pi t}}$	$2\sqrt{\frac{t}{\pi}}$	$\ln t$	$\frac{1}{2\sqrt{\pi t}} e^{-\frac{r^2}{4t}}$
$\tilde{f}(p)$	1	$\frac{p}{p-\alpha}$	$\frac{1}{p^{\alpha}}$	$\frac{1}{p^n}$	$\sqrt{p}$	$\frac{1}{\sqrt{p}}$	$\ln\frac{1}{p} + \Gamma'(1)$	$\frac{\sqrt{p}}{2} e^{-r\sqrt{p}}$

The significant feature here is the fact that in shifting from the class M of original functions to the class $\tilde{M}$ of their images, certain mathematical operations are replaced by other mathematical operations. This replacement of certain mathematical operations by others is indicated by the basic rules of operational calculus.

The simplest basic rules of operation calculus can be written in the form of the following correspondences:

$$1.\quad \sum_{i=1}^{n} f_i(t) \longleftrightarrow \sum_{i=1}^{n} \tilde{f}_i(p), \tag{39}$$

i.e., the image of a sum is equal to the sum of the images.

$$2.\quad \int_0^t f(t)\,dt \longleftrightarrow \frac{1}{p}\tilde{f}(p), \tag{40}$$

i.e., the operation of integration of the original function corresponds to division of its image by p.

$$3.\quad f^{(n)}(t) \longleftrightarrow p^n\left[\tilde{f}(p) - f(0) - \frac{1}{p}f'(0) - \ldots - \frac{1}{p^{n-1}}f^{(n-1)}(0)\right]. \tag{41}$$

This is the so-called rule for differentiating the original functions.

$$4.\quad f(\alpha t) \longleftrightarrow \tilde{f}\left(\frac{p}{\alpha}\right),\ \alpha = \text{const} > 0. \tag{42}$$

This relation is called the similarity rule.

$$5.\quad e^{-p\tau}\tilde{f}(p) \longleftrightarrow \begin{cases} 0, & \text{if } t < \tau, \\ f(t-\tau), & \text{if } t \geqslant \tau, \end{cases} \qquad \tau = \text{const} > 0. \tag{43}$$

This is the so-called time lag theorem.

$$6.\quad e^{-\lambda t}f(t) \longleftrightarrow \frac{p}{p+\lambda}\tilde{f}(p+\lambda), \qquad \lambda = \text{const}. \tag{44}$$

This equation is called the displacement theorem.

$$7.\quad \frac{\tilde{f}(p)\tilde{f}_1(p)}{p} \longleftrightarrow \int_0^t f(\tau) f_1(t-\tau)\, d\tau. \tag{45}$$

This equation is called Borel's convolution theorem.

8. If $f_1(t, \tau) \longleftrightarrow u(p) e^{-\tau\zeta(p)}$, then

$$\int_0^\infty f_1(t,\ \tau) f(\tau)\, d\tau \longleftrightarrow \frac{u(p)}{\zeta(p)} \tilde{f}(\zeta(p)). \tag{46}$$

This is the so-called Efros-Danilevskiy transformation.

$$9.\quad t^n f(t) \longleftrightarrow (-1)^n p \frac{d^n}{dp^n}\left[\frac{\tilde{f}(p)}{p}\right]. \tag{47}$$

This equation is the rule for differentiating the images.

The validity of rules 1-6 and 9 follows immediately from Eq. (28) by means of integration by parts and change of variables. To prove rule 7, let us calculate the image of the so-called convolution of the functions $f(t)$ and $f_1(t)$:

$$F(t) = \int_0^t f(\tau) f_1(t-\tau)\, d\tau.$$

We have

$$\tilde{F}(p) = p \int_0^\infty e^{pt}\, dt \int_0^t f(\tau) f_1(t-\tau)\, d\tau = p \iint_D e^{pt} f(\tau) f_1(t-\tau)\, d\tau\, dt,$$

where D is the domain $0<t<\infty, 0<\tau<t$ (Fig. 41). Now, the integral over the domain D can be represented as an iterated integral first with respect to t, then with respect to τ. We obtain

$$\tilde{F}(p) = p \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty f_1(t-\tau) e^{-pt}\, dt.$$

If we now set $t-\tau=t'$, we obtain

$$\tilde{F}(p) = p \int_0^\infty f(\tau) e^{-p\tau}\, d\tau \int_0^\infty f_1(t') e^{-pt'}\, dt' = \frac{\tilde{f}(p)\tilde{f}_1(p)}{p}.$$

This proves rule 7.

The validity of rule 8 follows from the following obvious transformations:

$$p\int_0^\infty e^{-pt}\left[\int_0^\infty f_1(t,\tau)f(\tau)\,d\tau\right]dt = p\int_0^\infty f(\tau)\,d\tau\int_0^\infty e^{-pt}f_1(t,\tau)\,dt =$$

$$= \int_0^\infty f(\tau)\,u(p)\,e^{-\tau\zeta(p)}\,d\tau = \frac{u(p)}{\zeta(p)}\zeta(p)\int_0^\infty f(\tau)\,e^{-\tau\zeta(p)}\,d\tau = \frac{u(p)}{\zeta(p)}\tilde{f}(\zeta(p)).$$

These basic rules of operational calculus enable us in many cases to find the original function if we know its image and conversely, to find the image if we know the original function.

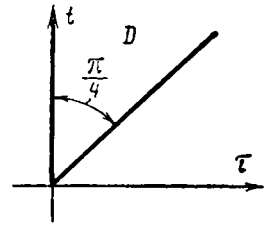

Fig. 41.

For example, let us find the image of the instantaneous unit source function for the heat-flow equation $u_{xx}-u_t=0$:

$$\delta(x, x', t-t') = \frac{1}{2\sqrt{\pi(t-t')}}e^{-\frac{r^2}{4(t-t')}} \qquad (r=|x-x'|)$$

(see Chapter IV, Sect. 4, subsection 2). From the table presented earlier, we have

$$\frac{1}{2\sqrt{\pi t}}e^{-\frac{r^2}{4t}} \longleftrightarrow \frac{\sqrt{p}}{2}e^{-r\sqrt{p}}.$$

Consequently, on the basis of the lag theorem, we have

$$\frac{1}{2\sqrt{\pi(t-t')}}e^{-\frac{r^2}{4(t-t')}} \longleftrightarrow \frac{\sqrt{p}}{2}e^{-r\sqrt{p}-t'p}. \tag{48}$$

However, to find the initial function from its image, it is generally necessary to use formula (29), i.e., the theorem on the inverse of the Carson-Laplace integral (28). Here, it proves very convenient to use residue theory. In connection with this, the following lemma is often useful.

LEMMA (Jordan). *If a function $F(z)$ is analytic to the left of the imaginary axis except at certain isolated singular points and if the condition*

$$\lim_{n\to\infty}\max_{z\subset C_n}|F(z)|=0,$$

holds on a sequence of infinitely expanding semicircles C_n: $|z|=R_n$, where $\operatorname{Re} z<0$, then for every positive number t,

$$\lim_{n\to\infty}\int_{C_n}e^{tz}F(z)\,dz = 0. \tag{49}$$

Proof: For any arbitrarily small number $\varepsilon>0$, we have, for sufficiently large n,

$$A=\left|\int_{C_n} e^{tz}F(z)\,dz\right|\leqslant \varepsilon\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR_n\cos\theta}R_n\,d\theta=\varepsilon\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R_n e^{-tR_n\cos\varphi}\,d\varphi=$$

$$=2\varepsilon\int_0^{\frac{\pi}{2}} R_n e^{-tR_n\cos\varphi}\,d\varphi.$$

If we set $\varphi=\pi-\psi$, we obtain

$$A\leqslant 2\varepsilon\int_0^{\frac{\pi}{2}} R_n e^{-tR_n\sin\psi}\,d\psi.$$

Making use of

$$\frac{\sin\psi}{\psi}\geqslant\frac{2}{\pi}\quad\text{for } 0\leqslant\psi\leqslant\frac{\pi}{2},$$

we obtain

$$A\leqslant 2\varepsilon\int_0^{\frac{\pi}{2}} e^{-tR_n\frac{2\psi}{\pi}}\,d\psi=\varepsilon\frac{\pi}{t}(1-e^{-tR_n}).$$

This completes the proof of the lemma.

Now, with the purpose of making clear the basic principle of the method of operational calculus, let us first consider a few very simple examples not connected with partial differential equations.

Suppose that we wish to solve the Cauchy problem for the ordinary differential equation

$$\frac{df}{dt}+f=f_1,\qquad f(0)=k. \tag{50}$$

Keeping the initial condition $f(0)=k$ in mind, we obtain from the rule for differentiating the original functions,

$$\frac{df}{dt}\longleftrightarrow p\tilde{f}(p)-kp.$$

Consequently, the Cauchy problem (58) in the class of image functions takes the form

$$p\tilde{f}(p)-kp+\tilde{f}(p)=\tilde{f}_1(p). \tag{51}$$

From this, we obtain

$$\tilde{f}(p) = \frac{\tilde{f}_1(p) + kp}{p+1} = k\frac{p}{p+1} + \frac{1}{p+1}\tilde{f}_1(p). \qquad (51')$$

Now, it only remains to find the original function $f(t)$ from the image $\tilde{f}(p)$. From the table, we have

$$k\frac{p}{p+1} \longleftrightarrow ke^{-t}.$$

From the convolution theorem, we obtain

$$\frac{1}{p+1}\tilde{f}_1(p) = \frac{1}{p}\frac{p}{p+1}\tilde{f}_1(p) \longleftrightarrow \int_0^t f_1(\tau)\, e^{-(t-\tau)}\, d\tau.$$

Thus, the solution of problem (50) may be written in the form

$$f(t) = ke^{-t} + \int_0^t f_1(\tau)\, e^{-(t-\tau)}\, d\tau. \qquad (50')$$

Let us now solve the integral equation

$$f(t) = f_1(t) + \lambda \int_0^t K(t-\tau) f(\tau)\, d\tau. \qquad (52)$$

According to the convolution theorem, this equation may be written in terms of the images in the form

$$\tilde{f}(p) = \tilde{f}_1(p) + \lambda \frac{\tilde{K}(p)\tilde{f}(p)}{p}. \qquad (53)$$

From this, we obtain the solution of the problem in the image space

$$\tilde{f}(p) = \frac{p\tilde{f}_1(p)}{p + \lambda\tilde{K}(p)}. \qquad (53')$$

Turning from the images to the original functions, we obtain the solution of the integral equation (52). For example, if $K(t-\tau) = 1$, then

$$\tilde{f}(p) = \frac{p}{p+\lambda}\tilde{f}_1(p)$$

and the solution may be written in the form

$$f(t) = \frac{d}{dt}\int_0^t f_1(\tau)\, e^{-\lambda(t-\tau)}\, d\tau.$$

In the case of Abel's integral equation,

$$f_1(t) = \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\, d\tau, \tag{54}$$

we obtain, by virtue of $\frac{1}{\sqrt{t}} \leftrightarrow \sqrt{\pi p}$,

$$\tilde{f}(p) = \frac{1}{\sqrt{\pi}} \frac{p\tilde{f}_1(p)\sqrt{p}}{p} = \frac{1}{\sqrt{\pi}} \frac{[p\tilde{f}_1(p) - pf_1(0)]}{p} + \frac{1}{\sqrt{\pi}} f_1(0)\sqrt{p}. \tag{55}$$

From this, remembering that

$$\sqrt{p} \leftrightarrow \frac{1}{\sqrt{\pi t}},$$

$$p\tilde{f}_1(p) - pf_1(0) \leftrightarrow f_1'(t),$$

we obtain

$$f(t) = \frac{f_1(0)}{\pi\sqrt{t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{f_1'(\tau)}{\sqrt{t-\tau}}\, d\tau. \tag{54'}$$

Let us now give an example illustrating the application of operational calculus to the solution of boundary-value problems for the heat-flow equation.

Suppose that we wish to solve the first boundary-value problem for the heat-flow equation for the one-dimensional half-space $x>0$

$$u_{xx} - u_t = 0,\ u|_{t=0} = 0,\ u|_{x=0} = \lambda = \text{const},\ u|_{x=\infty} = 0. \tag{56}$$

Using the initial conditions of this problem, we can write it in terms of the image functions in the following form:

$$\frac{d^2\tilde{u}(x, p)}{dx^2} - p\tilde{u}(x, p) = 0,\ \tilde{u}(x, p)|_{x=0} = \lambda,\ \tilde{u}(x, p)|_{x=\infty} = 0. \tag{57}$$

The general solution of this last ordinary differential equation may be written in the form

$$u(x, p) = Ae^{-\sqrt{p}x} + Be^{\sqrt{p}x} \quad (A, B = \text{const}). \tag{58}$$

The condition $\tilde{u}(x, 0)|_{x=\infty} = 0$ yields $B=0$ and, consequently, $A=\lambda$. Thus, the solution of the problem (57) in the class of image functions is

$$\tilde{u}(x, p) = \lambda e^{-x\sqrt{p}}. \tag{59}$$

Now, we only need to find the original function $u(x, t)$, representing the solution of problem (56) from the image (59). From the table, we have

$$\sqrt{p}e^{-x\sqrt{p}} \leftrightarrow \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}, \quad \frac{1}{\sqrt{p}} \leftrightarrow 2\sqrt{\frac{t}{\pi}}.$$

Therefore, from the convolution theorem, we have

$$e^{-x\sqrt{p}} = \sqrt{p}e^{-x\sqrt{p}} \frac{1}{\sqrt{p}} \leftrightarrow \frac{d}{dt} \int_0^t 2\sqrt{\frac{t-\tau}{\pi}} \frac{e^{-\frac{x^2}{4\tau}}}{\sqrt{\pi\tau}} d\tau =$$

$$= \frac{1}{\pi} \int_0^t e^{-\frac{x^2}{4\tau}} \frac{d\tau}{\sqrt{(t-\tau)\tau}}.$$

Thus, the desired solution of the problem (64) can be written in the form*

$$u(x, t) = \frac{\lambda}{\pi} \int_0^t e^{-\frac{x^2}{4\tau}} \frac{d\tau}{\sqrt{(t-\tau)\tau}}. \tag{60}$$

The method of operational calculus enables us to solve many boundary-value problems associated with parabolic equations and also hyperbolic equations. Here, the solution of partial differential equations is essentially reduced, as was indicated in the example chosen, to the solution of ordinary differential equations involving the image functions. The solution of the corresponding problems involving the images does not as a rule involve any sort of difficulty but considerable difficulties do arise in many cases in shifting back from the images to the original functions although the answer to this question is given formally by the inverse formula (29). However, the solution in terms of the images, if it can be expressed in terms of fairly simple functions, is of interest in its own right for the following reason. Even when we are not completely successful in finding the desired solution in terms of the original functions in a simple form from this image solution, we can frequently, and in a very simple manner, establish various properties of the solution of the boundary-value problem which are of considerable interest.

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* Other examples of the application of operational calculus to the solution of problems for the heat-flow equation and also for the wave equation can be found, for example, in the book by H. S. Carslaw and J. C. Jaeger, *Operational methods in Applied Mathematics*, 2nd ed., London, Oxford University Press, 1948.

3. Methods of Finite Integral Transformations

Suppose that $f(x)$ is a function defined in a finite interval (a, b). An expression of the form

$$\tilde{f}(\lambda) = \int_a^b K(x, \lambda)\,\rho(x)\,f(x)\,dx, \tag{61}$$

where $K(x, \lambda)$ is a function of x depending on a parameter λ, where λ assumes a discrete set of values, and where $\rho(x)$ is a function of x that is continuous and positive on the interval $[a, b]$, is called a *finite integral transformation* of the function $f(x)$ with *kernel* $K(x, \lambda)$ and with *weight* $\rho(x)$.*

The following are examples of finite integral transformations:

The *finite Fourier sine transformation*

$$\tilde{f}(\lambda) = \int_0^{\pi} \sin \lambda x f(x)\,dx, \tag{62}$$

the *finite Fourier cosine transformation*

$$\tilde{f}(\lambda) = \int_0^{\pi} \cos \lambda x f(x)\,dx, \tag{63}$$

the *finite Hankel transformation*

$$\tilde{f}(\lambda) = \int_0^1 J_\nu(\lambda x)\,x f(x)\,dx, \tag{64}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν (for $\nu \geqslant 0$), the *finite Legendre transformation*

$$\tilde{f}(\lambda) = \int_{-1}^1 P_\lambda(x)\,f(x)\,dx, \tag{65}$$

where $P_\lambda(x)$ is the Legendre polynomial of degree λ, with λ taking the values 0, 1, 2, ...

To arrive at the methods of finite integral transformations in the most natural manner and to make clear the basic idea behind them in as complete a manner as possible in the most general case, let us consider the boundary-value problem for the equation

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* We introduce the concept of integral transformations with weight $\rho(x)$ in connection with the necessity of using them in subsequent calculations.

$$Lu \pm \rho \frac{\partial^k u}{\partial t^k} = \rho f(x, t), \quad (a < x < b, t > 0, k = 1, 2), \tag{66}$$

where

$$Lu = \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) - qu \tag{66'}$$

with the boundary conditions

$$\left.\begin{aligned} R_0'(u) &= \sqrt{p(a)}\left[u \cos\alpha + \frac{\partial u}{\partial x} \sin\alpha\right]_{x=a} = \varphi_0(t) \\ R_1'(u) &= \sqrt{p(b)}\left[u \cos\beta + \frac{\partial u}{\partial x} \sin\beta\right]_{x=b} = \varphi_1(t) \end{aligned}\right\} \tag{67}$$

and the initial conditions

$$\left.\begin{aligned} u|_{t=0} &= \psi_0(x), \quad \text{if} \quad k = 1, \\ u|_{t=0} &= \psi_0(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x), \quad \text{if} \quad k = 2. \end{aligned}\right\} \tag{68}$$

Here, p, q, and ρ are given functions of x that are continuous on the interval $[a, b]$. On this interval, the derivative dp/dx is assumed continuous and p and ρ are assumed positive in the interval (a, b). Furthermore, $f(x, t)$, $\varphi_0(t)$, $\varphi_1(t)$, $\psi_0(x)$, and $\psi_1(x)$ are given continuous functions of their arguments in the corresponding closed regions, and α and β are constants such that $0 \leqslant \alpha$ and $\beta < 2\pi$.*

If we multiply both sides of Eq. (66) by $K(x, \lambda)$ and integrate with respect to x from a to b, we obtain

$$\int_a^b K(x, \lambda) Lu(x, t)\, dx \pm \frac{\partial^k \tilde{u}(\lambda, t)}{\partial t^k} = \tilde{f}(\lambda, t), \tag{69}$$

where $\tilde{u}(\lambda, t)$ and $\tilde{f}(\lambda, t)$ are finite integral transforms with kernel $K(x, \lambda)$ and weight $\rho(x)$ of the functions $u(x, t)$ and $f(x, t)$.

Integrating by parts, we obtain

$$\int_a^b K(x, \lambda) \frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) dx = K(x, \lambda) p \frac{\partial u}{\partial x}\Big|_a^b - \int_a^b p \frac{\partial u}{\partial x} \frac{\partial K(x, \lambda)}{\partial x} dx =$$

$$= \left[K(x, \lambda) p \frac{\partial u}{\partial x} - \frac{\partial K(x, \lambda)}{\partial x} pu\right]_a^b + \int_a^b u \frac{\partial}{\partial x}\left(p \frac{\partial K(x, \lambda)}{\partial x}\right) dx.$$

* We mention that for $k = 2$ and a plus sign in front of $\partial^2 u/\partial t^2$, the initial conditions (68) may be replaced by boundary conditions of some kind for $t = 0$ and $t = T$, where T is a given positive number.

Therefore,

$$\int_a^b K(x, \lambda) Lu(x, t)\, dx = \left[p\left(K(x, \lambda) \frac{\partial u}{\partial x} - \frac{\partial K(x, \lambda)}{\partial x} u \right) \right]_a^b + \\ + \int_a^b u(x, t) LK(x, \lambda)\, dx. \tag{70}$$

Let us now look at the Sturm–Liouville problem

$$\frac{d}{dx}\left(p \frac{dU}{dx} \right) - qU + \lambda \rho U = 0 \tag{71}$$

$$\left.\begin{aligned} R_0' U &= \sqrt{p(a)}\,[U \cos \alpha + U' \sin \alpha]_{x=a} = 0, \\ R_1' U &= \sqrt{p(b)}\,[U \cos \beta + U' \sin \beta]_{x=b} = 0. \end{aligned}\right\} \tag{72}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n \ldots$ denote the eigenvalues of the problem (71)–(72) and let $U_1(x), U_2(x), \ldots, U_n(x), \ldots$ denote the eigenfunctions corresponding to them. Let us take $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ for the discrete set of values of the parameter λ in the finite integral transformation (61) and let us take $K(x, \lambda_n) = U_n(x)$ (for $n = 1, 2, \ldots$). Then, Eq. (70) takes the form

$$\int_a^b K(x, \lambda_n) Lu(x, t)\, dx = -\lambda_n \tilde{u}(\lambda_n, t) + \left[p\left(U_n \frac{\partial u}{\partial x} - \frac{\partial U_n}{\partial x} u \right) \right]_a^b. \tag{70'}$$

By virtue of the boundary conditions (72), we can write

$$\left.\begin{aligned} \sqrt{p(a)}\, U_n(a) &= A_n \sin \alpha, \quad \sqrt{p(a)}\, U_n'(a) = -A_n \cos \alpha, \\ \sqrt{p(b)}\, U_n(b) &= B_n \sin \beta, \quad \sqrt{p(b)}\, U_n'(b) = -B_n \cos \alpha, \end{aligned}\right\} \tag{73}$$

where the A_n and B_n are nonzero constants, which we shall assume known since they are expressed in terms of the values of the eigenfunctions or their derivatives at the end point of the interval $[a, b]$. If we substitute (73) into (70′) and take account of the boundary conditions (67), we obtain

$$\int_a^b K(x, \lambda_n) Lu(x, t)\, dx = -\lambda_n \tilde{u}(\lambda_n, t) + B_n \varphi_1(t) - A_n \varphi_0(t). \tag{70''}$$

If we substitute (70″) into (69), we arrive at the ordinary differential equation

$$\pm \frac{d^k \widetilde{u}(\lambda_n, t)}{dt^k} - \lambda_n \widetilde{u}(\lambda_n, t) = \widetilde{f}(\lambda_n, t) + A_n \varphi_0(t) - B_n \varphi_1(t) \quad (k = 1, 2). \tag{74}$$

The initial conditions (68) yield

$$\left.\begin{aligned} &\widetilde{u}(\lambda_n, 0) = \int_a^b K(x, \lambda_n)\, \rho(x)\, \psi_0(x)\, dx = C_n, \quad \text{if} \quad k = 1, \\ &\widetilde{u}(\lambda_n, 0) = C_n, \left.\frac{d\widetilde{u}(\lambda_n, t)}{dt}\right|_{t=0} = \int_a^b K(x, \lambda_n)\, \rho(x)\, \psi_1(x)\, dx = D_n, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad k = 2, \end{aligned}\right\} \tag{75}$$

where C_n and D_n are known constants. Thus, the finite integral transformation

$$\widetilde{u}(\lambda_n, t) = \int_a^b K(x, \lambda_n)\, \rho(x)\, u(x, t)\, dx \quad (n = 1, 2, \ldots) \tag{61'}$$

is completely defined as the solution of the ordinary differential equation (74) with initial conditions (75). It is convenient to obtain this solution in explicit form, for example, by the method of operational calculus or by some other method.

The question now arises as to how we can find the function $u(x, t)$ representing the solution of the boundary-value problem (66)–(68) if we know its finite integral transform (61) or (61'). In other words, we need to find the inverse formula for this finite integral transformation. To answer this question, we recall that the eigenfunctions $U_1(x), U_2(x), \ldots, U_n(x), \ldots$ of the Sturm–Liouville problem (71), (72) are orthogonal with weight $\rho(x)$ and that they constitute a complete system of functions on the interval $[a, b]$.* This means that, for every function $F(x)$ which is square integrable on the interval $[a, b]$,

$$F(x) = \sum_{n=1}^{\infty} a_n U_n(x), \tag{76}$$

where

$$a_n = \frac{1}{\int_a^b U_n^2(x)\, \rho(x)\, dx} \int_a^b F(x)\, U_n(x)\, \rho(x)\, dx, \tag{77'}$$

*This last assertion was proven by us in Chapter VI under the condition that $p(a) \neq 0$ and $p(b) \neq 0$, but it remains valid in all the particular cases that we shall encounter even when this condition is not satisfied.

Equation (76) should be understood in the sense that the partial sums on the right converge to the function $F(x)$ in the mean square.

Equations (76) and (76′) for the function $u(x, t)$ may be written in the form

$$u(x, t) = \sum_{n=1}^{\infty} c_n K(x, \lambda_n), \tag{77}$$

$$c_n = \frac{1}{\int_a^b K^2(x, \lambda_n)\rho(x)\,dx} \int_a^b u(x, t) K(x, \lambda_n)\rho(x)\,dx. \tag{77′}$$

Thus, we obtain a formula for the inverse of the finite integral transformation (61) or (61′) in the form

$$u(x, t) = \sum_{n=1}^{\infty} \frac{\tilde{u}(\lambda_n, t)}{\int_a^b K^2(x, \lambda_n)\rho(x)\,dx} K(x, \lambda_n). \tag{78}$$

The function $u(x, t)$ defined by this series is the desired solution of the boundary-value problem (66)–(68) whenever the given series admits two termwise differentiations with respect to x and t.

This answers the question of the solution of the boundary-value problem (66)–(68) by the method of finite integral transformations and of the construction of inverse formulas for the finite integral transformations.

Let us consider a few examples.

1. Let $Lu = \partial^2 u/\partial x^2$, $\rho(x) = 1$, $a = 0$, $b = \pi$. In this case, the Sturm–Liouville problem (71), (72) takes the form

$$\frac{d^2U}{dx^2} + \lambda U = 0, \tag{79}$$

$$[U\cos\alpha + U'\sin\alpha]_{x=0} = 0, \quad [U\cos\beta + U'\sin\beta]_{x=\pi} = 0. \tag{80}$$

If $\cos\alpha = 1$ and $\cos\beta = 1$, i.e., in the case of boundary conditions of the first kind, we have $\lambda_n = n^2$ and $U_n(x) = \sin nx$. The finite integral transformation (61) coincides with the finite Fourier sine transformation

$$\tilde{u}(n^2, t) = \int_0^{\pi} \sin nx\, u(x, t)\,dx \quad (n = 1, 2, \ldots), \tag{81}$$

and the inverse formula (78) which gives the solution of the problem (66)–(68), in this case takes the form

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \tilde{u}(n^2, t) \sin nx. \tag{81'}$$

If $\sin \alpha = 1$ and $\sin \beta = 1$, i.e., in the case of boundary conditions of the second kind, we have $\lambda_n = n^2$ and $U_n(x) = \cos nx$ (for $n = 0, 1, \ldots$). Then, (61) coincides with the Fourier cosine transformation

$$\tilde{u}(n^2, t) = \int_0^{\pi} \cos nx\, u(x, t)\, dx. \tag{82}$$

The inverse formula (78), giving the solution of the problem (66)–(68) now takes the form

$$u(x, t) = \frac{1}{\pi} \tilde{u}(0, t) + \frac{2}{\pi} \sum_{n=1}^{\infty} \tilde{u}(n^2, t) \cos nx. \tag{82'}$$

In the most general case of the boundary conditions (80), the eigenfunctions of the Sturm–Liouville problem (79), (80) can be written in the form

$$U_n(x) = \cos(\lambda_n x + \delta_n) \qquad (n = 1, 2, \ldots),$$

where the λ_n and δ_n are constants determined by the boundary conditions (80), i.e., they represent all possible roots of the system of equations

$$\left.\begin{aligned} &\cos\delta \cos\alpha - \lambda \sin\delta \sin\alpha = 0, \\ &\cos(\lambda\pi + \delta) \cos\beta - \lambda \sin(\lambda\pi + \delta) \sin\beta = 0. \end{aligned}\right\} \tag{83}$$

In this case, the finite integral transformation (61) takes the form

$$\tilde{u}(\lambda_n, t) = \int_a^b \cos(\lambda_n x + \delta_n)\, u(x, t)\, dx \quad (n = 1, 2, \ldots), \tag{83'}$$

where the λ_n and δ_n are roots of the system of equations (83) and the inverse formula (78) takes the form

$$u(x, t) = \sum_{n=1}^{\infty} \frac{\tilde{u}(\lambda_n, t)}{\int_a^b \cos^2(\lambda_n x + \delta_n)\, dx} \cos(\lambda_n x + \delta_n). \tag{83''}$$

2. Let

$$Lu = \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) - \frac{\nu^2}{x} u,$$

where ν is a real constant, $\rho(x) = x$, $a = 0$, $b = 1$. In this case, the Sturm–Liouville problem (71), (72) takes the form

$$\frac{d}{dx}\left(x\frac{dU}{dx}\right) - \frac{\nu^2}{x}U + \lambda x U = 0, \tag{84}$$

$$[\sqrt{x}(U\cos\alpha + U'\sin\alpha)]_{x=0} = 0, \quad [U\cos\beta + U'\sin\beta]_{x=1} = 0. \tag{85}$$

Multiplication of Eq. (84) by x and performing the differentiation yields

$$x^2\frac{d^2U}{dx^2} + x\frac{dU}{dx} - (\nu^2 - \lambda x^2)U = 0. \tag{84'}$$

If we make the change of variable $y = x\sqrt{\lambda}$ we get Bessel's equation of order ν:

$$y^2\frac{d^2U}{dy^2} + y\frac{dU}{dy} - (\nu^2 - y^2)U = 0. \tag{84''}$$

Consequently, one solution of (84) is the Bessel function of the first kind of order ν:

$$U(x) = J_\nu(x\sqrt{\lambda}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\nu)}\left(\frac{x\sqrt{\lambda}}{2}\right)^{2n+\nu}. \tag{86}$$

Suppose that $\sin\alpha = 0$. Then, the function (86) satisfies the first of boundary conditions (85). Let us consider the two possibilities:

(a) $\sin\beta = 0$: Here, the eigenvalues of the problem (84), (85), namely, $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are defined as the squares of the non-negative roots of the equation

$$J_\nu(\mu) = 0; \tag{87}$$

The eigenfunctions corresponding to them are

$$U_n(x) = J_\nu(x\sqrt{\lambda_n}). \tag{88}$$

In this case, the finite integral transformation (61) coincides with the finite Hankel transformation

$$\tilde{u}(\lambda_n, t) = \int_0^1 J_\nu(x\sqrt{\lambda_n})\, x u(x, t)\, dx \quad (n = 1, 2, \ldots). \tag{89}$$

And the inverse formula (78), giving the solution of the problem (66)–(68) is the expansion of the function $u(x, t)$ in a Fourier–Bessel series.

$$u(x, t) = \sum_{n=1}^{\infty} \frac{\widetilde{u}(\lambda_n, t)}{\int_0^1 J_\nu^2(x\sqrt{\lambda_n})\, x\, dx} J_\nu(x\sqrt{\lambda_n}); \tag{90}$$

(b) $\sin\beta \neq 0$: Here, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ of the problem (84)–(85) are defined as the squares of the nonnegative roots of the equation

$$J_\nu(\mu)\cos\beta + J_\nu'(\mu)\sin\beta = 0, \tag{91}$$

and the eigenfunctions are

$$U_n(x) = J_\nu(x\sqrt{\lambda_n}) \quad (n = 1, 2, \ldots). \tag{92}$$

The finite integral transformation takes the form (89) and the inverse formula takes the form (90). The only thing that is different from case (a) is the fact that the eigenvalues λ_n (for $n = 1, 2, \ldots$) are now defined by Eq. (91).

Let us now suppose that $\sin\alpha \neq 0$. In this case, the situation will be the same as in the case of $\sin\alpha = 0$ if we impose on the constant ν the restriction that the first of boundary conditions (85) be satisfied by the function (86) automatically. For example, it will be sufficient to assume that $\nu \geqslant 0$ but that ν does not assume values in the half-open interval $0 < \nu \leqslant 1/2$.

We note that if $a \neq 0$ (b arbitrary), the Sturm–Liouville problem (84)–(85) takes the same form except that the boundary conditions are given for $x = a$ and $x = b$. The eigenfunctions represent definite combinations of νth-order Bessel functions of the first and second kinds. Otherwise, the determination of the eigenvalues, the construction of the finite transformation, and its inverse formulas giving the solution of the problem (66)–(68) remain the same as in the preceding cases.

3. Let

$$Lu = \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right],$$

with $\rho(x) = 1$, $a = -1$, and $b = 1$. In this case, the Sturm–Liouville problem (71), (72) takes the form

$$\frac{d}{dx}\left[(1 - x^2)\frac{dU}{dx}\right] + \lambda U = 0, \tag{93}$$

$$\left.\begin{aligned} &[\sqrt{1 - x^2}\,(U\cos\alpha + U'\sin\alpha)]_{x=-1} = 0, \\ &[\sqrt{1 - x^2}\,(U\cos\beta + U'\sin\beta)]_{x=1} = 0. \end{aligned}\right\} \tag{94}$$

If we use the differential equation for the Legendre polynomials (see Chapter VI, Sect. 4, subsection 1), we see easily that the eigenvalues of the problem (93), (94) are determined by the equations

$$\lambda_n = n(n+1) \quad (n = 0, 1, \ldots), \tag{95}$$

and that the eigenfunctions corresponding to them are the Legendre polynomials

$$U_n(x) = P_n(x) \quad (n = 0, 1, \ldots). \tag{96}$$

Therefore, the finite integral transformation in the given case is found to be the finite integral Legendre transformation

$$\tilde{u}(n(n+1), t) = \int_{-1}^{1} P_n(x)\, u(x, t)\, dx, \tag{97}$$

and the inverse formula, which in this case gives the solution of the problem (66)–(68), coincides with the expansion of the function $u(x, t)$ in a series of Legendre polynomials

$$u(x, t) = \sum_{n=0}^{\infty} \frac{\tilde{u}(n(n+1), t)}{\int_{-1}^{1} P_n^2(x)\, dx} P_n(x) \tag{98}$$

or

$$u(x, t) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1)\, \tilde{u}(n(n+1), t)\, P_n(x). \tag{98'}$$

Here, we shall confine ourselves to a consideration of examples of the construction of finite integral transformations and their inverse formulas. However, it is easy to see that the number of such examples can be considerably extended by using this method, which is based on examination of the generalized Sturm–Liouville problem. We note that in many cases, when there are several space coordinates in the differential equation, it is expedient to make several successive finite integral transformations, one for each of the space coordinates. Several examples of repeated application of the Fourier cosine and sine transformations may be found in the book by C. J. Tranter, *Integral Transforms in Mathematical Physics*, 2nd ed., London, Methuen and New York, Wiley, 1956 and in the book by I. N. Sneddon, *Fourier Transforms*, New York, McGraw-Hill, 1951. These books also contain a rather extensive bibliography on the methods of integral transformations.

In conclusion, we note that the methods of integral transformations deal with methods that were developed in mathematical physics rather recently (cf. the book by Tranter referred to). The possibilities of these methods have not yet been studied to the same extent as have, for example, the possibilities of the method of separation of variables. Furthermore, the construction of integral transformations may be done not just with the purpose of eliminating differentiation with respect to one of the variables from a differential equation, as we have done in this chapter, but also with the purpose of simplifying this operation. For example, this may be done for the integral transformation*

$$u(x, y) + iv(x, y) = \frac{2}{\pi}\left\{\widetilde{u}(0, y) + \right.$$

$$\left. + \int_0^x \left[x \frac{\partial \widetilde{u}(\xi, y)}{\partial \xi} + i \frac{\partial \widetilde{v}(\xi, y)}{\partial \xi}\right] \frac{d\xi}{\sqrt{x^2 - \xi^2}}\right\}. \tag{99}$$

The integral operator (99) maps the functions $\widetilde{u}(x, y)$ and $\widetilde{v}(x, y)$, which satisfy the system of equations

$$\frac{\partial \widetilde{u}}{\partial x} = \frac{1}{x} \frac{\partial \widetilde{v}}{\partial y}, \quad \frac{\partial \widetilde{u}}{\partial y} = -\frac{1}{x} \frac{\partial \widetilde{v}}{\partial x}$$

and the boundary condition $\widetilde{v}|_{x=0} = 0$, into the functions $u(x, y)$ and $v(x, y)$, which satisfy the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and the boundary condition $v|_{x=0} = 0$. The inverse of the integral transformation (99) is given by the equation

$$\widetilde{u}(x, y) + i\widetilde{v}(x, y) = \int_0^x [u(\xi, y) + i\xi v(\xi, y)] \frac{d\xi}{\sqrt{x^2 - \xi^2}}. \tag{100}$$

In particular, the integral transformation (99) transforms three-dimensional harmonic functions $\widetilde{u}(x, y)$ possessing axial symmetry (about the y-axis) into harmonic functions of two variables $u(x, y)$ in the xy-plane ($x > 0$).

*See G. M. Polozhiy, "Pro odne integral'ne peretvorennya uzagal'nenykh analitychnykh funktsii" (On an integral transformation of generalized analytic functions), *Visnyk Kyyivs'kogo universytetu*, No. 2, 1959, seriya astronoiyi, matematyky ta mekhaniky, part I.

Furthermore, suppose that $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ that satisfies the condition $v|_{x=0} = 0$ (in the region in question) and suppose that

$$\tilde{f}(z) = \tilde{u}(x, y) + i\tilde{v}(x, y) = \frac{1}{2}\int_{-\bar{z}}^{z} f(\zeta)(x^{1-k} + \zeta - iy)(z-\zeta)^{\frac{k}{2}-1} \times$$

$$\times (\bar{z} + \zeta)^{\frac{k}{2}-1} d\zeta, \tag{101}$$

where k is a positive number, where $\zeta = \xi + i\eta$, where the integration with respect to ζ from the point $-\bar{z}$ to the point z is taken over an arbitrary path joining these two points that lies in the region of analyticity of the function $f(z)$.*

Then, the function $\tilde{f}(z)$ defined by Eq. (101) is the function x^k, which is analytic in the region in question, i.e.,

$$\frac{\partial \tilde{u}}{\partial x} = \frac{1}{x^k}\frac{\partial \tilde{v}}{\partial y}, \quad \frac{\partial \tilde{u}}{\partial y} = -\frac{1}{x^k}\frac{\partial \tilde{v}}{\partial x} \tag{102}$$

and satisfies the condition $\tilde{v}|_{x=0} = 0$.

Inverse formulas for the integral transformation (101) are: For k equal to an even integer,

$$f(z) = u(x, y) + iv(x, y) = \frac{2}{\Gamma(m)}\left\{x\frac{d^m}{(dx^2)^m}[\tilde{u}(x, y)\, x^{2m-1}] + \right.$$

$$\left. + i\frac{d^m}{(dx^2)^m}\tilde{v}(x, y)\right\}, \tag{103}$$

for k not equal to an even integer,

$$f(z) = u(x, y) + iv(x, y) =$$

$$= \frac{2}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m - \frac{k}{2} + 1\right)}\left\{\frac{d}{dx}\int_0^x \frac{d^m\tilde{u}(\xi, y)\,\xi^{k-1}}{(d\xi^2)^m}(x^2 - \xi^2)^{m-\frac{k}{2}}\xi\, d\xi + \right.$$

$$\left. + i2\int_0^x \frac{d^{m+1}\tilde{v}(\xi, y)}{(d\xi^2)^{m+1}}(x^2 - \xi^2)^{m-\frac{k}{2}}\xi\, d\xi\right\} \tag{104}$$

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*See G. N. Polozhiy, "O predel'nykh znacheniyakh i formulakh obrashcheniya vdol' razrezov osnovnogo integral'nogo predstavleniya p-analitichnykh funktsiy s kharakteristikoy $p = x^k$" (On the limiting values and inverse formulas along branch cuts in the basic integral representation of p-analytic functions with characteristic $p = x^k$), Parts I and II, *Ukrainskiy matematicheskiy zhurnal*, 16, No. 5, 6, 1964.

$$f(z) = u(x, y) + iv(x, y) = \frac{4}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(m - \frac{k}{2} + 1\right)} \times$$

$$\times \left\{ \frac{d^m \widetilde{u}(\xi, y)\, \xi^{k-1}}{(d\xi^2)^m} \Bigg|_{\xi=0} \frac{x^{2m-k+1}}{2} + \int_0^x \frac{d^{m+1}}{(d\xi^2)^{m+1}} [x\widetilde{u}(\xi, y)\, \xi^{k-1} + \right.$$

$$\left. + i\widetilde{v}(\xi, y)] \cdot (x^2 - \xi^2)^{m-\frac{k}{2}} \xi\, d\xi \right\}, \qquad (104')$$

where m is the integral part of the number $k/2$.

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